Solutions. The solutions below for the exam problems refer to pictures on the green handout given out in class on 10/19/11. There are some extra copies outside my office, PDL C338 (in a bin on the wall next to the door).

Problem 1. (a) \( \int_0^2 \int_0^{x^2} f(x, y) \, dy \, dx \). (See picture on green handout.)

\[ \bar{x} = \frac{1}{m} \int_D x f(x, y) \, dA = \frac{k}{m} \int_D xe^y \, dA. \]

The integral is easiest using the set up from part (a):

\[ \int_0^2 \int_0^{x^2} xe^y \, dy \, dx = \int_0^2 xe^y \bigg|_{y=0}^{y=x^2} \, dx = \int_0^2 x(e^{x^2} - 1) \, dx = \frac{e^{x^2}}{2} - \frac{x^2}{2} \bigg|_0^2 = \frac{e^4 - 5}{2}. \]

(It can also be done in the original order:

\[ \int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} xe^y \, dx \, dy = \int_0^4 \frac{x^2}{2} e^y \bigg|_{x=-\sqrt{y}}^{x=\sqrt{y}} \, dy = \int_0^4 (2e^y - y e^y) \, dy = 2e^y - ye^y + e^y \bigg|_0^4 = \frac{e^4 - 5}{2}, \]

using integration by parts to get \( \int ye^y \, dy = ye^y - e^y + C. \) Either way, \( \bar{x} = \frac{k(e^4 - 5)}{2m} \).

Problem 2. See picture on handout. If you don’t know the polar equation for this circle, find it as follows. The \( xy \)-equation for the circle is \((x - 2)^2 + y^2 = 4\), or \(x^2 + y^2 = 4x\), so \(r^2 = 4r \cos \theta\), or \(r = 4 \cos \theta\).

\[ \int_{\pi/4}^{\pi/2} \int_0^{4 \cos \theta} \int_0^{\cos \theta} \theta \, dr \, d\theta = k \int_{\pi/4}^{\pi/2} \int_0^{4 \cos \theta} \cos \theta \, d\theta = 4k \int_{\pi/4}^{\pi/2} \cos \theta \, d\theta = 4k \sin \theta \bigg|_{\pi/4}^{\pi/2} = 4k(1 - \frac{1}{\sqrt{2}}). \]

(It’s not too hard to set up the integral for this mass in \( xy \)-coordinates but the integrand is \((x^2 + y^2)^{-1/2}\), which is rather complicated to integrate.)

Problem 3. (a) I’m giving a detailed explanation of how to find the limits – more than was required for full credit – because this problem seemed to be the hardest one on the test.

To find the limits \( y \) and \( z \), look at region \( D \) in the \( yz \)-plane which is the projection of the region \( E \) to this plane and is bounded by \( y = z \), \( y^2 + z^2 = 1 \), and the \( y \)-axis. (See picture of \( D \) on handout.) In \( D \), the largest value of \( z \) occurs where the surfaces \( y = z \) and \( y^2 + z^2 = 1 \) intersect, so we get \( 2z^2 = 1 \), or \( z = 1/\sqrt{2} \), and the lower limit for \( z \) is \( 0 \). Still considering \( D \), for fixed \( z \), the lowest value of \( y \) occurs on \( y = z \) and the highest value on the circle \( y^2 + z^2 = 1 \). Finally, for the limits for \( x \) we must think about the three dimensional region \( E \), pictured on the handout. For any point \((y, z)\) in \( D \), the \( x \) values are bounded below by the \( yz \)-plane, and bounded above by the sphere. Putting all this together, we get

\[ \int_0^{1/\sqrt{2}} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_0^{\sqrt{1-z^2-y^2}} f(x, y, z) \, dx \, dy \, dz. \]

Originally, I was going to give you this iterated integral, and ask you to rewrite it as an iterated integral with respect to \( dy \, dz \, dx \). Try this, and for even more practice, try several other orders of integration; answers for all but one at the end of this document.
(b) If we integrate first with respect to $z$, then we have to set limits for $z$ for each choice of $(x, y)$ possible in $E$. The lower limit for $z$ is always 0, but the upper limit is on the sphere some places and on the plane $y = z$ others. We can see this even just looking at the intersection of $E$ with the $yz$-plane, which happens to be the same as the projection onto the $yz$-plane in this example: for $x = 0$ and $y \leq 1/\sqrt{2}$, the upper bound is $z = y$, and for $x = 0$ and larger $y$ it’s $z = \sqrt{1 - y^2}$. More precisely, we have an upper limit of $z = y$ in region $R_1$ in the $xy$-plane (see picture on handout) and an upper limit $z = \sqrt{1 - x^2 - y^2}$ in region $R_2$.

**Problem 4.** The sphere $x^2 + y^2 + z^2 = 2$ is centered at $z = 1$ on the $z$-axis and has radius 1. (On the handout there is a picture of the cross section of the two spheres.) In spherical coordinates, it’s

$$
\rho^2 = x^2 + y^2 + z^2 = 2z = 2\rho \cos \phi,
$$

and we can safely cancel one factor of $\rho$, because the origin remains a solution of the resulting equation $\rho = 2 \cos \phi$.

Next we find the $\rho$-coordinate where the spheres intersect. The other sphere has the equation $\rho = 1$, so they intersect where $\rho = 1 = 2 \cos \phi$. This implies $\phi = \pi/3$. Thus the integral is

$$
\int_0^{2\pi} \int_0^{\pi/3} \int_1^{2 \cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.
$$

(You can change the order of integration by putting $d\theta$ last, first, or in the middle, but must put $d\rho$ before $d\phi$ unless you want to deal with very messy limits of integration.)

I was going to ask you to compute the volume, but deleted that question because of time. In case you want to try it for practice, here’s the final answer: $\frac{11}{12} \pi$.

**Problem 5.** By the change of variable formula in §15.9,

$$
\iint_R \left( \left( \frac{x}{2} \right)^2 + \left( \frac{y}{3} \right)^2 \right) \, dA = \iint_S \left( \left( \frac{2u}{2} \right)^2 + \left( \frac{3v}{3} \right)^2 \right) \left| \frac{\partial (x, y)}{\partial (u, v)} \right| \, dA,
$$

where we need to figure out what region $S$ is in the $uv$-plane and compute the Jacobian. Substituing, we find $S$ is bounded by

$$
1 = \left( \frac{x}{2} \right)^2 + \left( \frac{y}{3} \right)^2 = \left( \frac{2u}{2} \right)^2 + \left( \frac{3v}{3} \right)^2 = u^2 + v^2
$$

so $S$ is the interior of the unit circle centered at the origin. Pictures of $R$ and $S$ are on the handout. Because they are different regions, you should not use the same letter for both of them! The Jacobian is

$$
\frac{\partial (x, y)}{\partial (u, v)} = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = 6.
$$

Because the transformation is so simple, I did not require that you show the matrix. If you just wrote that $dA = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v}$, be sure you know how to handle cases where none of the entries in the matrix are zero.
To do the integral on $S$ in the $uv$-plane, it is easiest to use polar coordinates:

$$\int\int_S (u^2 + v^2)6du dv = \int_0^{2\pi} \int_0^1 r^2 r d\theta dr = 6 \int_0^{2\pi} d\theta \int_0^1 r^3 dr = 12\pi r^4 \Big|_0^1 = 3\pi.$$ 

Other orders of integration for #3.

$$\int_0^{1/\sqrt{2}} \int_0^y \int_0^{\sqrt{1-x^2-y^2}} f(x, y, z) \, dx \, dz \, dy + \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{\sqrt{1-z^2-y^2}} f(x, y, z) \, dx \, dz \, dy$$

$$\int_0^1 \int_0^{\sqrt{1-x^2/2}} \int_z^{\sqrt{1-x^2-z^2}} f(x, y, z) \, dy \, dz \, dx$$

$$\int_0^{1/\sqrt{2}} \int_0^{\sqrt{1-x^2}} \int_z^{\sqrt{1-x^2-z^2}} f(x, y, z) \, dy \, dz \, dx$$

$$\int_0^1 \int_0^{\sqrt{1-x^2/2}} \int_z^y f(x, y, z) \, dz \, dy \, dx + \int_0^1 \int_0^{\sqrt{1-x^2}} \int_z^{\sqrt{1-x^2-z^2}} f(x, y, z) \, dz \, dy \, dx$$

The remaining possibility would have to be written as a sum of three integrals; can you see why?