Deriving the Annuity Formulas

There are four varients of the annuity formulas that we will discuss in this class. Before we can discuss where they come from, we need the following pattern.

An important Pattern

By expanding you can see:

$$\begin{aligned}(1+x+x^2)(x-1) &= x^3-1\\(1+x+x^2+x^3)(x-1) &= x^4-1\\(1+x+x^2+x^3+x^4)(x-1) &= x^5-1\end{aligned}$$

Dividing by $(x-1)$ in each case gives:

$$\begin{aligned}1+x+x^2 &= \frac{x^3-1}{x-1}\\1+x+x^2+x^3 &= \frac{x^4-1}{x-1}\\1+x+x^2+x^3+x^4 &= \frac{x^5-1}{x-1}\end{aligned}$$

In general, we get what is called the geometric identity

$$1+x+x^2+\cdots+x^n = \frac{x^{n+1}-1}{x-1}.$$

For example: $1+(1.02)+(1.02)^2+\cdots+(1.02)^{20} = \frac{(1.02)^{21}-1}{1.02-1} \approx 25.783.$

SEVERAL EXAMPLES EXPLAINING WHERE ANNUITY FORMULAS COME FROM:

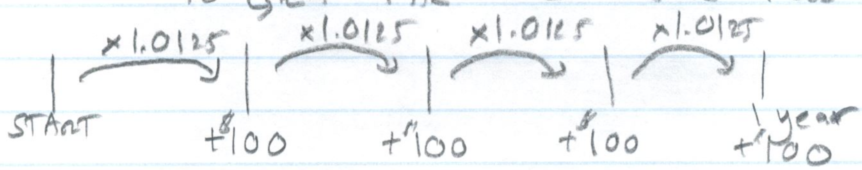
Ex 1) You deposit $100 at the end of each quarter in an account that earns ~~5% annually compounded quarterly.~~

$\boxed{\text{WHAT IS THE BALANCE IN A YEAR?}}$

STEP 1: QUARTERLY RATE $= i = \frac{r}{m} = \frac{0.05}{4} = 0.0125$

SO EVERY QUARTER THE BALANCE IS MULTIPLIED BY $1+i = 1.0125$

TO GET THE NEW VALUE PLUS INTEREST



STEP 2: THERE ARE 4 PAYMENTS OF $100.

LET'S TALK ABOUT EACH ONE:

1st PAYMENT: WILL EARN INTEREST 3 TIMES!
IT WILL GROW TO $100(1.0125)^3$

2nd PAYMENT: WILL EARN INTEREST 2 TIMES!
IT WILL GROW TO $100(1.0125)^2$

3rd PAYMENT: WILL GROW TO $100(1.0125)^1$

4th PAYMENT: NO INTEREST YET WILL JUST BE 100.

$$\text{ANSWER} = 100 + 100(1.0125) + 100(1.0125)^2 + 100(1.0125)^3$$

$$= 100\underbrace{\left(1 + (1.0125) + (1.0125)^2 + (1.0125)^3\right)}_{\text{same as}}$$

$$= 100\,\frac{(1.0125)^4 - 1}{1.0125 - 1} = 100\,\frac{(1.0125)^4 - 1}{0.0125}$$

$$\approx 407.562695$$

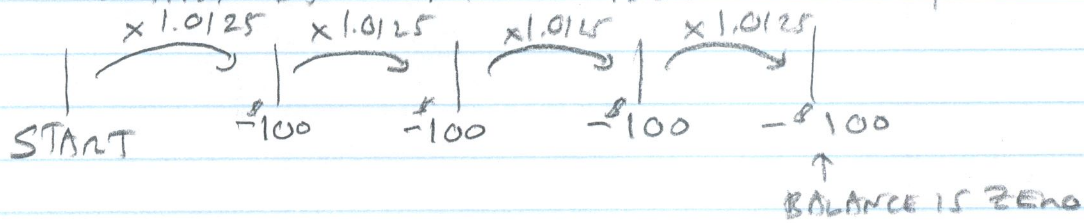
$$= \boxed{\$407.56}$$

Ex 2 You withdraw \$100 at the end of each quarter from an account that earns 5% annually, compounded quarterly AND the balance is zero at the end of the year.

$\boxed{\text{WHAT WAS THE STARTING BALANCE?}}$

STEP 1: Again $i = \frac{r}{m} = \frac{0.05}{4} = 0.0125$.

MULTIPLY BY $1+i = 1.0125$ each quarter



STEP 2: WORK BACKWARD! LET'S TALK ABOUT EACH WITHDRAWAL

1st withdrawl: What amount grew to give \$100?

That is, $100 = P(1.0125)$

So $P = \frac{100}{1.0125} = 100(1.0125)^{-1}$

$\approx \$98.7654$

Thus, $100(1.0125)^{-1} \approx \$98.7654$ of the original balance is what gave the 1st \$100 withdrawl.

2nd withdrawl: $100 = P(1.0125)^2$

So $P = \frac{100}{(1.0125)^2} = 100(1.0125)^{-2}$

$\approx \$97.54611$ became the 2nd withdraw

3rd: $P = 100(1.0125)^{-3}$

4th: $P = 100(1.0125)^{-4}$

TOTAL START BALANCE $= 100(1.0125)^{-1} + 100(1.0125)^{-2} + 100(1.0125)^{-3} + 100(1.0125)^{-4}$

$= 100\left((1.0125)^{-1} + (1.0125)^{-2} + (1.0125)^{-3} + (1.0125)^{-4}\right)$

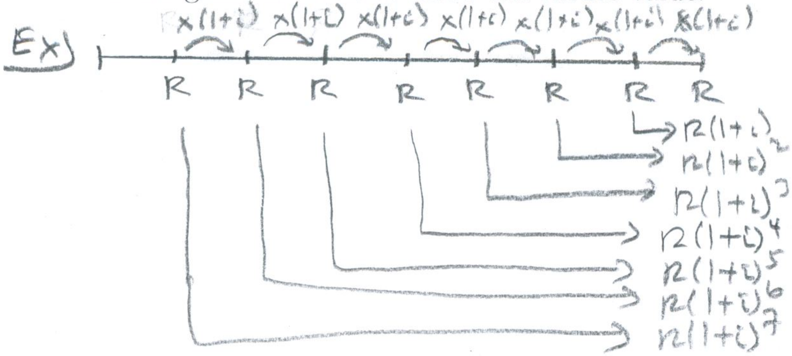
$= 100\left[\frac{1-(1.0125)^{-4}}{0.0125}\right] = 387.805798$

$\boxed{\$387.81}$

$$i = \frac{r}{m}$$

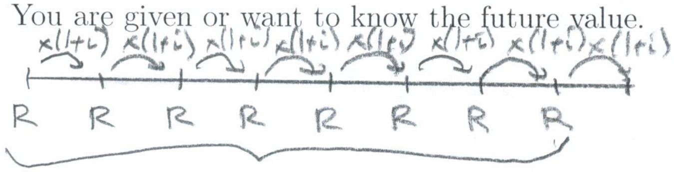
The four scenarios: $n = mt$

1. **Ordinary Annuity Future Value**: Payments at the END of each compounding period. You are given or want to know the future value.



$$\text{TOTAL} = R\left[1+(1+i)+(1+i)^2+\cdots+(1+i)^7\right]$$
$$= R\,\frac{(1+i)^8-1}{(1+i)-1}$$
$$= R\,\frac{(1+i)^8-1}{i}$$
$$\boxed{F = R\left[\frac{(1+i)^n-1}{i}\right]}$$

2. **Annuity Due Future Value**: Payments at the BEGINNING of each compounding period. You are given or want to know the future value.

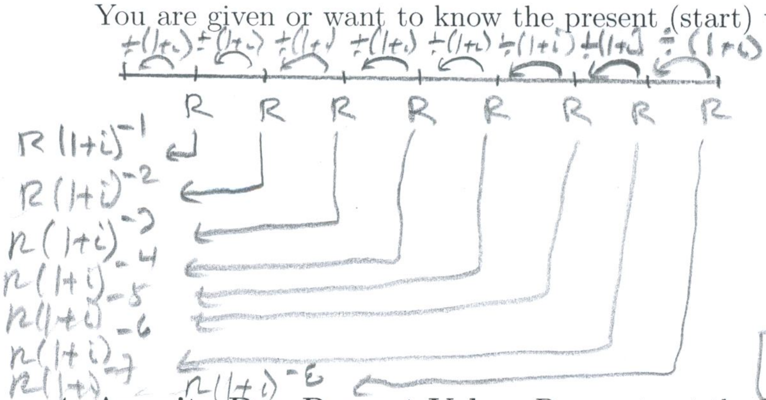


$$\text{TOTAL} = R\,\frac{(1+i)^8-1}{i}\,(1+i)$$

SAME AS ABOVE BUT WITH ONE EXTRA COMPOUNDING PERIOD AT THE END

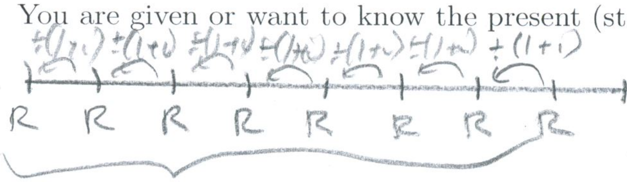
$$\boxed{F = R\left[\frac{(1+i)^n-1}{i}\right](1+i)}$$

3. **Ordinary Annuity Present Value**: Payments at the END of each compounding period. You are given or want to know the present (start) value.



$$\text{TOTAL} = R\left[\frac{1}{1+i}+\frac{1}{(1+i)^2}+\frac{1}{(1+i)^3}+\cdots+\frac{1}{(1+i)^8}\right]$$
$$= \frac{R}{(1+i)}\left(1+(1+i)^{-1}+\left((1+i)^{-1}\right)^2+\cdots+\left((1+i)^{-1}\right)^7\right)$$
$$= \frac{R}{(1+i)}\,\frac{\left((1+i)^{-1}\right)^8-1}{(1+i)^{-1}-1}$$
$$= R\,\frac{(1+i)^{-8}-1}{1-(1+i)} = R\,\frac{1-(1+i)^{-8}}{i}$$
$$\boxed{P = R\left[\frac{1-(1+i)^{-n}}{i}\right]}$$

4. **Annuity Due Present Value**: Payments at the BEGINNING of each compounding period. You are given or want to know the present (start) value.



$$\text{TOTAL} = R\left(\frac{1-(1+i)^{-8}}{i}\right)(1+i)$$

SAME AS ABOVE BUT WITH ONE LESS DIVISION

$$\boxed{P = R\left[\frac{1-(1+i)^{-n}}{i}\right](1+i)}$$