

## Several annuity examples from your financial lifetime

**A Student Loan:** Let's say you accumulate \$40,000 in student loans and you have graduated so it is now time to repay the loan. Most student loans are paid over 10 years and the typical rate is around 4.5%, compounded monthly. What will your monthly payments be? (Assume the first payment is at the end of the first month after graduation)

*Answer:* First, note that  $i = \frac{r}{m} = \frac{0.045}{12} = 0.00375$  and there will be  $n = mt = 12 \cdot 10 = 120$  payments. This is an ordinary annuity, present value question (as loans typically are). The balance starts at \$40,000 and we will pay it down to zero. So

$$P = R \frac{1 - (1 + i)^{-n}}{i} \quad \text{becomes} \quad 40,000 = R \frac{1 - (1.00375)^{-120}}{0.00375} \quad \text{which becomes} \quad 40,000 = R \cdot 96.489323986$$

Dividing gives

$$R = \frac{40000}{96.489323986} \approx 414.553635 = \$414.55 \text{ monthly payment for 10 years .}$$

Other notes: In total you will make 120 payments of \$414.55, so

Total Payments =  $R \cdot n = 414.55 \cdot 120 = \$49,746$ , which means you will pay

Total Interest =  $R \cdot n - P = \$9,746$ .

**Saving for retirement:** Your company opens a 401(k) retirement account for you. Today (at the beginning of the first month), you got your first paycheck and it says there will be \$500 removed from your paycheck each month and placed in your 401(k) retirement. You plan to retire in 35 years and your retirement adviser conservative estimates that your 401(k) will earn approximately 6%, compounded monthly. How much will you have saved when you retire?

*Answer:* First, note that  $i = \frac{r}{m} = \frac{0.06}{12} = 0.005$  and there will be  $n = mt = 12 \cdot 35 = 420$  payments. This is an annuity due, future value question. So

$$F = R \frac{(1 + i)^n - 1}{i} (1 + i) \quad \text{becomes} \quad F = 500 \frac{(1.005)^{420} - 1}{0.005} (1.005) = 715916.92513 = \$715,916.93.$$

Other notes: In total you will make 420 payments of \$500, so

Total Payments =  $R \cdot n = 500 \cdot 420 = \$210,000$ , which means you earn

Total Interest =  $F - R \cdot n = \$505,916.93$ .

**Buying a Home:** You want to buy a home. You have saved enough money for a down payment of \$40,000 which your mortgage broker says allows you to get a 30-year loan that has a rate of 5%, compounded monthly. You know that you can only afford to make monthly loan payments of \$2,000. How big of a home loan can you afford (in other words, what is the most expensive home price you can afford)?

*Answer:* First, note that  $i = \frac{r}{m} = \frac{0.05}{12} = 0.004166\dots$  and there will be  $n = mt = 12 \cdot 30 = 360$  payments. This is an ordinary annuity (all loans are, the first payment is made at the END of the first month). And it is a present value question because you will be paying down the balance. So

$$P = R \frac{1 - (1 + i)^{-n}}{i} \quad \text{becomes} \quad P = 2000 \frac{1 - (1.004166\dots)^{-360}}{0.004166\dots} \approx 372,563.2341 = \$372,563.23$$

You can't afford a home that gives you a loan above \$372,563.23 (after your down payment). Thus, the largest home value you can afford is  $\$372,563.23 + \$40,000 = \$412,563.23$ .

Other notes: If you get a loan of this size, then in total you will make 360 payments of \$2000, so Total Payments =  $R \cdot n = 2000 \cdot 360 = \$720,000$ , which means you will pay Total Interest =  $R \cdot n - P = \$346,436.77$ .

**Spending your Retirement Savings:** You are 65 and you just retired. You were able to save \$1,600,000 in your retirement account which earns 5%, compounded monthly. You plan to withdraw \$8,000 at the end of every month for the rest of your life. How old will you be when the money runs out?

*Answer:* First, note that  $i = \frac{r}{m} = \frac{0.05}{12} = 0.004166\dots$  and there will be  $n = mt = 12t$  payments (we don't know  $t$ ). This is a ordinary, present value question. So

$$P = R \frac{1 - (1 + i)^{-n}}{i} \quad \text{becomes} \quad 1,600,000 = 8000 \frac{1 - (1.004166\dots)^{-12t}}{0.004166\dots}$$

Now we solve:

$200 = \frac{1 - (1.004166\dots)^{-12t}}{0.004166\dots}$	divided by 8000
$0.833\dots = 1 - (1.004166\dots)^{-12t}$	multiplied by 0.004166\dots
$-0.166\dots = -(1.004166\dots)^{-12t}$	subtracted 1
$0.166\dots = (1.004166\dots)^{-12t}$	multiplied by -1
$\ln(0.166\dots) = \ln((1.004166\dots)^{-12t})$	took $\ln( )$ of both sides
$\ln(0.166\dots) = -12t \ln(1.004166\dots)$	brought down power
$t = \frac{\ln(0.166\dots)}{-12 \ln(1.004166\dots)} \approx 35.91$	divided

Thus, the money will be gone in 35.91 years, so you will be  $65 + 35.91 = 100.91$  years old (so you'll have enough money to live in this way to about an age of 101).

Other notes: There are  $n = 12t = 12 \cdot 35.91 = 430.92$  or about 431 payments, so Total Payments  $\approx R \cdot n = 8000 \cdot 431 = \$3,448,000$ , which means you earn about Total Interest =  $R \cdot n - P = \$1,848,000$  (over the 36 years you have the account).