Problem 1. (a) \( \nabla f = (\cos \pi y, -\pi x \sin \pi y + e^z, ye^z) \), so \( \nabla f(2,3,1) = -i + ej + 3ek. \)

(b) Let \( \mathbf{w} \) be the vector from (2,3,1) to (5,3,5). Then \( \mathbf{r}'(0) = 2\frac{\mathbf{w}}{||\mathbf{w}||} = 2\frac{(3,0,4)}{\sqrt{9+0+16}} = (\frac{6}{5},0,\frac{8}{5}), \)

so \( \frac{d}{dt} f(\mathbf{r}(t)) = \nabla f \cdot \mathbf{r}'(0) = (-1,e,3e) \cdot (\frac{6}{5},0,\frac{8}{5}) = \frac{24e - 6}{5}. \)

Problem 2. Let \( C_1 \) and \( C_2 \) be the two line segments.

Parametrize: \( C_1, \mathbf{r}(t) = (t,t,t) \) for \( 0 \leq t \leq 1 \), and \( C_2, \mathbf{r}(t) = (1,1-t,1) \) for \( 0 \leq t \leq 1. \)

\[ \text{mass} = \int_C (2-z)ds = \int_{C_1} (2-t)\sqrt{1+1+1}dt + \int_{C_2} (2-1)\sqrt{0+1+0}dt = ... = \frac{\sqrt{3}}{2} + 1. \]

Problem 3. (a) The components \( P = x^3 - 2xy^3 \) and \( Q = -3x^2y^2 \) are defined and continuously differentiable everywhere on the \( xy \)-plane, which is simply connected (has “no holes”). So \( \mathbf{F} \) will be conservative if \( \partial P/\partial y = \partial Q/\partial x. \) Both these partials are \(-6xy^2\), so \( \mathbf{F} \) is conservative.

(b) If \( f \) is a potential for \( \mathbf{F} \), then \( \frac{\partial f}{\partial x} = P = x^3 - 2xy^3. \) Integrating with respect to \( x, f(x,y) = \frac{x^4}{4} - x^2y^3 + g(y). \) Then computing \( \frac{\partial f}{\partial y} \) from this and setting it equal to \( Q = -3x^2y^2, \) we get \( \frac{\partial f}{\partial y} = -3x^2y^2 + g'(y) = -3x^2y^2. \) Thus \( g(y) \) is constant, and \( f(x,y) = \frac{x^4}{4} - x^2y^3 + k \) is a potential for \( \mathbf{F}. \) (One potential function suffices as an answer, but more generally, \( f(x,y) = \frac{x^4}{4} - x^2y^3 + k \) for any constant \( k \) is also a potential for \( \mathbf{F}. \))

(c) Let \( C \) be the given parametrized curve, \( \mathbf{r} = (\cos^3 t, \sin^3 t). \) Because we know a potential for \( \mathbf{F} \) from (b), we can use the Fundamental Theorem for Line Integrals and compute

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(\pi/2)) - f(\mathbf{r}(0)) = f(0,1) - f(1,0) = (0 - 0) - \left( \frac{1}{4} - 0 \right) = -\frac{1}{4}. \]

Problem 4. (a) Idea: The equation for the plane, \( y + z = 5, \) will give me the formula for \( z \) if I have one for \( y \) (or vice versa). So I should pick formulas for \( x \) and \( y \) that make sense for the cylinder, going completely around it; for instance, \( x(t) = 3\cos t \) and \( y(t) = 3\sin t. \) Thus \( \mathbf{r}(t) = (3\cos t,3\sin t,5 - 3\sin t) \) for \( 0 \leq t \leq 2\pi \) will work.

(b) \( \mathbf{r}'(t) = (-3\sin t,3\cos t,-3\cos t) \)

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (3\cos t,6\sin t,-4) \cdot (-3\sin t,3\cos t,-3\cos t)dt \]

\[ = \int_0^{2\pi} (-9\cos t\sin t + 18\cos t\sin t + 12\cos t)dt \]

\[ = \int_0^{2\pi} (9\cos t\sin t + 12\cos t)dt = \frac{9}{2}\sin^2 t - 12\sin t \bigg|_0^{2\pi} = 0 \]

(If I hadn’t said “Use your parametrization to compute,” what other way might you have done this integral?)