1. (10 pts) Consider the vector field \( \mathbf{F}(x, y, z) = x^2y \mathbf{i} + zy \mathbf{j} + yx^3 \mathbf{k} \) on \( \mathbb{R}^3 \).

(a) (6 pts) Compute the following:

i. \( \text{curl} \, \mathbf{F} \):

\[
\begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
2x^2y & zy & yx^3 \\
x^2y & 2y & yx^3
\end{vmatrix} = (x^3 - yz) \mathbf{i} - (3x^2 - 0) \mathbf{j} + (0 - x^2) \mathbf{k} = (x^3 - yz) \mathbf{i} - 3x^2 \mathbf{j} - x^2 \mathbf{k}
\]

ii. \( \nabla(\text{div} \, \mathbf{F}) \):

\[
\text{div} \, \mathbf{F} = 2xy + z + 0
\]
\[
\nabla(\text{div} \, \mathbf{F}) = \left< 2y, 2x, 1 \right>
\]

iii. \( \text{div} \, (\text{curl} \, \mathbf{F}) \):

\[
= 0 \quad \text{Always}
\]

(b) (4 pts) Give a short one sentence answer to each of the two questions below:

i. What can you conclude for a vector field where \( \text{curl} \, \mathbf{F} \neq 0 \)?

\( \mathbf{F} \) is not conservative.

ii. What can you conclude for a vector field where \( \text{div} \, \mathbf{F} \neq 0 \)?

\( \mathbf{F} \) is not the curl of another vector field.
2. (8 pts)

(a) (4 pts) Give a parameterization for the part of surface \( y^2 + z^2 - x = 3 \) with \( 0 \leq x \leq 1 \). Include bounds on the parameters.

\[ y = \sqrt{\cos(u)} \]
\[ z = \sqrt{\sin(u)} \]
\[ x = \sqrt{y^2 - 3} \]

BECAUSE OF THE BOUNDS, EASIEST TO USE

\[ y^2 + z^2 - x = 3 \]  
\[ \sqrt{\cos(u)}^{2} + \sqrt{\sin(u)}^{2} - \sqrt{y^2 - 3} = 3 \]  
\[ \cos(u) + \sin(u) = 3 \]  
\[ \sqrt{\cos(u)} = 3 \]
\[ 0 \leq u \leq 2\pi \]

Bounds
\[ 0 \leq x \leq 1 \]
\[ 0 \leq \sqrt{y^2 - 3} \leq 1 \]
\[ 3 \leq y^2 \leq 4 \]
\[ \sqrt{3} \leq y \leq 2 \]
\[ 0 \leq u \leq 2\pi \]

(b) (4 pts) You are told that \( x = x(t) \), \( y = y(t) \), and \( z = z(t) \) is the parameterization for the motion of some particle along the curve \( C \) which is on the surface \( z = x^2 + \sin(y) - xy^2 \). If \( x(1) = 2 \), \( y(1) = 0 \), \( x'(1) = 3 \), and \( y'(1) = -5 \), then what is the value of \( z'(1) \)?

That is, find \( \frac{dz}{dt} \) at \( t = 1 \).

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = (2x-y^2) \frac{dx}{dt} + (\cos(y)-2xy) \frac{dy}{dt}
\]

\[
\frac{dz}{dt} \bigg|_{t=1} = (2(2)-(0)^2)(3) + (\cos(0)-2(2)(0))(-5)
\]

\[
= 12 - 5 = 7
\]
3. (9 pts) Consider the vector field \( \mathbf{F}(x, y, z) = (-z \sin(x) + y^2) \mathbf{i} + (2xy + e^{z^2}) \mathbf{j} + (\cos(x) + 2yz e^{z^2}) \mathbf{k} \) on \( \mathbb{R}^3 \). You are told that the vector field is conservative!

(a) (6 pts) Find a function \( f(x, y, z) \) such that \( \nabla f = \mathbf{F} \).

\[
\begin{align*}
  f_x(x, y, z) &= -z \sin(x) + y^2 \\
  f_y(x, y, z) &= 2xy + e^{z^2} \\
  f_z(x, y, z) &= \cos(x) + 2yz e^{z^2}
\end{align*}
\]

\[
\Rightarrow \begin{cases}
  f(x, y, z) = z \cos(x) + xy^2 + g(y, z) \\
  g_y(y, z) = e^{z^2} \\
  g(z) = y e^{z^2} + h(z)
\end{cases}
\]

\[
f(x, y, z) = z \cos(x) + xy^2 + ye^{z^2} + h(z)
\]

**GENERAL ANSWER:**

\[
f(x, y, z) = z \cos(x) + xy^2 + ye^{z^2} + C
\]

(b) (3 pts) Compute \( \int_C \mathbf{F} \cdot d\mathbf{r} \) over the curve, \( C \), given by \( \mathbf{r}(t) = (\pi t, 3 - 3t^4, \sin(\pi t) + 5t) \) for \( 0 \leq t \leq 1 \). (Please think about your options here.)

**START POINT:** \( A \) \( \mathbf{r}(0) = (0, 3, 0) \)

**END POINT:** \( B \) \( \mathbf{r}(1) = (\pi, 0, 5) \)

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A) = f(\pi, 0, 5) - f(0, 3, 0)
\]

\[
= \left[ (5 \cos(\pi) + (\pi)0^2 + (0) e^{(5)^2}) - \left[ (0) \cos(0) + (0)(0)^2 + 3 e^{(0)^2} \right] \right] - 3
\]

\[
= -5 - 3
\]

\[
= -8
\]
4. (8 pts) Use Green's Theorem to evaluate

\[ \oint_C \sin(x^3) \, dx + 4x^2 y \, dy \]

where \( C \) is the triangle with vertices (0, 0), (2, 0), and (2, 6).

\[
\begin{align*}
\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA \\
= \int_0^2 \int_0^{3x} (9xy^2 - 8xy) \, dy \, dx \\
= \int_0^2 9xy^3 - 8x^2 \, dx \\
= \frac{3}{4} x^4 \bigg|_0^2 = 9 \cdot 2^4 = 9 \cdot 16 = 144
\end{align*}
\]

5. (5 pts) Assume the temperature at each point on the \( xy \)-plane is given by

\[ T(x, y) = \frac{1}{3} x^2 y + 5\sqrt{x^2 + y^2} \] degrees Celsius,

where \( x \) and \( y \) are in feet. Find the directional derivative of \( T(x, y) \) at the point (3, 4) in the direction of \((-1, 2)\). Give the units for your answer.

\[ \nabla T(x, y) = \left( \frac{2}{3} x y + \frac{5x}{\sqrt{x^2 + y^2}}, \frac{1}{3} x + \frac{5y}{\sqrt{x^2 + y^2}} \right) \]

\[ \nabla T(3, 4) = \left( \frac{2 \cdot 3 \cdot 4}{3} + \frac{5 \cdot 3}{\sqrt{3^2 + 4^2}}, \frac{1}{3} \cdot 3 + \frac{5 \cdot 4}{\sqrt{3^2 + 4^2}} \right) = \langle 11, \ 7 \rangle \]

UNIT DIRECTION VECTOR: \( \mathbf{u} = \frac{1}{\sqrt{5}} \langle -1, \ 2 \rangle \)

\[ \nabla T(3, 4) \cdot \frac{1}{\sqrt{5}} \langle -1, \ 2 \rangle = \frac{1}{\sqrt{5}} (-11 + 14) = \frac{3}{\sqrt{5}} \frac{\circ C}{ft} \]
6. (10 pts) Assume, again, the temperature at each point on the xy-plane is given by 
\[ T(x, y) = \frac{1}{3}x^2 y + 5\sqrt{x^2 + y^2} \] 
degrees Celsius. You are told that the average temperature along a 
curve \( C \) is given by 
\[ \frac{1}{L} \int_C T(x, y) \, ds \], where \( L \) is the total length of \( C \).

Let \( C \) be the curve consisting of a straight line segment from the origin to \((0, 2)\), then one quarter of the circle \( x^2 + y^2 = 4 \) from \((0, 2)\) to \((2, 0)\). Compute the average temperature along \( C \). That is, compute 
\[ \frac{1}{L} \int_C T(x, y) \, ds \].

(Hint: Parameterize!)

\[ \text{NOTE:} \quad L = 2 + \pi \]

\[ \text{length of} \ C_1 \quad \text{length of} \ C_2 \quad \frac{1}{4}(\pi) \]

\[ C_1: \quad x = 0, \quad y = 2t, \quad 0 \leq t \leq 1 \]
\[ x' = 0, \quad y' = 2 \]
\[ ds = \sqrt{0^2 + 2^2} \, dt = 2 \, dt \]

\[ \int_{C_1} T(x, y) \, ds = \int_0^1 \left( \frac{1}{3} (0)^2 (2t) + 5 \sqrt{0^2 + (2t)^2} \right) 2 \, dt \]
\[ = \int_0^1 5(2t) \, 2 \, dt = 10 \left[ \frac{t^2}{2} \right]_0^1 = 10 \]

\[ C_2: \quad x = 2 \cos(t), \quad y = 2 \sin(t), \quad 0 \leq t \leq \frac{\pi}{2} \]
\[ x' = -2 \sin(t), \quad y' = 2 \cos(t) \]
\[ ds = \sqrt{(-2 \sin(t))^2 + (2 \cos(t))^2} \, dt = 2 \, dt \]

\[ \int_{C_2} T(x, y) \, ds = \int_0^{\frac{\pi}{2}} \left( \frac{1}{3} \cos^2(t) \sin(t) + 5 \sqrt{4 \cos^2(t) + 4 \sin^2(t)} \right) 2 \, dt \]
\[ = \int_0^{\frac{\pi}{2}} \frac{16}{3} \cos^2(t) \sin(t) \, dt + \int_0^{\frac{\pi}{2}} 20 \, dt \]
\[ = \frac{16}{3} \left[ \frac{u^2}{2} \right]_{u = \cos(t)}^{u = \cos(0)} + 10 \pi \]
\[ = \frac{16}{3} \left[ \frac{1}{2} \right]_{1}^{1} + 10 \pi = \frac{16}{3} + 10 \pi \]

\[ \text{Average} = \frac{10 + \frac{16}{3} + 10 \pi}{2 + \pi} = \frac{10 + \frac{90 \pi}{3}}{2 + \frac{9 \pi}{3}} \approx 8.40064 \, ^\circ \text{C} \]