1. (10 pts) Compute the following integrals:

(a) \( \int_C (y - 2) \, ds \) where \( C \) is the line segment from \( (0, 1) \) to \( (4, -2) \).

\[
\text{PARAMETERIZE} \quad x = 0 + 4t, \quad 0 \leq t \leq 1 \quad \hat{r}(t) = \langle 4t, 1-3t \rangle \\
y = 1 - 3t \\
ds = \sqrt{(4)^2 + (-3)^2} \, dt = 5 \, dt
\]

\[
\int_C (y - 2) \, ds = \int_0^1 (1 - 3t - 2) \cdot 5 \, dt \\
= 5 \int_0^1 (-1 - 3t) \, dt \\
= 5 \left[ -t - \frac{3}{2} t^2 \right]_0^1 \\
= 5 \left[ -\frac{5}{2} \right] = -\frac{25}{2} = -12.5
\]

(b) \( \iint_S 21z \, dS \), where \( S \) is the surface of the sphere \( x^2 + y^2 + z^2 = 1 \) in the first octant.

\[
\text{PARAMETERIZE} \quad \hat{r}(\phi, \theta) = \langle \sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi \rangle \\
|\hat{r}_\phi \times \hat{r}_\theta| = 1^2 \sin \phi \\
0 \leq \phi \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq \frac{\pi}{2}
\]

\[
\iint_S 21z \, dS = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 21 \cos \phi \sin \phi \, d\phi \, d\theta \\
= 21 \int_0^{\frac{\pi}{2}} \cos \phi \, d\phi \int_0^{\frac{\pi}{2}} \sin \phi \, d\phi \\
= 21 \left[ \frac{\phi}{2} \right]_0^{\frac{\pi}{2}} \left[ -\cos \phi \right]_0^{\frac{\pi}{2}} \\
= \frac{21\pi}{2} \left( -\cos \frac{\pi}{2} \right) - \left( -\cos 0 \right) \\
= \frac{21\pi}{2} \cdot 1 = \frac{21\pi}{2}
\]
2. (10 points) Consider the region, \( R \), in the xy-plane bounded by \( y = 8 - x \), \( x = 0 \) and \( y = 0 \). You are given the transformation (assume \( u \geq 0 \)):

\[
x = u^2 + v \quad \text{and} \quad y = u^2 - v.
\]

(a) Find, sketch, and label a region in the uv-plane whose image is \( R \).

\[
\begin{align*}
x + y &= 2u^2 \quad \Rightarrow \quad u^2 = \frac{1}{2}(x+y) \quad \Rightarrow \quad u = \sqrt{\frac{1}{2}(x+y)} \\
x - y &= 2v \quad \Rightarrow \quad v = \frac{1}{2}(x-y)
\end{align*}
\]

\( C_1 : y = 0, \, 0 \leq x \leq 8 \Rightarrow 0 = u^2 - v \Rightarrow v = u^2 \quad \Rightarrow \quad 0 \leq u \leq 2 \)

\( C_2 : y = 8 - x, \, 0 \leq x \leq 8 \Rightarrow x + y = 8 \Rightarrow u = \frac{\sqrt{2}}{2}x = 2 \quad v = \frac{1}{2}(2x-8) \Rightarrow -4 \leq v \leq 4 \)

\( C_3 : x = 0, \, 0 \leq y \leq 8 \Rightarrow 0 = u^2 + v \Rightarrow v = -u^2 \quad \Rightarrow \quad 0 \leq u \leq 2 \)

(b) Write out the result of this change of variable for the integral:

\[
\iint_{R} f(x,y) \, dA = \int_{?}^{?} \int_{?}^{??} ?? \, dv \, du
\]

(There is no evaluation here, just tell me exactly what goes in the question mark locations).

\[
\iint_{R} f(x,y) \, dA = \iint_{-u^2}^{u^2} f(u^2+v, u^2-v) \, 4u \, dv \, du
\]

\[
\frac{\partial (x,y)}{\partial (u,v)} = \begin{vmatrix} 2u & 2u \\ 1 & -1 \end{vmatrix} = -4u
\]

\[
\frac{\partial (x,y)}{\partial (u,v)} = \begin{vmatrix} \frac{\partial (x,y)}{\partial (u,v)} \end{vmatrix} = \begin{vmatrix} -4u \end{vmatrix}
\]
3. (10 pts) Consider the vector field $\mathbf{F}(x, y, z) = (y^2 + 2, 2xy, 6z^2)$ on $\mathbb{R}^3$. Note that $\text{curl} \mathbf{F} = 0$.
Let $C$ be the curve parameterized by $\mathbf{r}(t) = (3t^10, \cos(\pi t), 2t^3 - t - 1)$ for $0 \leq t \leq 1$.
Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.
(Please use the consequences of the fact that $\text{curl} \mathbf{F} = 0$).

(1) $f_x(x, y, z) = y^2 + 2$
$f(x, y, z) = \int y^2 + 2 \, dx = y^2x + 2x + g(y, z)$

(2) $f_y(x, y, z) = 2xy$
$2yx + 0 + g_y(y, z) = 2xy$
$g_y(y, z) = 0$
$g(y, z) = h(z) \Rightarrow f(x, y, z) = y^2x + 2x + h(z)$

(3) $f_z(x, y, z) = 6z^2$
$0 + 0 + h'(z) = 6z^2$
$h(z) = \int 6z^2 \, dz = 2z^3 + k$ \quad (a constant)
$f(x, y, z) = y^2x + 2x + 2z^3 + k$

$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0))$
$= f(3, -1, 0) - f(0, \pi, -1)$
$= [(1)^2(-1) + 2(1) + 2(0)^3] - [(1)^2(0) + 2(0) + 2(-1)^3]$
$= -2 + 2 = 0$

= 0 - 2 = [1]$
4. (10 pts) Set up (DO NOT EVALUATE) two triple integrals that represent the volume of the solid bounded by the planes $2x + 3y + z = 6$, $z = 0$, $y = 1$, and $x = 0$. You must give two answer in the orders specified.

(a) In the order $dydzdx$:

Inner Bounds: $1 \leq y \leq \frac{6 - 2x - z}{3}$

Projection:

$y = 1$ \Rightarrow $2x + 3 + 2 = 6$

$z = 3 - 2x$

$0 \leq z \leq 3 - 2x$

$0 \leq x \leq 3$

\[ \int \int \int_1^3 S_1 S_2 S_3 \, 1 \, dydzdx \]

(b) In the order $dzdxdy$:

Inner Bounds: $0 \leq z \leq 6 - 2x - 3y$

Projection:

$z = 0$ \Rightarrow $2x + 3y = 6$

$y = 2 - \frac{2}{3}x$

$x = 3 - \frac{3}{2}y$

$0 \leq x \leq 3 - \frac{3}{2}y$

$1 \leq y \leq 2$

\[ \int \int \int_1^2 S_1 S_2 S_3 \, 1 \, dzdxdy \]
5. (10 pts) You impose a coordinate system on a sand beach and find the temperature at each point is given by \( T(x, y) = x^2 + y^2 + 2y + 90 \) degrees Fahrenheit, where \( x \) and \( y \) are in feet.
Assume you walk barefoot half way around a circular path, \( C \), from \((3, 0)\) to \((-3, 0)\) in such a way that your motion is parameterized by \( r(t) = (3\cos(t), 3\sin(t)) \) where \( t \) is in seconds with \( 0 \leq t \leq \pi \).

GIVE UNITS FOR ALL YOUR ANSWERS.

(a) Give the direction and magnitude of the greatest rate of change at the point \((3,0)\).

\[
\nabla T(x, y) = \langle 2x, 2y + 2 \rangle
\]

\[
\nabla T(3, 0) = \langle 6, 2 \rangle
\]

\[
|\nabla T(3, 0)| = \sqrt{6^2 + 2^2} = \sqrt{40} \text{ OF ft}
\]

(b) As you walk along the curve \( C \), what is the rate of change of temperature with respect to time at \( t = \pi/4 \) seconds?

\[
\frac{dT}{dt} = \frac{dT}{dx} \frac{dx}{dt} + \frac{dT}{dy} \frac{dy}{dt} = 2x(-3\sin(\theta)) + (2y + 2)(3\cos(\theta))
\]

\[
x(\frac{\pi}{4}) = 3\sqrt{2}, \ y(\frac{\pi}{4}) = 3\sqrt{2} \Rightarrow
\]

\[
= -9 + (3\sqrt{2} + 2)3\sqrt{2}/\pi
\]

\[
= -9 + 9 + 3\sqrt{2} = 3\sqrt{2} \text{ OF sec}
\]

(c) Compute \( \frac{1}{3\pi} \int_C T(x, y)ds \). (This is the average temperature along \( C \)).

\[
\frac{1}{3\pi} \int_0^\pi (9 + 6\sin(t) + 90) \sqrt{(-3\sin(t))^2 + (3\cos(t))^2} \, dt
\]

\[
\frac{3}{3\pi} \int_0^\pi 99 + 6\sin(t) \, dt
\]

\[
\frac{1}{3\pi} \left[ 99t - 6\cos(t) \right]_0^\pi = \frac{1}{3\pi} \left[ (99\pi + 12) - (0 - 6) \right]
\]

\[
= \frac{1}{3\pi} \left[ 99\pi + 12 \right] \text{ OF}
\]

\[
= \frac{99 + 12}{\pi} \text{ OF}
\]
6. (10 pts) Use Stokes' theorem to evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F}(x, y, z) = \langle z^2, 3x, 2y \rangle \) and \( C \) is the \text{CLOSED} curve of intersection of the cylinder \( x^2 + y^2 = 1 \) and the plane \( z = 5 - x \) with counterclockwise orientation when viewed from above.

\[
\text{curl} \mathbf{F} = \left| \begin{array}{ccc}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
0 & z & 2 \\
-2 & 3 & 0
\end{array} \right| = \langle 2, 2z, 3 \rangle
\]

\[S: \quad z = 5 - x \quad \mathbf{r}(x, y) = \langle x, y, 5 - x \rangle \]

\[\mathbf{r}_x \times \mathbf{r}_y = \langle 1, 0, 1 \rangle \quad \text{upward} - \]

\[
\int_S \text{curl} \mathbf{F} \cdot d\mathbf{S} = \int_S \langle 2, 2z - (5 - x), 3 \rangle \cdot \langle 1, 0, 1 \rangle \ dA
\]

\[= \int_D 5 \ dA \]

\[\int_0^{2\pi} \int_0^1 5r \ dr \ d\theta = \int_0^{2\pi} 5 \theta \bigg|_0^1 = 5\pi \]

\[\int_0^{2\pi} \int_0^1 5r^2 \ d\theta \ d\alpha = \int_0^{2\pi} \frac{1}{2} r^2 \bigg|_0^1 = \frac{5\pi}{2} \]

\[S\pi \]
7. (10 pts) Consider the vector field \( \mathbf{F}(x, y, z) = (1+3y^2, -6x, 2z^2+x) \) on \( \mathbb{R}^3 \). Let \( S \) be the CLOSED surface that consists of the cylinder \( x^2 + y^2 = 9 \) for \( 0 \leq z \leq 1 \) and the parts of the planes \( z = 0 \) and \( z = 1 \) that are inside the cylinder. Find the flux of \( \mathbf{F} \) across \( S \).

That is, compute \( \iint_S \mathbf{F} \cdot d\mathbf{S} \).

You may pick either the outward or inward orientation for \( S \), but in the end I want you to tell me if the net flux of \( \mathbf{F} \) across \( S \) is outward or inward.

\[
\text{NET FLUX OF } \mathbf{F} \text{ ACROSS } S \text{ (circle one): } \text{INWARD} \text{ OR } \text{OUTWARD}
\]