Final answers and sketches of some solutions for the sample final.

CORRECTIONS made in limits in #3 and last line of answer to #6.

1. (a) Scalar, vector, nonsense, nonsense, vector, scalar. (Note that the last one is the length of a vector, not a vector.)
   (b) Only $\text{curl}(\text{grad}(f))$ vanishes no matter what the fields are.

2. Many choices of the parametrizations are possible, but the final answers should be the same for any choice.

Let $C_1$ be the straight section and $C_2$ the curved section of $C$. Parametrize $C_1$: $(t, 0)$, $0 \leq t \leq 2$. For $C_2$, $\langle 2\cos(t), 2\sin(t) \rangle$, $-\pi/2 \leq t \leq 0$.

(a) $\int_{C_1} x \, ds = \int_{0}^{2} t \, dt = \ldots = 2$. On $C_2$, $ds = 2dt$ and $\int_{C_2} x \, ds = \int_{-\pi/2}^{0} 2\cos(t) \, 2dt = \ldots = 4$. So $\int_{C} x \, ds = \int_{C_1} x \, ds + \int_{C_2} x \, ds = 6$.

(b) $dy = 0$ on $C_1$, so $\int_{C_1} x \, dy = 0$ and therefore

$$\int_{C} x \, dy = \int_{C_1} x \, dy = \int_{0}^{\pi/2} 2\cos(t) \, \frac{d}{dt}(2\sin(t)) \, dt = \ldots = -\pi$$

3. (a) $\int_{-\pi}^{\pi} \int_{0}^{\sqrt{25-x^2}} \int_{-\sqrt{25-x^2-y^2}}^{\sqrt{25-x^2-y^2}} (e^x + z^2) \, dz \, dy \, dx$

(b) $\int_{0}^{2\pi} \int_{0}^{5} \int_{-5}^{5} (e^{r\cos(\theta)} + z^2) r \, dz \, dr \, d\theta$ or

$$\int_{0}^{2\pi} \int_{0}^{3\pi/4} \int_{0}^{5} (e^{\rho\sin(\psi)\cos(\theta)} + \rho^2 \cos^2(\psi)) \rho^2 \sin(\psi) \, d\rho \, d\psi \, d\theta$$

4. Many choices of the parametrizations are possible.

(a) $\mathbf{r}(u, v) = \langle 4\cos(v), 0, 2\sin(v) \rangle$ for $0 \leq u \leq 10$ and $0 \leq v \leq 2\pi$. Then $\mathbf{r}_u \times \mathbf{r}_v = \langle 2\cos(v), 0, 4\sin(v) \rangle$, which points outward and so this parametrization gives the same orientation as the one specified.

(b) For $C_1$, parametrize by $\langle 4\cos(t), 0, 2\sin(t) \rangle$ with $0 \leq t \leq 2\pi$. For $C_2$, just change $y$ from $0$ to $10$: $\langle 4\cos(t), 10, 2\sin(t) \rangle$ with $0 \leq t \leq 2\pi$. For both of these, the curve starts at a point in the $xy$-plane where $x > 0$ and moves upward (increasing $z$). For $C_1$, the surface is on the right when going in this direction, so the parametrization gives the opposite orientation to the one consistent with the specified one on $S$. For $C_2$, the surface is on the left when going in this direction, so this orientation for $C_2$ is consistent with the specified one on $S$.

5. Choosing a simple parametrization, $\mathbf{r}(u, v) = \langle 1 - u^2 - v^2, u, v \rangle$, with $u^2 + v^2 \leq 1$ and $u \leq v$ (an eighth of a unit disk). Then $\mathbf{r}_u \times \mathbf{r}_v = \langle 1, 2u, 2v \rangle$, which points away from the origin. So I’ll have to put a minus sign in front of the integral.

$$- \int_{0}^{1/\sqrt{2}} \int_{u}^{\sqrt{1-u^2}} (1 - u^2 - v^2, u, v) \cdot (1, 2u, 2v) \, dv \, du = - \int_{0}^{1/\sqrt{2}} \int_{u}^{\sqrt{1-u^2}} (1 + u^2) \, du \, dv.$$

Change to polar coordinates to get simpler limits of integration and evaluate to get $-3\pi/16$. By using polar coordinate ideas in the parametrization, you can get the simpler limits more directly: $\mathbf{r}(u, v) = \langle 1 - u^2, u\cos(v), u\sin(v) \rangle$, with $0 \leq u \leq 1$ and $\pi/4 \leq v \leq \pi/2$. The orientation is still opposite to the given one. The integral becomes

$$- \int_{\pi/4}^{\pi/2} \int_{0}^{1} (u + u^3) \, du \, dv$$

(with of course the same final answer of $-3\pi/16$).
6. Computing directly, you should get $9/2$, $-17/2$, and $1$ for the integrals along the three line segments, for a total integral around $C$ of $-3$.

**OR**, notice, in addition to the the hint that you may use a theorem, that the curl of $F$ will be a constant vector field, so Stokes’s Theorem seems like a good idea. Compute $\text{curl}(F) = 5i - 3j + 7k$. Fill in $C$ with a planar surface $T$: it’s a triangle with unit normal $\mathbf{n} = (i - k)/\sqrt{2}$ giving the compatible orientation. You can parametrize and integrate as usual, but $F \cdot \mathbf{n} = (5 - 7)/\sqrt{2} = -\sqrt{2}$, a constant. So we can compute the integral by multiplying this constant times the area of $T$. The perpendicular sides of the triangle have lengths $3$ and $\sqrt{2}$, so altogether we get $\iint_T F \cdot \mathbf{n} dA = -\sqrt{2}(3\sqrt{2}/2) = -3$.

7. (a) The boundary of $E$ is the union of $S_1$ and $S_2$ together, but with the reversed orientation on $S_2$ (because “downward” is “outward” on the bottom). Therefore the Divergence Theorem tells us that

$$
\iiint_E \text{div}(F) dV = \iint_{S_1} F \cdot dS - \iint_{S_2} F \cdot dS
$$

(where the integrals over $S_1$ and $S_2$ mean with the original orientations of those surfaces). Adding the integral over $S_2$ to both sides gives us the desired result.

(b) Use part (a). Compute $\text{div}(F) = 2$, so $\iiint_E \text{div}(F) dV = 2($volume of hemisphere of radius $3)$ = $36\pi$ (using remembered formula for the volume of a sphere, or computing using spherical coordinates). On $S_2$, the unit normal is $k$, so we only need the $k$-component of $F$, which is $x + z = x$ on $S_2$. By symmetry (or calculation), $\iint_{S_2} x dx dy = 0$. So $\iint_{S_1} F \cdot dS = 36\pi + 0 = 36\pi$.

**Final answers and sketches of some solutions for the additional review problems:**

1. (a) Compute the curl of each vector field. For $\mathbf{F}$, it’s the zero vector field. As $\mathbf{F}$ and the partials of its components are defined on all of space, we know that $\mathbf{F}$ is conservative. A potential function is given by $f(x, y, z) = \frac{x^2}{2} + y + xz$. (It’s a good idea to check by computing $\text{grad}(f)$ and make sure it gives you $\mathbf{F}$ back again.)

The curl $\mathbf{G} = (0, 0, -2)$ is not zero, so $\mathbf{G}$ is not conservative. (If you made a sign error and thought the curl was zero, then you should discover your error when you try to find the potential function: you won’t be able to solve the equations, or if you think you do, when you compute the gradient you won’t get $\mathbf{G}$.)

(b) $\int_C \mathbf{F} \cdot d\mathbf{r} = f(4, 2, 20) - f(0, 0, 0) = ... = 90$

(Or, compute using the parametrization below.)

To get a parametrization of $C$, notice that the first equation gives $z$ in terms of $x$ and $y$, and the second gives $x$ in terms of $y$. So let $\mathbf{r}(t) = (2t, t, 5t^2)$, with $0 \leq t \leq 2$. Then compute $\int_C \mathbf{G} \cdot d\mathbf{r} = ... = e^{20} - 1$.

2. (a) $\nabla g = \langle z^2, 1, 2xz \rangle$, so $\nabla g(\mathbf{r}(5)) = \langle 9, 1, 12 \rangle$.

$h'(5) = \nabla g(\mathbf{r}(5)) \cdot \mathbf{r}'(5) = \langle 9, 1, 12 \rangle \cdot \langle -1, \pi, 2 \rangle = 15 + \pi$

(b) $9x + y + 12z = 47$

(c) We want $\mathbf{u} = \langle a, b, c \rangle$ so that $0 = \nabla g(2, -7, 3) \cdot \mathbf{u} = 9a + b + 12c$. One answer is $\mathbf{u} = \langle 1, 3, -1 \rangle / 11$. (Technically, “direction” means unit vector; one may also say “in the direction of $\langle 1, 3, -1 \rangle$.” But I’m not very concerned about this technicality.)
3. (a) \( \mathbf{r}(u,v) = (u, v^2, v) \) with \( u^2 + v^2 \leq 4 \), is the easiest one to write down.
\( \mathbf{r}(u,v) = (u \cos(v), u^2 \sin^2(v), u \sin(v)) \) with \( 0 \leq u \leq 2, 0 \leq v \leq 2\pi \) also works.

For both of these, \( \mathbf{r}_u \times \mathbf{r}_v \) has negative \( \mathbf{j} \) component, so the orientation given by the parametrization is opposite to the specified orientation.

(b) I'll use \( \mathbf{q}(t) \) for the curve, because we're already using \( \mathbf{r} \) for the surface:
\( \mathbf{q}(t) = (2 \cos(t), 4 \sin^2(t), 2 \sin(t)), \) for \( 0 \leq t \leq 2\pi \).

The orientation consistent with the given orientation of \( S \) is counterclockwise as viewed from the positive \( y \)-axis, which is opposite to the orientation of the parametrization \( \mathbf{q} \).

(c) Using Stokes' Theorem,
\[
\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = -\int_0^{2\pi} \mathbf{F}(\mathbf{q}(t)) \cdot \mathbf{q}'(t) dt = \ldots = 8\pi.
\]
(The minus sign in front of the integral comes from orientation discussion in part (b).)

4. (a) \( f(x,y,z) = 2z \) is constant around each horizontal circular cross-section. The length of such a cross-section is \( 6\pi \), so the surface integral is equal to \( 6\pi \int_0^5 2z dz = 150\pi \).

(b) doesn’t make sense, because we can’t take a dot product of a scalar function with the vector \( d\mathbf{S} = \mathbf{n} dS \).

(c) This integral is the flux of \( \mathbf{F} \) through \( S \). But \( \mathbf{F} \) is constant, so the flux in one side of \( S \) is equal to the flux out the other side. Thus the integral is zero.

Same idea, described slightly differently: By symmetry of \( S \), we have \( \mathbf{F} \cdot \mathbf{n} \) with the same magnitude and opposite sign at the points \((a,b,c)\) and \((-a,-b,c)\) in \( S \), so the integral will be zero.

I didn’t ask you about the fourth variation, \( \iint_S \mathbf{F} d\mathbf{S} \). This expression “makes sense,” its meaning is to compute the integral of each of the components of \( \mathbf{F} \), resulting in three numbers that are the components of a single vector. But it’s not a type of calculation we’ve done, because it’s not a concept that is particularly useful.