Section 2.3 Review

In this section, we revisit our business applications but now we can use quadratic function and functional notation facts. See the 2.1 review for quadratic facts (*i.e.* solving with the quadratic formula and finding the vertex) and see the 2.2 review for functional notation facts (*i.e.* how to get AR, AC, AVC, MR, and MC from TR and TC).

Applications:

1. **Graph Shape and Vertex Questions**: If a problem asks you about the shape/features of a function, then sketch a picture! (Anything about maximum/minimum or increasing/decreasing) If the function happens to be a quadratic, then you must find and label the **vertex** in order to answer the question. But always sketch a picture before you put your final answer down. Here are examples from homework:
   - Find the maximum revenue ...
     (Sketch a graph of the revenue function, and label the vertex if it's quadratic).
   - Give the longest interval on which total revenue and profit are both increasing.
     (Sketch TR, label the vertex, figure out when it is increasing. Then sketch Profit, label the vertex, figure out when it is increasing.)
   - Find the longest interval on which $f(x) - g(x)$ is increasing.
     (Find the formula for $f(x) - g(x)$, then sketch this new function, label the vertex).
   - Find the maximum profit ...
     (Sketch a graph of the profit function, and label the vertex if it's quadratic).
     Note: You can also find when $MR = MC$.

2. **Solving and Quadratic Equation Questions**: Questions that ask about a particular output value are questions where you must solve an equation. If the equation is quadratic, you can use the quadratic formula. Otherwise, you can solve by getting x by itself on one side. It still is wise to sketch a picture of the function if you don't know how to interpret your answers. Here are examples from homework:
   - Find all values when $g(x) - f(x) = 4$.
     (Set up and solve)
   - For what range of quantities is average variable cost at most \$0.55 per bag?
     (First, solve $AVC(x) = 0.55$, then use a sketch of $AVC(x)$ to figure out what to do with your answers).
   - For what range of quantities is VC less than or equal to TR.
     (First, solve $TR = VC$, then use a sketch of TR and VC to decide what to do with your answers).
   - Find the quantity at which MR exceeds MC by \$4.25.
     (First, give the correct translation into an equation, I'll let you do this. Then solve).
   - Give the largest quantity at which profit is not negative.
     (This is just a fancy way to say, we want to know the largest quantity when profit is greater than or equal to 0. So first solve for when Profit = 0, then sketch a graph to figure out what to do with your numbers).

3. **Special Applications:**
   - The **break even points** are the quantities at which Profit = 0, which is the same as the quantities where $TR(x) = TC(x)$. In these problems you are solving! If it is a quadratic problem, then you will be using the quadratic formula.
   - To find the **quantity at which profit is maximum**:
     METHOD 1: Find the function for Profit. If is is quadratic, then use the vertex formula.
     METHOD 2: Find MR and MC. Then solve MR = MC (if there are multiple intersection points, then you want the answer when it switches from $MR > MC$ to $MR < MC$).
     Note: If a question asks you to find the 'price' that corresponds to maximum revenue or maximum profit, then, at the very end of the problem, you plug the quantity you found into the price formula to get the corresponding price (in such a problem, you would have a price formula at some point earlier in the problem).
   - The **Break Even Price (BEP)** is the lowest y-coordinate of $AC(x)$. It is also the y-coordinate where $AC(x)$ and $MC(x)$ intersect. Typically, this second fact is easier to use. Meaning if you are asked to find the **Break Even Price (BEP)**, then you should solve $AC(x) = MC(x)$ (likely using the quadratic formula). Then plug the value you get for x back into $AC(x)$ or $MC(x)$ to get the y-value (both should give you the same y-value).
   - The **Shutdown Price (SDP)** is the lowest y-coordinate of $AVC(x)$. It is also the y-coordinate where $AVC(x)$ and $MC(x)$ intersect. If $AVC(x)$ is a quadratic, then you can find the y-coordinate of the vertex to get the SDP. Otherwise, you can solve when $AVC(x) = MC(x)$.

OLD EXAM PROBLEMS: The six problems below all come directly from old exams (some midterms and some finals). You should try these completely on your own and you should be able to tell if you are right. Full answers are on the next page, but if you peek you are wasting this great opportunity to practice. Do them on your own first!

1. Suppose you are given $f(x) = x^2 - 10x + 36$ and $g(x) = -0.25x^2 + 4x + 24$.
   (a) Find the values of x at which the graphs cross.
   (b) What is the largest value of $g(x) - f(x)$?
   (c) Find the largest interval on which $f(x)$ and $g(x) - f(x)$ are both increasing.
   (d) Compute and simplify $\frac{f(8+h)-f(8)}{h}$

2. You sell Trinkets and you have the following information:
   - For an order of q Trinkets, the selling price (in dollars per Trinket) is given by $p = -16q + 1000$.
   - Total cost is a **linear** function of quantity q.
   - Fixed costs are \$3484.
   - The total cost to produce 20 Trinkets is \$4004.

   (a) Find formulas for total revenue and total cost.
   (b) For what quantities is profit exactly \$3000.
   (c) What selling price leads to the largest possible profit?
   (d) For what quantity is average cost exactly \$243.75 per Trinket?
   (e) Find formulas for $MR(q)$ and $MC(q)$.

3. You sell calculators. Your fixed costs are \$240, and the **average variable cost** of producing q calculators is given by the function
$$AVC(q) = 0.01q^2 - 0.5q + 26.$$
   (a) Find the **total cost** of producing 30 calculators, $TC(30)$.
   (b) Find a formula for the **average cost** to produce q calculators, $AC(q)$.
   (c) Compute the **shutdown price**.
   (d) For **what value(s)** of q is the average variable cost \$28 per calculator?

4. You make and sell marionettes. The marginal cost and the average cost for producing marionettes are given by the following functions, where q is the number of marionettes and MC and AC are in dollars per marionette.
$$MC(q) = 0.012q^2 - 0.84q + 16.7 \qquad\qquad AC(q) = 0.004q^2 - 0.42q + 16.7 + \frac{20}{q}$$
   (a) Find the variable cost of producing 25 marionettes.
   (b) At what quantities does the average variable cost equal \$12.90 per marionette?
   (c) If you sell each marionette for \$26.30, what is your maximum profit?

5. You sell Things. The total cost to produce q **hundred Things** is given by the formula
$$TC(q) = 1.5q^2 + 11.25q + 10 \quad \text{hundred dollars.}$$
   If you sell q **hundred Things**, then selling price is given by the formula
$$p = -3q + 90 \quad \text{dollars per Thing.}$$
   (a) Give the formula in terms of q for total revenue for selling q hundred Things, $TR(q)$.
   (b) Find all quantities at which you break even.
   (c) What is the maximum possible total revenue?
   (d) What is the **selling price** when you maximize **profit**?

6. The weight in ounces of a newborn puppy t days after it's born is given by the function
$$w(t) = 0.5t^2 - 2t + 6.$$
   (a) For what range of time is the puppy's weight at most 5 ounces?
   (b) Compute, simplify and translate the meaning of
$$\frac{w(5+h) - w(5)}{h}.$$

Fully Explained ANSWERS to the old exam questions from previous page:

1. (a) We want to find the values of x when $f(x) = g(x)$.

$x^2 - 10x + 36$	$= -0.25x^2 + 4x + 24$	
$1.25x^2 - 14x + 12$	$= 0$	make one side zero!
x	$= \frac{-(-14) \pm \sqrt{(-14)^2 - 4(1.25)(12)}}{(2(1.25))}$	quad. form. $a = 1.25$, $b = -14$, $c = 12$
x	$= \frac{14 \pm \sqrt{136}}{2.5} \approx \frac{14 \pm 11.661904}{2.5}$	simplifying

Thus, $x \approx \frac{2.338096}{2.5} = 0.9353$ or $x \approx \frac{25.661904}{2.5} = 10.2648$.

(b) First find the function $g(x) - f(x) = (-0.25x^2 + 4x + 24) - (x^2 - 10x + 36) = -0.25x^2 + 4x + 24 - x^2 + 10x - 36$, so $g(x) - f(x) = -1.25x^2 + 14x - 12$.

The questions asks for the largest value of this function. Let's think about what the graph looks like (make a sketch)! The function $y = -1.25x^2 + 14x - 12$ is a parabola that opens downward, so the largest value is at the **vertex**.

The vertex occurs at $x = -\frac{b}{(2a)} = -\frac{14}{2(-1.25)} = \frac{14}{2.5} = 5.6$. And the value of $g(x) - f(x)$ at this location is $g(5.6) - f(5.6) = -1.25(5.6)^2 + 14(5.6) - 12 = 27.2$.

(c) We know the two functions in question $f(x) = x^2 - 10x + 36$ and $g(x) - f(x) = -1.25x^2 + 14x - 12$. The question asks when they both are increasing. Let's think about what the graphs look like (make a sketch)!

- $f(x)$ is a parabola that opens upward. So it is increasing **after its vertex**. The x-coordinate of its vertex is $x = -\frac{-10}{2(1)} = 5$. Thus, $f(x)$ is increasing from $x = 5$ and on (*i.e.* on the interval $x \geq 5$).
- $g(x) - f(x)$ is a parabola that opens downward. So it is increasing **before its vertex**. The x-coordinate of its vertex is $x = -\frac{14}{2(-1.25)} = 5.6$. Hence, $g(x) - f(x)$ is increasing before and up to $x = 5.6$ (*i.e.* on the interval $x \leq 5.6$).

Therefore, they both are increasing on the interval $5 \leq x \leq 5.6$.

(d) A standard algebraic problem of getting the rate formula. Let's go through that algebra again:

$\frac{f(8+h)-f(8)}{h}$	$= \frac{((8+h)^2 - 10(8+h)+36) - ((8)^2 - 10(8)+36)}{h}$	using the function def'n
$\frac{f(8+h)-f(8)}{h}$	$= \frac{(64+16h+h^2-80-10h+36)-(64-80+36)}{h}$	expanding
$\frac{f(8+h)-f(8)}{h}$	$= \frac{64+16h+h^2-80-10h+36-64+80-36}{h}$	still expanding, drop the parentheses (dist. negative!)
$\frac{f(8+h)-f(8)}{h}$	$= \frac{16h+h^2-10h}{h}$	cancel terms in the numerator
$\frac{f(8+h)-f(8)}{h}$	$= 16 + h - 10 = 6 + h$	simplify

2. (a) Since $TR = $ (Price per item)(Quantity), we have $TR(x) = (-16q + 1000)q = -16q^2 + 1000q$.

The problem says TC is linear, so that means it has the form $TC = m(q - q_1) + y_1$. We are given two points $(q, TC) = (0, 3484)$ and $(q, TC) = (20, 4004)$. Find the slope: $m = \frac{4004-3484}{20-0} = \frac{520}{20} = 26$ and given the line: $TC(q) = 26(q - 0) + 3484 = 26q + 3484$.

(b) We want to know when profit is \$3000. Let's first find a formula for profit.

$P(q) = TR(q) - TC(q) = (-16q^2 + 1000q) - (26q + 3484) = -16q^2 + 1000q - 26q - 3484 = -16q^2 + 974q - 3484$.

We are asked to solve:

$-16q^2 + 974q - 3484$	$= 3000$	
$-16q^2 + 974q - 6484$	$= 0$	make one side zero!
q	$= \frac{-(974) \pm \sqrt{(974)^2 - 4(-16)(-6484)}}{(2(-16))}$	quad. form. $a = -16$, $b = 974$, $c = -6484$
q	$= \frac{-974 \pm \sqrt{533700}}{-32} \approx \frac{-974 \pm 730.54774}{-32}$	simplifying

Thus, $q \approx \frac{-1704.5477}{-32} = 53.267$ or $x \approx \frac{-243.4523}{-32} = 7.608$.

(c) The question is about when profit is maximum. The profit function is $P(q) = -16q^2 + 974q - 3484$. Let's think about what the graph looks like (make a sketch)! It is a parabola that opens downward, so the maximum occurs at the **vertex**.

The vertex occurs when $q = -\frac{974}{2(-16)} = 30.4375$.

This quantity corresponds to a **price** of $p = -16(30.4375) + 1000 = 531$ dollars per Trinket.

(d) Since $AC(q) = \frac{TC(q)}{q} = \frac{26q+3484}{q} = 26 + \frac{3484}{q}$, we are being asked to solve the following equation:

$26 + \frac{3484}{q}$	$= 243.75$	
$26q + 3484$	$= 243.75q$	clear denominator! (mult. both sides by q)
3484	$= 217.75q$	this is a linear equation, get x by itself.
$\frac{3484}{217.75} = 16$	$= q$	simplifying

(e) Since $MR(q) = \frac{TR(q+1)-TR(q)}{1}$, we have some algebra to do:

$MR(q)$	$= (-16(q+1)^2 + 1000(q+1)) - (-16q^2 + 1000q)$	using the function def'n
$MR(q)$	$= (-16(q^2+2q+1) + 1000q + 1000) - (-16q^2 + 1000q)$	expanding
$MR(q)$	$= -16q^2 - 32q - 16 + 1000q + 1000 + 16q^2 - 1000q$	drop the parentheses (dist. negative!)
$MR(q)$	$= -32q - 16 + 1000 = -32q + 984$	cancel terms

Since $MC(q) = \frac{TC(q+1)-TC(q)}{1}$, we can do this with algebra (and there is a faster way):

$MC(q)$	$= (26(q+1) + 3484) - (26q + 3484)$	using the function def'n
$MC(q)$	$= (26q + 26 + 3484) - (26q + 3484)$	expanding
$MC(q)$	$= 26q + 26 + 3484 - 26q - 3484 = 26$	drop the parentheses (dist. negative!)

We should have already known this because TC is linear so each additional item costs the same amount to produce (in this case \$26).

3. (a) Since $AVC(q) = \frac{VC(q)}{q}$ and you are told that $AVC(q) = 0.01q^2 - 0.5q + 26$, you know that $\frac{VC(q)}{q} = 0.01q^2 - 0.5q + 26$. Multiplying both sides by q gives $VC(q) = 0.01q^3 - 0.5q^2 + 26q$.
And since $FC = 240$, you know that $TC(q) = 0.01q^3 - 0.5q^2 + 26q + 240$.
The problem asks for the value of $TC(30)$, so you need to compute
$TC(30) = 0.01(30)^3 - 0.5(30)^2 + 26(30) + 240 = \$840$.

(b) Since $AC(q) = \frac{TC(q)}{q}$ and from what I showed in the last part, we have
$AC(q) = \frac{0.01q^3 - 0.5q^2 + 26q + 240}{q} = 0.01q^2 - 0.5q + 26 + \frac{240}{q}$.

(c) The **shutdown price** is the minimum y-value of $AVC(q)$. Think of the graph of $AVC(q)$ (sketch it)! Since $AVC(q) = 0.01q^2 - 0.5q + 26$ it is a quadratic that opens upward. So the minimum occurs at the **vertex**.
We know the vertex occurs at $q = -\frac{-0.5}{2(0.01)} = 25$ and the y value of $AVC(q)$ at the vertex is
$AVC(25) = 19.75$ dollars per item.

(d) We are being asked to solve when $AVC(q) = 28$:

$0.01q^2 - 0.5q + 26$	$= 28$	
$0.01q^2 - 0.5q - 2$	$= 0$	make one side zero!
q	$= \frac{-(-0.5)\pm\sqrt{(-0.5)^2-4(0.01)(-2)}}{(2(0.01))}$	quad. form. $a = 0.01$, $b = -0.5$, $c = -2$
q	$= \frac{0.5\pm\sqrt{0.33}}{0.02} \approx \frac{0.5\pm0.574456}{0.02}$	simplifying

Thus, $q \approx \frac{-0.074456}{0.02} = -3.723$ or $q \approx \frac{1.074456}{0.02} = 53.72$. It can't be negative, so $q = 53.72$.

4. (a) Since $AC(q) = \frac{TC(q)}{q}$ and you are given $AC(q) = 0.004q^2 - 0.42q + 16.7 + \frac{20}{q}$, you know that $\frac{TC(q)}{q} = 0.004q^2 - 0.42q + 16.7 + \frac{20}{q}$. Multiplying both sides by q gives: $TC(q) = 0.004q^3 - 0.42q^2 + 16.7q + 20$. Thus, we can see that $FC = TC(0) = 20$ and we can see that $VC(q) = 0.004q^3 - 0.42q^2 + 16.7q$. The question asks for $VC(25) = 0.004(25)^3 - 0.42(25)^2 + 16.7(25) = \$217.50$.

(b) Since $AVC(q) = \frac{VC(q)}{q}$ and using what we did in the previous part we have:
$AVC(q) = \frac{0.004q^3 - 0.42q^2 + 16.7q}{q} = 0.004q^2 - 0.42q + 16.7$. We are asked to find when this function is equal to 12.90:

$0.004q^2 - 0.42q + 16.7$	$= 12.9$	
$0.004q^2 - 0.42q + 3.8$	$= 0$	make one side zero!
q	$= \frac{-(-0.42)\pm\sqrt{(-0.42)^2-4(0.004)(3.8)}}{(2(0.004))}$	quad. form. $a = 0.004$, $b = -0.42$, $c = 3.8$
q	$= \frac{0.42\pm\sqrt{0.1156}}{0.008} = \frac{0.42\pm0.34}{0.008}$	simplifying

Thus, $q = \frac{0.08}{0.008} = 10$ or $q = \frac{0.76}{0.008} = 95$.

(c) If the price per item is a constant \$26.30, then you know that $TR(q) = 26.30q$ and $MR(q) = 26.30$. The easiest way to find maximum profit here is to solve when $MR(q) = MC(q)$:

$0.012q^2 - 0.84q + 16.7$	$= 26.30$	
$0.012q^2 - 0.84q - 9.6$	$= 0$	make one side zero!
q	$= \frac{-(-0.84)\pm\sqrt{(-0.84)^2-4(0.012)(-9.6)}}{(2(0.012))}$	quad. form. $a = 0.012$, $b = -0.84$, $c = -9.6$
q	$= \frac{0.84\pm\sqrt{1.1664}}{0.024} = \frac{0.84\pm1.08}{0.024}$	simplifying

Thus, $q = \frac{-0.24}{0.024} = -10$ or $q = \frac{1.92}{0.024} = 80$. Note that if you sketch the picture you see that we switch from $MR > MC$ to $MR < MC$ at $q = 80$. So this is the quantity that maximizes profit.
To get the maximum profit, we now need to compute $TR(80) - TC(80)$.

- Since the market price is \$26.30, we have $TR(q) = 26.30q$. So $TR(80) = 26.30 \cdot 80 = \$2104$
- In a previous problem we used the AC formula to deduce the formula for TC. We found $TC(q) = 0.004q^3 - 0.42q^2 + 16.7q + 20$. So $TC(80) = 0.004(80)^3 - 0.42(80)^2 + 16.7(80) + 20 = \$716$.

Therefore, the maximum profit is $P(80) = TR(80) - TC(80) = 2104 - 716 = \$1388$.

5. (a) Since TR = (Price per item)(Quantity), we have $TR(q) = (-3q + 90)q = -3q^2 + 90q$. Note, since q is in hundreds of things and price is in dollars per thing, we get TR in hundreds of dollars.

(b) The quantities at which you break even are the quantities when profit is equal to zero. This is the same as asking when $TR(q) = TC(q)$:

$1.5q^2 + 11.25q + 10 = -3q^2 + 90q$	
$4.5q^2 - 78.75q + 10 = 0$	make one side zero!
$q = \frac{-(-78.75) \pm \sqrt{(-78.75)^2 - 4(4.5)(10)}}{(2(4.5))}$	quad. form. $a = 4.5$, $b = -78.75$, $c = 10$
$q = \frac{78.75 \pm \sqrt{6021.5625}}{9} \approx \frac{78.75 \pm 77.59873}{9}$	simplifying

Thus, $q \approx \frac{1.15127}{9} = 0.13$ or $q \approx \frac{156.34873}{9} = 17.37$ hundred items (you must keep two digits of accuracy since we are in hundreds).

(c) We already know that $TR(q) = -3q^2 + 90q$. We are asked to find the maximum value of this function. Let's think about what the picture looks like (make a sketch)! The graph of $TR(q)$ is a parabola that opens downward, so the highest point is at the **vertex**.

The vertex occurs where $q = -\frac{90}{2(-3)} = 15$ hundred Things. And the value of TR at the vertex is $TR(15) = -3(15)^2 + 90(15) = \$675$.

(d) The question is about when profit is maximum. The profit function is $P(q) = TR(q) - TC(q) = (-3q^2 + 90q) - (1.5q^2 + 11.25q + 10) = -3q^2 + 90q - 1.5q^2 - 11.25q - 10$. So the function for profit is $P(q) = -4.5q^2 + 78.75q - 10$. Let's think about what the graph looks like (make a sketch)! It is a parabola that opens downward, so the maximum occurs at the **vertex**.

The vertex occurs when $q = -\frac{78.75}{2(-4.5)} = 8.75$ hundred Things.

This quantity corresponds to a **price** of $p = -3(8.75) + 90 = 63.75$ dollars per Thing.

6. (a) The question is asking when the weight is less than or equal to 5 ounces. First, find when the weight is equal to 5 ounces, then use a sketch of the picture to figure out your answer (note that $w(t)$ is a parabola that opens upward). Here is the algebra:

$0.5t^2 - 2t + 6 = 5$	
$0.5t^2 - 2t + 1 = 0$	make one side zero!
$t = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(0.5)(1)}}{(2(0.5))}$	quad. form. $a = 0.5$, $b = -2$, $c = 1$
$t = \frac{2 \pm \sqrt{2}}{1} \approx 2 \pm 1.4152$	simplifying

Thus, $t \approx 0.5858$ or $t \approx 3.4142$ days.

Since $w(t)$ is a parabola that opens upward, it must be below 5 ounces for all the days between these two times. So the puppy's weight is less than or equal to 5 ounces from $t = 0.59$ to $t = 3.41$ days.

(b) The expression $\frac{w(5+h) - w(5)}{h}$ is the rate of change of the puppy's weight from $t = 5$ days to $t = 5 + h$ days (*i.e.* the rate of change from $t = 5$ days to h days later). The units will be in ounces per day. Now let's do the algebra work:

$\frac{w(5+h)-w(5)}{h} = \frac{(0.5(5+h)^2 - 2(5+h) + 6) - (0.5(5)^2 - 2(5) + 6)}{h}$	using the function def'n
$\frac{w(5+h)-w(5)}{h} = \frac{(0.5(25 + 10h + h^2) - 10 - 2h + 6) - (12.5 - 10 + 6)}{h}$	expanding
$\frac{w(5+h)-w(5)}{h} = \frac{12.5 + 5h + 0.5h^2 - 10 - 2h + 6 - 12.5 + 10 - 6)}{h}$	still expanding, drop the parentheses (dist. negative!)
$\frac{w(5+h)-w(5)}{h} = \frac{5h + 0.5h^2 - 2h}{h}$	cancel terms in the numerator
$\frac{w(5+h)-w(5)}{h} = 5 + 0.5h - 2 = 3 + 0.5h$	simplify