Section 2.2 Review

In this section we discussed how to find rates and rate formulas when given a function. You need to be able to work with functional notation well to understand this section. Make sure you check out my Functional notation review (posted earlier in the quarter) for more practice with functional notation.

Rates from Functions:

1. Given a function $f(x)$, recall that we defined:

$$\frac{f(x)-f(0)}{x} = \text{'overall rate of change of } f \text{ from 0 to } x\text{'.}$$

Note in many cases, $f(0)=0$, and in other cases, we want the slope of the diagonal line. So we also have

$$\frac{f(x)}{x} = \text{'slope of the diagonal line to } f \text{ at } x\text{.' (Examples: } ATS,\ AR,\ AC,\ AVC)$$

$$\frac{f(b)-f(a)}{b-a} = \text{ rate of change from } a \text{ to } b. \text{ (Examples: } AS,\ MR,\ MC)$$

Intervals can be described in a few different ways, but we always just replace $a =$ START, and $b =$ END. For example,

'the interval of length h that starts at x' would be $a = x$ to $b = x+h$.

2. Overall Rate Examples:

$D(t) = 3t - 10t^2 =$ Distance	$ATS(t) = \frac{D(t)}{t} = \frac{3t-10t^2}{t} = 3 - 10t =$ Average Trip Speed
$R(x) = 10x + 3x^2 =$ Total Revenue	$AR(x) = \frac{TR(x)}{x} = \frac{10x+3x^2}{x} = 10 + 3x =$ Average Revenue
$C(x) = 4 - 7x + 30x^2 =$ Total Cost	$AC(x) = \frac{TC(x)}{x} = \frac{4-7x+30x^2}{x} = \frac{4}{x} - 7 + 30x =$ Average Cost
$V(x) = -7x + 30x^2 =$ Variable Cost	$AVC(x) = \frac{VC(x)}{x} = \frac{-7x+30x^2}{x} = -7 + 30x =$ Average Variable Cost

3. Incremental Rates: There is more algebraic manipulation in finding an incremental rate. It takes about 5-6 lines of work. But it is always the same steps and with a bit of care and practice you can quickly become good at working with functions in this way. The most common rate we need to find is

$$\frac{f(x+h)-f(x)}{h} = \text{'the rate of change of } f \text{ from } x \text{ to } x+h\text{'}$$

Let's do a detailed example: Consider the function $f(x) = 5 + 4x + 3x^2$. Let's find and simplify $\frac{f(x+h)-f(x)}{h}$ for this function.

(a) First, do the substitution and simplify $f(x+h)$.

$f(x+h)$	$= 5 + 4(x+h) + 3(x+h)^2$	
$f(x+h)$	$= 5 + 4x + 4h + 3(x^2 + 2xh + h^2)$	expanding
$f(x+h)$	$= 5 + 4x + 4h + 3x^2 + 6xh + 3h^2$	still expanding

(b) Second, substitute into the full expression and simplify.

$\frac{f(x+h)-f(x)}{h}$	$= \frac{(5+4x+4h+3x^2+6xh+3h^2)-(5+4x+3x^2)}{h}$	using the previous part and the function def'n
$\frac{f(x+h)-f(x)}{h}$	$= \frac{5+4x+4h+3x^2+6xh+3h^2-5-4x-3x^2}{h}$	drop the parentheses and distribute the negative.
$\frac{f(x+h)-f(x)}{h}$	$= \frac{4h+6xh+3h^2}{h}$	cancel terms in the numerator.
$\frac{f(x+h)-f(x)}{h}$	$= 4 + 6x + 3h$	cancel the h from the denominator.

So if $f(x) = 5 + 4x + 3x^2$, then we just found that $\frac{f(x+h)-f(x)}{h} = 4 + 6x + 3h$ is the average rate from x to $x+h$.

Let's see another example:
Assume $R(x) = 42x - x^2$ is the formula for total revenue and assume x is in hundreds of items and revenue is in hundreds of items. Remember, in this situation, that marginal revenue is defined by $MR(x) = \frac{TR(x+0.01)-TR(x)}{0.01}$. Let's compute this new formula using algebra.

1. First, do the substitution and simplify $R(x + 0.01)$.

$R(x+0.01)$	$= 42(x+0.01) - (x+0.01)^2$	
$R(x+0.01)$	$= 42x + 0.42 - (x^2 + 0.02x + 0.0001)$	expanding
$R(x+0.01)$	$= 42x + 0.42 - x^2 - 0.02x - 0.0001$	still expanding

2. Second, substitute into the full expression and simplify.

$\frac{R(x+0.01)-R(x)}{0.01}$	$= \frac{(42x+0.42-x^2-0.02x-0.0001)-(42x-x^2)}{0.01}$	using the previous part and the function def'n
$\frac{R(x+0.01)-R(x)}{0.01}$	$= \frac{42x+0.42-x^2-0.02x-0.0001-42x+x^2}{0.01}$	drop parentheses and distribute negative.
$\frac{R(x+0.01)-R(x)}{0.01}$	$= \frac{0.42-0.02x-0.0001}{0.01}$	cancel terms in the numerator.
$\frac{R(x+0.01)-R(x)}{0.01}$	$= 42 - 2x - 0.01$	dividing by 0.01.
$\frac{R(x+0.01)-R(x)}{0.01}$	$= 41.99 - 2x$	simplify.

So if $R(x) = 42x - x^2$, then we just found that $MR(x) = 41.99 - 2x$.
(Aside just for your own intersest: In Math 112, you'll learn methods to quickly get a good approximation to this formula. Those shortcuts will give an answer of $42 - 2x$ which is very close to what we got. More to come in Math 112.)

Additional Exercises (Try these problems on your own, answers are on the next page):

1. Let $D(t) = 1 + 2t^2$ (t in hours and D in miles). Find and simplify the formula for the average speed over a 2-hour interval starting at t.
2. Let $C(x) = 2 - 3x + x^2$ (x is in Items and C is in dollars). Find formulas for AC and MC.
3. Let $f(x) = 3x + 5x^2$. Find and simplify $\frac{f(x+h)-f(x)}{h}$.
4. Let $g(x) = 3 - 6x$. Find and simplify $\frac{g(x+h)-g(x)}{h}$.

1. We are given $D(t) = 1 + 2t^2$ and we want to find $\frac{D(t+2)-D(t)}{2}$. Here I do all the work together:

$\frac{D(t+2)-D(t)}{2} = \frac{(1+2(t+2)^2)-(1+2t^2)}{2}$	using the function def'n
$\frac{D(t+2)-D(t)}{2} = \frac{(1+2(t^2+4t+4))-(1+2t^2)}{2}$	expanding
$\frac{D(t+2)-D(t)}{2} = \frac{1+2t^2+8t+8-1-2t^2}{2}$	still expanding and dropping the paranthesis (dist. negative!)
$\frac{D(t+2)-D(t)}{2} = \frac{8t+8}{2}$	cancel terms in the numerator
$\frac{D(t+2)-D(t)}{2} = 4t + 4$	simplify

2. We are given $C(x) = 2 - 3x + x^2$ and we want $AC(x) = \frac{TC(x)}{x}$ and $MC(x) = \frac{TC(x+1)-TC(x)}{1}$.

$AC(x) = \frac{2-3x+x^2}{x}$	using the function def'n
$AC(x) = \frac{2}{x} - 3 + x$	simplifying

$MC(x) = \frac{(2-3(x+1)+(x+1)^2)-(2-3x+x^2)}{1}$	using the function def'n
$MC(x) = \frac{(2-3x-3+x^2+2x+1)-(2-3x+x^2)}{1}$	expanding
$MC(x) = \frac{2-3x-3+x^2+2x+1-2+3x-x^2}{1}$	dropping the paranthesis (dist. negative!)
$MC(x) = \frac{-3+2x+1}{1}$	cancel terms in the numerator
$MC(x) = 2x - 2$	simplify

3. We are given $f(x) = 3x + 5x^2$ and we want to find $\frac{f(x+h)-f(x)}{h}$. Here I do all the work together:

$\frac{f(x+h)-f(x)}{h} = \frac{(3(x+h)+5(x+h)^2)-(3x+5x^2)}{h}$	using the function def'n
$\frac{f(x+h)-f(x)}{h} = \frac{(3x+3h+5(x^2+2xh+h^2)-(3x+5x^2)}{h}$	expanding
$\frac{f(x+h)-f(x)}{h} = \frac{3x+3h+5x^2+10xh+5h^2-3x-5x^2}{h}$	still expanding and dropping the paranthesis (dist. negative!)
$\frac{f(x+h)-f(x)}{h} = \frac{3h+10xh+5h^2}{h}$	cancel terms in the numerator
$\frac{f(x+h)-f(x)}{h} = 3 + 10x + 5h$	simplify

4. We are given $g(x) = 3 - 6x$ and we want to find $\frac{g(x+h)-g(x)}{h}$. Here I do all the work together:

$\frac{g(x+h)-g(x)}{h} = \frac{(3-6(x+h))-(3-6x)}{h}$	using the function def'n
$\frac{g(x+h)-g(x)}{h} = \frac{(3-6x-6h)-(3-6x)}{h}$	expanding
$\frac{g(x+h)-g(x)}{h} = \frac{3-6x-6h-3+6x}{h}$	dropping the paranthesis (dist. negative!)
$\frac{g(x+h)-g(x)}{h} = \frac{-6h}{h}$	cancel terms in the numerator
$\frac{g(x+h)-g(x)}{h} = -6$	simplify