The Robinson-Schensted-Knuth Correspondence: Properties, Applications, and Generalizations

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Abstract

Since it was originally defined by Schensted [19] in 1961 to study increasing and decreasing subsequences of permutations, the Robinson-Schensted-Knuth correspondence has been generalized and applied in various ways to answer different questions in the fields of algebra, combinatorics, and representation theory. It and its generalizations have been used to study identities for symmetric functions [18] [16], reduced words in the symmetric group [21] [2], Kazhdan-Luzstig cells in some Coxeter groups [1] [13] [6], and asympotic distribution of Young tableaux [17].

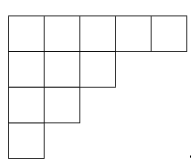
1 Definitions and Notation

1.1 Defining RSK

The main goal of this section is to define the Robinson-Schensted-Knuth correspondence, hereby refered to as RSK. First, we need a few preliminaries. We assume basic knowledge of the symmetric group S_n, but we review background on partitions and Young tableau here. We take weakly increasing (resp. decreasing) to allow for equality between consecutive elements in a sequence, whereas (strictly) increasing (resp. decreasing) does not allow for equality. The same idea applies to weakly (strictly) left or right. Let $\mathbb{P}$ denote the set of positive integers.

Definition 1.1. *A* **partition** $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ *is a weakly decreasing sequence of positive integers. We say* λ *is a partition of* n*, denoted* $\lambda \vdash n$*, if* $\sum_{j=1}^{m} \lambda_j = n$*. For example, the partitions of* 4 *are* $(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$.

Definition 1.2. *The* **Young diagram** *for a partition* $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_m)$*, is a left-justified set of cells with* λ_k *cells in the* k*-th row for* $k = 1, \ldots, n$*. For example, the Young diagram for the partition (5,3,2,1) is*



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Definition 1.3. *For a partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$*, a* **Semistandard Young Tableau (SSYT)***,* T *of shape* λ *is a filling of the Young diagram for* λ *with positive integers so that the entries weakly increase along rows left to right and strictly increase along columns top to* *bottom, and the set of these is denoted by* $\mathrm{SSYT}(\lambda)$. *A* **Standard Young Tableau (SYT)** *is a SSYT where* $1, 2, \ldots n$*, each appear exactly once, and the set of these of shape* λ *is denoted by* $\mathrm{SYT}(\lambda)$. *When we say tableau, we implicitly mean a SSYT for a partition shape. For example,*

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 \\ \hline 2 & 3 & 3 & 3 \\ \cline{1-4} 4 & 4 & 5 \\ \cline{1-3} \end{array}$$

is a (semistandard Young) tableau, and the elements of $\mathrm{SYT}((3,2))$ *are*

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}\ .$$

We also define a generalization of tableaux, called skew tableaux, for later use.

Definition 1.4. *Suppose* λ, μ *are partitions with* $\mu \subset \lambda$ *in the sense of the Young diagrams (or* $\mu_k \le \lambda_k$ *for all* k*). The skew shape* λ/μ *is simply the Young diagram for* λ *remove the Young diagram for* μ*. A skew tableau of shape* λ/μ *is a filling of the skew shape* λ/μ *with positive integers so that the rows are weakly increasing and columns are strictly increasing. We refer to such objects as skew tableaux. For example, the following is a skew tableau with shape (5,4,3,2,1) / (3,2,1,1).*

$$\begin{array}{ccccc} & & & \boxed{1} & \boxed{1} \\ & & \boxed{1} & \boxed{2} & \\ & \boxed{2} & \boxed{3} & & \\ & \boxed{3} & & & \\ \boxed{4} & & & & \end{array}$$

In general, let $\mathrm{sh}(T)$ *denote the shape of a (skew) tableau* T.

Definition 1.5. *Suppose we are given a tableau* T *and an element* $x \in \mathbb{P}$*. The insertion of* x *into* T*, denoted* $T \leftarrow x$*, is defined as follows.*

1. *If* x *is larger than every element in the first row of* T*, we terminate by appending* x *to the end of the first row.*

2. *Else, there is a leftmost element in this row which is the smallest element larger than* x*, call it* y*, where we break ties from left to right. Then, replace* y *by* x*, and insert* y *into the next row of* T *in the same way. Here, we say that* x **bumps** y.

3. *Continue to let the element entering a row bump down the leftmost element larger than it, and insert that element into the next row, until we are as in step 1, where the element is larger than every element the row into which it is inserting, and append it to the end of that row. This includes the case where that row is empty, where it becomes a new row.*

For example,

$$\begin{array}{ccc} 1 & 4 & 6 \\ 2 & 5 & \\ 7 & 8 & \end{array} \leftarrow 3 \Longrightarrow \begin{array}{ccc} 1 & 3 & 6 \\ 2 & 5 & \\ 7 & 8 & \end{array} \Longrightarrow \begin{array}{ccc} 1 & 3 & 6 \\ 2 & 4 & \\ 7 & 8 & \end{array} \Longrightarrow \begin{array}{ccc} 1 & 3 & 6 \\ 2 & 4 & \\ 5 & 8 & \end{array} \Longrightarrow \begin{array}{ccc} 1 & 3 & 6 \\ 2 & 4 & \\ 5 & 8 & \\ 7 & & \end{array}.$$

where 3 *bumps 4, which bumps 5, which bumps 7, which forms a new row.*

If x is inserted into cell C_1, which bumps an element into cell C_2, ..., which bumps an element into cell C_m, we say that the bumping chain of x is $C_1, C_2, \ldots, C_m$. Whenever we bump down an element y, the element strictly below y is already larger than y, so bumping chains progess weakly from right to left. The bumping chain for insertion in the example (1,2), (2,2), (3,1), (4,1) does in fact proceed weakly from right to left. Notice that insertion changes the shape of a tableau by adding a single cell to the shape. Now we are ready to define RSK.

Definition 1.6. *Suppose we are given a word $w = w_1 \ldots w_n$, with $w_i \in \mathbb{P}$ for all i. Form a sequence of tableaux (P_k, Q_k) for $k = 0, \ldots, n$ as follows. Initially, set $P_0 = Q_0 = \emptyset$, the empty tableau. Then, recursively define P_k by*

$$P_k = P_{k-1} \leftarrow w_k,$$

and form Q_k from Q_{k-1} by adding the unique cell in $\mathrm{sh}(P_k)$ but not in $\mathrm{sh}(P_{k-1})$ with entry k. Finally, we terminate with $P := P_n$ and $Q := Q_n$. Let

$$RSK(w) = (P, Q).$$

That is, we insert the sequence of elements $w_1, \ldots, w_n$ in that order to form tableau P, and then we record where the k-th insertion terminated by the position of k in Q. Here, P is called the insertion tableau, and Q the recording tableau. We will refer to these for a given word w by $P = P(w)$ and $Q = Q(w)$. For example, for $w = 623514$,

$$(P_1, Q_1) = \begin{array}{c} 6 \end{array}, \begin{array}{c} 1 \end{array} \Longrightarrow (P_2, Q_2) = \begin{array}{c} 2 \\ 6 \end{array}, \begin{array}{c} 1 \\ 2 \end{array} \Longrightarrow (P_3, Q_3) = \begin{array}{cc} 2 & 3 \\ 6 & \end{array}, \begin{array}{cc} 1 & 3 \\ 2 & \end{array}$$

$$\Longrightarrow (P_4, Q_4) = \begin{array}{ccc} 2 & 3 & 5 \\ 6 & & \end{array}, \begin{array}{ccc} 1 & 3 & 4 \\ 2 & & \end{array} \Longrightarrow (P_5, Q_5) = \begin{array}{ccc} 1 & 3 & 5 \\ 2 & & \\ 6 & & \end{array}, \begin{array}{ccc} 1 & 3 & 4 \\ 2 & & \\ 5 & & \end{array}$$

$$\Longrightarrow RSK(w) = (P, Q) = \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 & \\ 6 & & \end{array}, \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 6 & \\ 5 & & \end{array}.$$

If the word has repeats, we break ties from left to right as illustrated in the following insertion:

$$\begin{array}{ccccc} 1 & 1 & 2 & 2 & 2 \\ 2 & 3 & 3 & & \\ 4 & 4 & & & \end{array} \leftarrow 1 = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & & \\ 3 & 4 & & & \\ 4 & & & & \end{array}.$$

So, RSK can take any word on $\mathbb{P}$ as input.

1.2 The Hook Length Formula

A natural question any combinatorialist would ask is can we count the number of SYT of a given shape, and it turns out there is a remarkably simple answer, originally discovered by Frame, Robinson, Thrall [4]. For a partition λ, let $f^\lambda = \#\,\mathrm{SYT}(\lambda)$.

Definition 1.7. *For a given cell C in a Young diagram, its* **hook length** *is the number of cells directly left of C, below C, or C itself. For example, for the shapes below, we fill each cell with its hook length.*

7	6	4	3	1
5	4	2	1	
2	1			

and

4	3	1
2	1	

Labelling cells by (i, j) by their row and column number respectively, let $h_{i,j}$ denote the hook length for cell (i, j).

Theorem 1.8. *The hook length formula states that for $\lambda \vdash n$,*

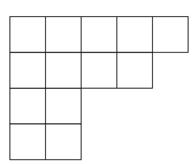
$$f^\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}.$$

For example, if $\lambda = (3, 2)$, looking at hook lengths in the second example above,

$$f^{(3,2)} = \frac{5!}{4 \cdot 3 \cdot 1 \cdot 2 \cdot 1} = 5,$$

which agrees with our list of SYT((3, 2)) in the SYT example. Although the formula is relatively simple, its proofs are more complex. A particularly elegant proof is given in [9] by Greene, Nijenhuis, and Wilf using a probabalistic argument, which we summarize below.

Definition 1.9. *We say a cell (i, j) is an* **inner corner** *of a shape λ if $(i, j) \in \lambda$ but $(i + 1, j) \notin \lambda$ and $(i, j + 1) \notin \lambda$. For example, ther inner corners of $(4, 4, 2, 2)$, shown below, are $(1, 5), (2, 4), (4, 2)$.*



Because the largest entry n must appear in an inner corner in a SYT, we have the recurrence

$$f^\lambda = \sum_{C \text{ corner of } \lambda} f^{\lambda \setminus C},$$

where the sum is over all inner corners C of λ, and $\lambda \setminus C$ represents λ with cell C removed. Greene, Nijenhuis, and Wilf defined an algorithm by picking a uniformly random cell, then picking a cell on its hook uniformly at random, and terminating when we reach an inner corner. They showed that the probability of hitting a particular inner corner C is the same as the probability that n appears in C in a uniformly random SYT of shape λ. They use this fact to show that f^λ and $\frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}$ each obey the same initial conditions and recurrence, and so must be the same. Furthermore, their algorithm can be used to uniformly sample from $\mathrm{SYT}(\lambda)$.

2 Basic Properties

Why would anyone be interested the RSK algorithm? Why are we bumping elements in this way? One of the purposes of this paper is to convince the reader that RSK is not at all arbitrary, which we do by exploring its basic properties. This insertion construction is far from an arbitrary one, for one would not expect an arbitrary insertion algorithm to have all of these wonderful properties. Here, we describe some key properties of this algorithm. We try to give a vague idea why each property holds or at least give the key technique used in the proof, but do not go into any of the details. The interested reader may seek more details in [18], Chapter 3.

2.1 Bijection

Proposition 2.1. *The map*

$$RSK : S_n \to \bigcup_{\lambda \vdash n} \mathrm{SYT}(\lambda)^2$$

is a bijection.

First, we describe why P, Q are actually SYT. By construction, they both have entries $1, \ldots, n$. P_k remains an SSYT at each step, as we insert into rows to preserve rows increasing, and bumping chains proceed form right to left, so the columns remain increasing. This explains why Schensted may have defined insertion in this way, so it preserves being a tableau. Also, Q is a SYT since we continue to add largest entry to an innder corner. Also, for inverting RSK, given any $\lambda \vdash n$ and $P, Q \in SYT(\lambda)$, we can undo each step in RSK in a unique way. We must have ended with the last bump ending in the cell where n appears in Q. Then, we can reverse the bumping procedure in P by "unbumping up" the largest entry that is smaller then the given "unbumped" entry starting from this cell where n appeared in Q, which must unbump an element out of the tableau P. Note that if applicable, we break ties between largest entries from right to left. From this bijection, we get the enumerative result that

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = |S_n| = n!$$

This looks similar to the result

$$\sum_{j=1}^{m} (\dim(\rho_j))^2 = |G|,$$

where $\{\rho_j\}_{j=1}^m$ are the irreducible representations of a finite group G. In fact, the irreducible representations of S_n, $\{\rho_\lambda\}_{\lambda\vdash n}$ are indexed by partitions of n and have dimensions $\dim(\rho_\lambda) = f^\lambda$. The representation theory of the symmetric groups is a beautiful theory, which the interested reader may learn more about in [18] or [5]. We simply state that Young tableaux are the main tool used in most constructions of these representations. In fact, Young invented Young tableaux for this purpose.

2.2 Symmetry

Proposition 2.2. *For $w \in S_n$, we have*

$$P(w^{-1}) = Q(w), \qquad Q(w^{-1}) = P(w).$$

This can be seen by observing that the bumping process for determining where n ends up in $P(w)$ and the process for determining where n ends up in $Q(w^{-1})$ result in the same cell and then inducting. For example, we have $(526143)^{-1} = 426513$,

$$RSK(526143) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline\end{array}\,, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline\end{array} \quad \text{and} \quad RSK(426513) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline\end{array}\,, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline\end{array}$$

In particular, if w is an involution, then $w = w^{-1}$, so $P(w) = Q(w)$. For example,

$$P(361452) = Q(361452) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}\,, \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}\,.$$

Therefore, RSK restricts to a bijection between involutions in S_n and the set of all SYT of size n.

2.3 Descents

We begin with an observation about the interaction between order and bumping chains.

Proposition 2.3. *Suppose we take a tableau T, insert x, and then insert y. If $y < x$, then y's bumping chain is weakly left of x's bumping chain, meaning the new cell created from y's insertion is weakly left of the new cell created from x's inserttion. Else, if $y \geq x$, then y's bumping chain stays strictly right of x's bumping chain, meaning the new cell created from y's insertion is strictly right of the new cell created from x's insertion.*

For example, when we insert 2 then 5 into the tableau

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 4 & 7 \\ \cline{1-2}\end{array}\,,$$

we get

$$(T \leftarrow 2) = \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 7 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}\;, \quad ((T \leftarrow 2) \leftarrow 5 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 6 \\ \cline{1-2} 4 & 7 \\ \cline{1-2}\end{array}\;,$$

whose bumping chains $(1,2),(2,1),(3,1)$ and $(1,3),(2,2),(3,2)$ proceed strictly from left to right.

Definition 2.4. *We say a word $w = w_1 \ldots w_n$ has a descent at i if $w_i > w_{i+1}$. We say a SYT T has a descent at i if $i+1$ appears below i in T. We let D be the operator that takes the descent set of words or tableaux. For both words and tableaux, we let the major index (*maj*) be the sum of the positions of the descents, so*

$$\mathrm{maj}(w) = \sum_{i \in D(w)} (i), \qquad \mathrm{maj}(T) = \sum_{i \in D(T)} (i).$$

For example, the descent set of the following tableau is $\{2,5\}$.

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}\;.$$

Proposition 2.3 tells us that under RSK, $D(w) = D(Q(w))$. Note that this is consistent with the above example in which $D(361452) = \{2,5\}$ as well. This will be quite useful when proving identities involving symmetric functions.

2.4 Knuth Equivalence

A natural question to ask is can we characterize the words with a given P-tableau?

Definition 2.5. *A Knuth move is an involution on three letter sequences of the form*

$$\begin{aligned} xzy &\leftrightarrow zxy \quad \text{for } x \le y < z, \\ yxz &\leftrightarrow yzx \quad \text{for } x < y \le z. \end{aligned}$$

Then, we can form the Knuth equivalence relation $\sim_K$ by saying that two sequences are Knuth equivalent if they differ by a sequence of Knuth moves. For example, $25134 \sim_K 23154$ by the following seqeunce of Knuth moves.

$$25134 \sim_K 21534 \sim_K 21354 \sim_K 23154.$$

Theorem 2.6. *For any words x, y,*

$$P(x) = P(y) \text{ if and only if } x \sim_K y.$$

Since Knuth moves do not affect the P-tableau, $x \sim_K y$ implies $P(x) = P(y)$. To see the reverse implication, we define an element of each Knuth (equivalence) class that deserves special attention.

Definition 2.7. *The reading word π_T of a skew tableaux T is the entries of T listed row by row from left to right, listed from bottom to top. For example, the following tableaux has reading word* 86734125*.*

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array} \quad .$$

The reading word is constructed so that $P(\pi_T) = T$, since the rows bump each other down one by one. So, the Knuth class for a given tableau T is the reading word π_T together with all other sequences that can be reached from it by Knuth moves. Then, one can show that if $P(x) = T$, then x can be turned into π_T by a sequence of Knuth moves, by mirroring the bumping process in RSK by Knuth moves. We also have dual Knuth equivalence defined by $x \sim_{dK} y$ if and only if $Q(x) = Q(y)$. By symmetry, we have

$$x \sim_{dK} y \text{ if and only if } x^{-1} \sim_K y^{-1}.$$

2.5 Jeu de Taquin

The main other action that has been studied on Young tableaux is sliding, also called jeu de taquin, which we introduce here because of its relationship with RSK, It is a very powerful tool in the combinatorics of tableaux and even gives us another way to define the P-tableau.

Definition 2.8. *A backward slide on a skew tableau T of shape λ/μ starting from an inner corner C of μ is a process initiated by sliding a cell into C, then sliding a cell into the place vacated by the previous slide, and continuing the slide a new cell into the previous vacated cell, until the vacated cell is outside of the skew tableau. In a single move, letting V denote vacated cell,*

$$\begin{array}{|c|c|}\hline V & a \\ \hline b \\ \cline{1-1}\end{array} \quad \textit{becomes} \quad \begin{array}{|c|c|}\hline a & V \\ \hline b \\ \cline{1-1}\end{array} \quad \textit{if } a \le b,$$

$$\begin{array}{|c|c|}\hline V & a \\ \hline b \\ \cline{1-1}\end{array} \quad \textit{becomes} \quad \begin{array}{|c|c|}\hline b & a \\ \hline V \\ \cline{1-1}\end{array} \quad \textit{if } b < a.$$

For example, by performing a forward slide,

$$\begin{array}{|c|c|c|}\hline V & 2 & 5 \\ \hline 1 & 3 & 6 \\ \hline 4 & 7 & 8 \\ \hline\end{array} \quad \textit{becomes} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 6 & 8 \\ \hline 4 & 7 \\ \cline{1-2}\end{array}$$

We now state a wonderful relationship between sliding and Knuth equivalence.

Proposition 2.9. *If S, T are skew tableaux that are related by a sequence of slides, then $\pi_S \sim_K \pi_T$.*

In the previous example, observe that we went from reading word 47813625 to 47368125 and

$$P(47813625) = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 6 & 8 \\ \hline 4 & 7 \\ \cline{1-2} \end{array} = P(47368125),$$

which verifies reading words are Knuth equivalent in this case. Suppose we are given a skew tableau S and perform a sequence of slides until it has partition shape, say tableau T. As slides preserve Knuth equivalence, $\pi_S \sim_K \pi_T$, so as T has partition shape,

$$T = P(\pi_T) = P(\pi_S).$$

In particular, this means that there is only one possible result from sliding S to partition shape, no matter the sequence of slides! This means we can define $P(w)$ in terms of jeu de taquin as follows. Given a word w, put $w_1, \ldots, w_n$ into the cells $(n, 1), \ldots (1, n)$ to get skew tableau S, making $\pi_S = w$. Then, $P(w)$ is the unique result we get by sliding S to partition shape. For example, when we slide to normal shape

$$\begin{array}{cccc} \cline{4-4} & & & \multicolumn{1}{|c|}{1} \\ \cline{3-4} & & \multicolumn{1}{|c|}{4} \\ \cline{2-3} & \multicolumn{1}{|c|}{2} \\ \cline{1-2} \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array} \quad \text{becomes} \quad \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array} = P(3241).$$

3 Symmetric Function Identities

3.1 Background on Symmetric Function Theory

Symmetric function theory is a large area of knowledge and research, but we only mention a few key definitions and properties that we will refer to later. For more background, see [18], and for even more, see [16].

Definition 3.1. *Given a tableau T, we define its weight as the monomial in $\mathbb{Z}[x_1, x_2, \ldots]$ given by*

$$x^T = \prod_{k \geq 1} (x_k)^{\textit{Number of times } k \textit{ appears in } T}.$$

For example, for

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 \\ \hline 2 & 3 & 3 & 3 \\ \cline{1-4} 4 & 4 & 5 \\ \cline{1-3} \end{array}, \quad x^T = x_1^2 x_2^4 x_3^3 x_4^2 x_5.$$

We define the Schur function $s_\lambda(x)$ associated to a partition λ as the formal power series

$$s_\lambda(x) := \sum_{T \in SSYT(\lambda)} x^T$$

For example, with $\lambda = (2,1)$, the possible fillings using only 1,2,3 are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\ , \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\ , \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\ , \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\ , \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\ , \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\ , \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\ , \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\ .$$

So truncating $s_{(2,1)}$ to the variables x_1, x_2, x_3 gives

$$s_{2,1}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x^3 + x_1 x_2^2 + x_1 x_3^2 + 2x_1 x_2 x_3 + x_2^2 x^3 + x_2 x_3^2.$$

It is not clear from the definition that the Schur functions are symmetric, but in fact they are, which can be shown by, for each i, exhibiting an involution on $SSYT(\lambda)$ that swaps the number of i's and $(i+1)$'s. Furthermore, the Schur functions form a basis for the ring of symmetric functions, denoted SYM. The Schur functions are ubiquitous in the field of algebraic combinatorics, and they have applications to representation theory, but I will only mention one important fact here. One can define a representation ring R of the symmetric groups, where addition corresponds to direct summing the representations and multiplication corresponds to inducing the tensor product of the representations into a symmetric group. Then, the Frobenius characterisitic map

$$\text{ch} : \text{R} \to \text{SYM}$$

is a ring isomorphism that maps the irreducible representation associated to λ to s_λ. For more background, see [18].

Definition 3.2. *The fundamental quasisymmeric functions F_S^n, indexed by pairs (n, S) with $n \in \mathbb{N}$ and $S \subset [n-1]$, are given by*

$$F_S^n(x) := \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n, \\ i_j < i_{j+1} \text{ for all } j \in S}} x_{i_1} x_{i_2} \ldots x_{i_n}.$$

By a standardization argument in [7], Gessel showed that for $\lambda \vdash n$,

Theorem 3.3. *[7] For each $\lambda \vdash n$,*

$$s_\lambda(x) = \sum_{T \in SYT(\lambda)} F_{D(T)}^n(x)$$

3.2 The Littlewood-Richardson Rule

The Littlewood Richardson Rule (LRR) is like the Pythagorean Theorem of algebraic combinatorics in terms of the number of proofs that it has, one of which we present here. The LRR expresses the product of two Schur functions in terms of Schur functions, and since the Schur functions for a basis for the symmetric functions, governs the multiplication of symmetric functions. In order to state the LRR, we need a couple definitions.

Definition 3.4. *Given a skew tableau T, we say that its content is* $\operatorname{cont}(T) = (\nu_1, \ldots, \nu_m)$ *where ν_k is the number of k's in T, and m is the largest entry in T. For example, the following skew tableau has content (1,3,1,2,0,3):*

$$
\begin{array}{|c|c|c|c|c|}
\cline{3-5}
\multicolumn{2}{c|}{} & 1 & 2 & 4 \\
\cline{2-5}
\multicolumn{1}{c|}{} & 2 & 3 & 4 & 6 \\
\hline
2 & 6 & 6 & \multicolumn{2}{c}{} \\
\cline{1-3}
\end{array}\ .
$$

Definition 3.5. *For a word w, we say that w is a Yamanouchi word if at each point from left to right, we have weakly more k's than $(k+1)$'s for all k. For example,* 11213234 *is a Yamanouchi word, but* 1123432 *is not, as we have more 3's than 2's after the second 3.*

Theorem 3.6. *[5] For partitions μ, ν, we have*

$$
s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda
$$

where the sum is over all partitions λ and $c^\lambda_{\mu\nu}$ is the number of skew tableau of shape λ/μ with content ν whose reverse reading word is a Yamanouchi word.

Proof. Given $T \in \mathrm{SSYT}(\mu)$, $U \in \mathrm{SSYT}(\nu)$, what Young tableaux do we get when we insert the reading row of U into T using the rules of RSK? Let $\nu = (\nu_1, \ldots, \nu_m)$, so we can write

$$
w := \pi_U = R_m R_{m-1} \ldots R_2 R_1,
$$

where each R_k lists the elements of row k in U from left to right, so has length ν_k. We insert the reading word w into T, giving us P, but for the recording tableau, label the cells added in Q by which row they came from in U, making Q a filling of some shape λ/μ with content ν. We claim that this map φ is a bijection

$$
\begin{gathered}
\varphi : \mathrm{SSYT}(\mu) \times \mathrm{SSYT}(\nu) \to \bigcup_\lambda \mathrm{SSYT}(\lambda) \times RY(\lambda/\mu, \nu), \\
(T, U) \mapsto (P, Q),
\end{gathered}
$$

where $\mathrm{RY}(\lambda/\mu, \nu)$ represents the set of all reverse tableau - interchange increasing and decreasing in the definition - with skew shape λ/μ and content ν whose reading word is a Yamanouchi word. For example, letting

$$
T = \begin{array}{|c|c|}
\hline
3 & 8 \\
\hline
5 & \multicolumn{1}{c}{} \\
\cline{1-1}
\end{array}\ , \qquad
U = \begin{array}{|c|c|c|}
\hline
1 & 2 & 9 \\
\hline
4 & 6 & \multicolumn{1}{c}{} \\
\cline{1-2}
7 & \multicolumn{2}{c}{} \\
\cline{1-1}
\end{array}\ ,
$$

we have $w = \pi_U = 746129$, and then

$$
P = T \leftarrow w = \begin{array}{|c|c|c|c|}
\hline
1 & 2 & 6 & 9 \\
\hline
3 & 4 & \multicolumn{2}{c}{} \\
\cline{1-2}
5 & 7 & \multicolumn{2}{c}{} \\
\cline{1-2}
8 & \multicolumn{3}{c}{} \\
\cline{1-1}
\end{array}\ , \qquad
Q = \begin{array}{|c|c|c|c|}
\cline{3-4}
\multicolumn{2}{c|}{} & 2 & 1 \\
\cline{2-4}
\multicolumn{1}{c|}{} & 3 & \multicolumn{2}{c}{} \\
\cline{1-2}
2 & 1 & \multicolumn{2}{c}{} \\
\cline{1-2}
1 & \multicolumn{3}{c}{} \\
\cline{1-1}
\end{array}\ .
$$

Notice that $\pi_Q = 121321$ is a Yamanouchi word. When we insert each row R_k, the new cells are created from left to right as R_k is weakly increasing. Since the j-th element of R_k is smaller than the j-th element of R_{k+1}, the j-th element of R_k creates a new cell left of the j-th element of R_{k+1}. This relies on the fact that bumping chains of both rows proceed from left to right. Therefore, the Q-tableau described above is a reverse tableau of some shape λ/μ for some partition λ with content ν and reading word a Yamanouchi word.

We can undo φ by undoing the bumps from right to left for each value in Q, as the cells were created from left to right. Then, the Yamanouchi word condition on Q transforms into the condition that the word we get from inverting this map is a reading word for a tableau of shape ν. Finally, $c^{\lambda}_{\mu\nu}$ also counts the number of reverse tableau of shape λ/μ with content ν and reading word a Yamanouchi word, which relies on the fact that there is unique tableau (resp. reverse tableau) of each partition shape whose reverse reading word (resp. reverse reading word) is a Yamanouchi word. Thus, we have a bijection such that the content of T, U put together is the content of P, so by taking the weight generating function of each side,

$$s_\mu s_\nu = \sum_\lambda c^{\lambda}_{\mu\nu} s_\lambda.$$

□

One benefit of this proof of the LRR is that the Yamanouchi condition shows up in a natural way. For an alternative proofs, see [18], Section 4.9, or [5], Chapter 5.

3.3 The Cauchy Identity

Another important identity is the Cauchy Identity, [18], Theorem 4.8.4:

Theorem 3.7.

$$\prod_{\lambda \text{ partition}} s_\lambda(x) s_\lambda(y) = \prod_{i,j\geq 1} (1 - x_i y_j)^{-1}.$$

We will sketch a proof of this identity using RSK. Let M be the set of all matrix biwords, which are $\begin{pmatrix} i_1 & i_2 & \dots & i_n \\ j_1 & j_2 & \dots & j_n \end{pmatrix}$ such that the pairs $\binom{i_r}{j_r}$ are arranged lexicographically from left to right, and let the weight of such a word be $\prod_{r=1}^{n}(x_{i_r} y_{j_r})$. Since a matrix biword is determined uniquely by how many times the pair $\binom{i}{j}$ occurs for all $i, j \geq 1$, the weight generating function is

$$\prod_{i,j\geq 1} \sum_{k\geq 0} (x_i y_j)^k = \prod_{i,j\geq 1} (1 - x_i y_j)^{-1}.$$

Letting the weight of the pair (P, Q) of tableaux be $x^Q y^P$, we now give weight preserving bijection

$$\psi : M \to \bigcup_{\lambda \text{ partition}} \mathrm{SSYT}(\lambda) \times \mathrm{SSYT}(\lambda),$$

$$\begin{pmatrix} i_1 & i_2 & \dots & i_n \\ j_1 & j_2 & \dots & j_n \end{pmatrix} \leftrightarrow (P, Q).$$

Here, we insert $j_1, j_2, \dots j_n$ in that order to get P, and we label the new cell created in Q by j_r's insertion i_r. For example,

$$\begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 2 & 5 & 3 & 4 & 7 & 1 & 6 \end{pmatrix} \mapsto \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 6 \\ \hline 2 & 7 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}.$$

Since for fixed $i := i_a = i_{a+1} = \dots = i_b$, we have $j_a \leq j_{a+1} \leq \dots \leq j_b$, the cells labeled by a common value i in Q were created from left to right, so we can undo this by unbumping from right to left. Hence, ψ is a bijection. As $x^Q y^P = \prod_{r=1}^{n}(x_{i_r} y_{j_r})$, ψ is a weight preserving bijection, completing the proof.

3.4 Words and Necklaces

Next, suppose we want to find the Schur expansion for the generating on all words of length n on $\mathbb{P}$, that is

$$W_n = \sum_{w_1, \dots, w_n \in \mathbb{P}} x_{w_1} \dots x_{w_n}.$$

Notice that W is clearly symmetric, so it can be written in the basis of Schur functions. By standardizing, we can also write this as

$$W_n = \sum_{\sigma \in \mathbb{S}_n} F^n_{D(\sigma)}(x).$$

Applying RSK gives us a bijection

$$RSK : \mathbb{P}^n \to \bigcup_{\lambda \vdash n} \mathrm{SSYT}(\lambda) \times \mathrm{SYT}(\lambda),$$

where the content of the word is the same as the content of the P-tableau. Hence,

$$W_n = \sum_{\lambda \vdash n} f^\lambda s_\lambda.$$

Under the Frobenius characteristic map, W_n corresponds to the regular representation of S_n.

Suppose that instead we consider the sum

$$L_\lambda = \sum_{\sigma \in \mathbb{S}_n,\, \sigma \text{ has cycle type } \lambda} F^n_{D(\sigma)}.$$

L_λ has been shown to come from an S_n-representation via the Frobenius characterisitic map, so it is known to be Schur-positive [3]. In fact, in the case of $\lambda = (n)$,

$$L_{(n)} = \sum_{\lambda \vdash n} a_\lambda s_\lambda,$$

where $a_\lambda := \#\{T \in \mathrm{SYT}(\lambda) \mid \mathrm{maj}(T) \equiv 1 \pmod{n}\}$, which was first shown in [12]. Furthermore, Schocker gave a generalization to cycles type of the form $\lambda = (d^k)$, or k copies of d [20]. What is missing from the literature is a bijective proof of this Schur expansion analogous to the proof above for W_n. However, as W_n was a generating function for words, Gessel and Reutenauer showed L_λ is also a generating function for multisets of necklaces, explained below.

Definition 3.8. *A word is* **primitive** *if it is not a smaller word repeated multiple times. For example, 111, and 1212 are not primitive, but 112 and 1213 are. Then, a* **necklace** *is an equivalence class of primitive words under cycling the entries. For example,* $\{112, 211, 121\}$ *and* $\{1213, 3121, 1312, 2131\}$ *are necklaces, but* $\{111\}$ *and* $\{1212, 2121\}$ *are not.*

First, note that $F^n_{D(\sigma)}$ is the generating function for compatible sequences $(i_1, \ldots, i_n)$ for σ, or sequences which satisfy

$$i_1 \le i_2 \le \cdots \le i_n, \qquad i_j < i_{j+1}, \text{ for all } j \in D(\sigma).$$

Hence, $\sum_{\sigma \in \mathbb{S}_n, \sigma \text{ has cycle type } \lambda} F^n_{D(\sigma)}$ is the generating function for pairs of permutations and compatible sequences where the permutation has fixed cycle type λ. Gessel and Reutenaur constructed a bijection between these pairs and multisets of necklaces to show that L_λ is the generating function for multisets of necklaces where we have λ_k necklaces of size k [8]. The vague idea of the bijection is to go from a permutation, compatible sequence pair to a multisets of necklaces by inserting the compatible sequence into the cycle form of the permutation in increasing order. For going from a multiset of necklaces to a permutation, compatible sequence pair, we standardize according to the lexicographic order of the circular word starting at that place. I have tried to complete the bijective proof of the expansion for $L_{(n)}$, and although I have not solved the general case, I have found a relatively straightforward bijection in the case where n is prime. We begin with a definition and then the key lemma that allows my bijection to work.

Definition 3.9. *A circular descent in a word* $w_1 \ldots w_n$ *is a position* i *such that* $w_i < w_{i+1}$ *of* $i < n$ *or* $w_n < w_1$ *if* $i = n$. *For example, 12345 has 1 circular decsent at 5, and 162534. has 3 circular descents at 2,4,6.*

Lemma 3.10. *Let* C *be the operator which cycles the last entry to the front of the word, so*

$$C(w_1 \ldots w_{n-1} w_n) = w_n w_1 \ldots w_{n-1}.$$

Then, if $w = w_1 \ldots w_n$, *the sequence* $(\mathrm{maj}(w), \mathrm{maj}(C^1(w)), \ldots \mathrm{maj}(C^{n-1}(w)))$, *forms an arithmetic seqeunce modulo* n *with difference* k, *where* k *is the number of the circular descents of* w. *In particular, each residue class modulo* n *shows up exactly once in this sequence if* $\gcd(k, n) = 1$.

Proof. The operator C simply moves circular descents one to the right, and moves any circular descent at the end to the start, either way increasing its index by 1 modulo n. Thus,

$$\text{maj}(C(w)) \equiv \text{maj}(w) + k \pmod{n},$$

from which the first result is immediate. If $\gcd(k, n) = 1$, then an arithmetic sequence with difference k will hit each residue class modulo n exactly once. $\square$

Now, suppose that n is prime. Since the word $aa \dots a$ is not primitive, a word with n letters is primitive if and only if it has between 1 and $(n-1)$ circular descents. This relies heavily on the fact that n is prime. For $n = 4$, 1212 has 2 circular descents, but is not primitive. But again, as n is prime, each integer between 1 and $(n-1)$ is relatively prime to n, so by Lemma 3.10, each necklace of size n has a unique representative with major index congruent to 1 modulo n, and all words with major index 1 show up as representatives. Restricting RSK to these words with major index 1 modulo n, we get a bijection

$$RSK : \{\text{words } w : \text{maj}(w) \equiv 1 \pmod{n}\} \to \bigcup_{\lambda \vdash n} \text{SSYT}(\lambda) \times \{T \in \text{SYT}(\lambda) \mid \text{maj}(T) \equiv 1 \pmod{n}\},$$

since RSK preserves descents between the word and Q-tableau. Hence,

$$L_{(n)} = \sum_{\text{maj}(w_1 \dots w_n) \equiv 1 \pmod{n}} x_{w_1} \dots w_{w_n} = \sum_{\lambda \vdash n} a_\lambda s_\lambda.$$

We cannot perform a similar argument when n is not prime, as we do not have unique major index 1 representatives for each necklace. For example, the necklace $\{1213, 3121, 1312, 2131\}$ has major indices $2, 0, 2, 0$ modulo 4 respectively, whereas $\{1132, 2113, 3211, 1321\}$ has major indices $3, 1, 3, 1$ modulo 4 respectively. A further issue with the general case is having to account for words with major index 1 that are not primitive, like 231231.

4 The Edelmann-Greene Correspondence

Another problem that has been studied using a variation of RSK is the number and structure of reduced words for the symmetric group. We begin with the fact that S_n is a Coxeter group with generators $s_i = (i, i+1)$ for $i = 1, \dots, (n-1)$, and relations

$$s_i^2 = 1 \text{ for all } i, \qquad s_i s_j = s_j s_i \text{ for all } i, j, |i-j| > 1, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for all } i.$$

We say $s_{a_1} \dots s_{a_p}$ (or $(a_1, \dots, a_p)$) is reduced if $s_{a_1} \dots s_{a_p}$ is not equivalent to a smaller word by the above relations. For example, $s_1 s_2 s_1$ is reduced but $s_2 s_1 s_2 s_1$ is not as

$$(s_2 s_1 s_2) s_1 = s_1 s_2 s_1 s_1 = s_1 s_2.$$

Then, the set of reduced words for $w \in S_n$ is denoted

$$R(w) = \{(a_1, \dots a_p) \mid s_{a_1} \dots s_{a_p} \text{ is reduced, } s_{a_1} \dots s_{a_p} = w\}.$$

For $w \in S_n$, in [21], Stanley defined a symmetric function G_w by

$$G_w := \sum_{a \in R(w)} F^n_{D(a)}.$$

Edelmann and Greene were able to find a bijective proof that G_w was Schur-positive using a modified version of RSK. We define Edelmann-Greene-Insertion (EGI) exactly like insertion above, except that when i is inserted into a row where $i, i+1$ are present, we bump down $i+1$ as in RSK, but we do not change the entry $i+1$, unlike in RSK [2]. So, for example, under EGI,

$$\begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array} \leftarrow 3 = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 4 \\ \cline{1-1} \end{array}, \text{ instead of } \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 4 \\ \cline{1-1} \end{array}.$$

What is the image of $R(w)$ under EGI? We define a Coxeter-Knuth move as a move of the form

$$\begin{aligned} xzy &\leftrightarrow zxy \quad \text{for } x \leq y < z, \\ yxz &\leftrightarrow yzx \quad \text{for } x < y \leq z \\ (i+1)i(i+1) &\leftrightarrow i(i+1)i, \text{ for some } i, \end{aligned}$$

and we say two words are v, w are Coxeter-Knuth equivalent, denoted $v \sim_{CK} w$, if v, w differ by a sequence of Coxeter-Knuth moves. Note that Coxeter-Knuth moves include Knuth moves. For example, $232124 \sim_{CK} 321342$ by the sequence of Coxeter-Knuth moves

$$232124 \sim_{CK} 323124 \sim_{CK} 321324 \sim_{CK} 321342.$$

Then, because of how we modified RSK, we get that $P(a) = P(b)$ if and only if $a \sim_{CK} b$. But all Coxeter-Knuth moves keep us inside $R(w)$, because these moves can be done using the relations on S_n. For example, for all of the words above lie in $R(43152)$, and have P-tableau

$$P = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}.$$

Therefore, for each P-tableau that shows up in $EGI(R(w))$, all possible Q-tableau must show up exactly once [2]. So, letting $a_\lambda(w)$ be the number of distinct P-tableau of shape λ which show up in $EGI(R(w))$, we can combine the $F^n_{D(a)}$ into Schur functions using Gessel's result, and we get

$$G_w = \sum_\lambda a_\lambda(w) s_\lambda,$$

showing that G_w is Schur-positive. But what is known about the coefficients $a_\lambda(w)$?

Definition 4.1. *A permutation $w \in S_n$ is called* **vexillary** *if it avoids the pattern 2143. For example,* 143265 *is not vexillary as it contains the subsequence* 4265*, which has the same pattern in terms of order as 2143, but* 146235 *is vexillary.*

Definition 4.2. *The code $c(w)$ of a permutation $w \in S_n$ is simply the sequence $c_1, \ldots, c_{n-1}$ where $c_k = \#\{j > k \mid w_k > w_j\}$. For example, $c(52413) = (4,1,2,0)$.*

Richard Stanley orignally showed that if w is vexillary, then

$$G_w(x) = s_{\mathrm{sort}'(\mathrm{c(w)})}(x),$$

where sort′ simply puts the sequence in weakly decreasing order [21]. So, as 52413 is vexillary, $G_{52413}(x) = s_{(4,2,1)}(x)$. The Stanley symmetric functions also obey a recurrence which we need some notation to describe. Lascoux and Schutzenberger showed that the Stanley symmetric functions obeyed the recurrence

$$G_w = \sum_{w' \in T(w)} G_{w'},$$

where $T(w) = \{vt_{i,j} : i < r, inv(vt_{i,r}) = inv(w)\}$, and (r, s) is the lexographically largest pair such that $r < s$ but $w_r > w_s$ [15]. So, if $w = 17623458$, then $(r, s) = (3, 7)$. This recurrence will eventually terminate at a set of vexilary permutations, so one can use this to calculate G_w for any $w \in S_n$.

David Little give a bijective proof of this reucrrence via an algorithm now called the Little Bump algorithm, to which we refer the interested reader to the original paper [14]. For any reduced words v, w - not necessarily for the same permutation,

$$Q(v) = Q(w) \text{ if and only if } v, w \text{ differ by a sequence of Little Bumps}$$

under EGI [10].

5 Kazhdan-Luzstig Cells

The theory of Kazhdan-Lusztig cells and representations is again a large area of knowledge and research, so will just state the definition of right (left) cells. For background on Coxeter groups in general, see [1] or [11]. For background on Kazhdan-Luzstig polynomials and representations, see [1], Chapters 5 and 6. For background on the Hecke algebra, which underlies the construction of representations of Coxeter groups, see [11], Chapter 7. Let (W, S) be a Coxeter system. That is, W is a group generated by S with relations

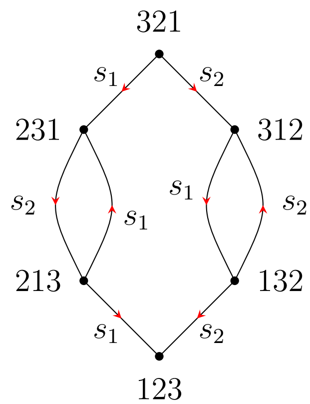
$$s^2 = 1, \qquad (ss')^{m(s,s')} = 1,$$

for all $s, s' \in S$, for some positive integers $m(s, s') \geq 2$ satisfying $m(s, s') = m(s', s)$. For example, $W = S_n, S = \{s_i = (i, i+1) \mid i = 1, \ldots n\}$ is a Coxeter system, with $m(s_1, s_j) = 2$ for $|i - j| > 1$, and $m(s_i, s_{i+1}) = 3$ for all i.

Definition 5.1. *The unlabelled (right) Kazhdan-Lusztig (K-L) graph is the directed graph* $\tilde{\Gamma}_{W,S} = (W, A)$, *where for all* $x \neq y$, *we have a directed edge* $x \to y$ *if and only if*

$$\overline{\mu}(x, y) \neq 0, \text{ and there exists } s \in S \text{ such that } xs < x, ys > y.$$

For example, below is the Kazhdan-Lusztig graph for S_3, *where we label each edge by an* $s \in S$ *satisfying the above requirement [13].*



Definition 5.2. *We say x, y are strongly connected in a directed graph G if there exists a path from x to y and from y to x in G. The strongly connected components of $\tilde{\Gamma}_{W,S}$ are called the right cells of W. For example, the right cells of S_3 are $\{123\}, \{213, 231\}, \{132, 312\}, \{321\}$.*

Similarly, we can define the left cells by using $sx < x, sy > y$ in the definition above instead. These left and right cells are an important concept to understand about any Coxeter system since they can be used to construct representations of W. What are the right cells for the symmetric group S_n? In fact, in S_n, the right cells are precisely the Knuth classes!

Theorem 5.3. *[13] Let $W = S_n$ and $S = \{(i, i+1) \mid i = 1, \dots n\}$. Then, two elements belong to the same right cell in $\tilde{\Gamma}_{W,S}$ if and only if they are Knuth equivelant. Equivalently, for each right cell C, there is a unique tableau T such that*

$$C = \{w \in S_n \mid P(w) = T\}.$$

Furthermore, the representation constructed from a left cell $C = P^{-1}(T)$ is the irreducible representation associated to $\mathrm{sh}(T)$. Similarly, the left cells are the dual Knuth classes $Q^{-1}(T)$ for some tableau T.

These amazing connections in S_n (Type A) do not carry over so easily to Type B, the group of signed permutations, but we can still characterize the left cells using an RSK-like algorithm.

Definition 5.4. *We define the group of signed permutations by*

$$S_n^B := \{ \text{ bijections } u : [-n, n] \to [-n, n] : u(x) = u(-x)\}.$$

Notice that how a signed permutation maps $[n]$ determines it by the condition $u(x) = -u(x)$, and so we can identify such maps with their restrictions to $[n]$. For example, $u = [-3, 2, 1, -4, 0, 4, -2, -1, 3] \in S_4^B$, reduces to $u = [4, -2, -1, 3]$ when restricted to $\{1, 2, 3, 4\}$. Then, RSK on Type B, hereby called Domino RSK, is an insertion algorithm taking signed permutations to pairs of standard domino tableaux, which has some similar properties to original RSK, like $P(w) = Q(w^{-1})$ [6].

Definition 5.5. *A domino tableau is a tableau the can be partitioned into dominos - 1×2 or 2×1 rectangles, so that each cell in a domino has the same entry. A standard domino tableau is one where the entries $1, 2, \ldots, n$ each appear exactly twice and occupying a domino shape, for some n. For example, the following are standard domino tableaux*

1	1	3	5	5
2	2	3		
4	4			

1	2	2
1	4	5
3	4	5
3		

,

but the following are not domino tableaux

1	1	2
2		

1	1	2
3	3	2

.

Domino RSK is similar in idea to ordinary RSK, where we insert each element one at a time, perform a bumping process to make it a tableau again, and label where each insertion terminated in a recording tableau. However, the actual algorithm is far too complicated to describe here, the complexity issues coming from the fact that we have both horizontal and vertical tiles to move as we insert (complex in standard sense, not run time sense). This algorithm was originally described in [6], but for a more concise description, see [22]. It should be noted that I wrote code for domino tableaux, Domino RSK, and its inverse which is in the process of being officially added to Sage.

Still, getting the left cells of S_n^B is not as simple as taking the inverse image of one of these tableaux, like for S_n. We must take the extra step of "specializing" them first, where specializing refers to a certain procedure of shifting the dominos around until the shape is special, which for Type B depends on parity conditions of the sizes of rows and columns. Again, for details, see [6]. Calling this specializing algorithm S, we then have

Theorem 5.6. *[6] For all $v, w \in S_n^B$,*

$$v, w \text{ lie in same right cell of } S_n^B \text{ if and only if } S(P(v)) = S(P(w)).$$

6 Asymptotic Distribution of Random Young Tableaux

In the sections above, we used RSK to go from words to tableaux, and used observations about the tableaux to tell us information about the orginal word. In this last section, we go from tableaux back to words, and prove a property of these words that tells us information about the tableaux. The following comes from [17], which the curious reader may see for more information. Fix a SYT T of shape $\lambda \vdash k$.

Theorem 6.1. *[17] Let $N(n, T) = \#\{\text{SYT on } n \text{ cells that contain } T \text{ as a subtableau}\}$. Let t_n be the number of SYT on n cells. Then,*

$$\lim_{n \to \infty} \frac{N(n, T)}{t_n} = \frac{f^\lambda}{k!}.$$

Let $\mathrm{Inv}(n)$ denote the set of involutions in S_n. Recall by restricting RSK to involutions that we get a bijection

$$RSK : \mathrm{Inv}(n) \to \text{SYT on } n \text{ cells}.$$

Also, since inserting an element $> k$ does not affect the entries $< k$ in the tableau, we have that the subtableau in which $1, 2 \dots k$ appear is precisely $P(\sigma)$, where $\sigma \in S_k$ lists the order in which $1, 2, \dots k$ appear. So, letting $Z(k) \subset S_k$ be the subset of orders in which $1, 2, \dots, k$ can appear to insert to T, and $F_n(\sigma)$ be the number of involutions in S_n in which $1, 2 \dots k$ appear in σ order, we can say that

$$\frac{N(n,T)}{t_n} = \sum_{\sigma \in Z_k} \frac{F_n(\sigma)}{t_n}.$$

To evaluate the limit as $n \to \infty$, we need the following theorem.

Theorem 6.2. *Let P be a family of permutations for which $P_n := P \cap S_n$ is a union of conjugacy classes and is nonempty for infinitely many n. Let f_n denote the average number of fixed points in P_n, and suppose*

$$\lim_{n\to\infty} \frac{f(n)}{n} = 0.$$

Then, for all $k \geq 1$, and $\tau \in S_k$, letting $p(n,\tau)$ be the probability that $1, 2, \dots, k$ appear in the order τ for a random element of P_n, we have

$$\lim_{n\to\infty} p(n,\tau) = \frac{1}{k!}.$$

The idea of the proof of Theorem 6.2 is showing the permutations π where $\pi([k]) \cap [k] \neq \emptyset$ become insignificant as $n \to \infty$, and then showing the rest split equally among the possible orders for $1, 2 \dots k$ using conjugation. In particular, the set of involutions is such a family. Applying this result for involutions means that

$$\lim_{n\to\infty} \frac{F_n(\sigma)}{t_n} = \frac{1}{k!}.$$

But also, the set of orders $Z(k)$ on $1, 2, \dots k$ that insert to T are in bijection with the possible Q-tableaux for inserting $1, 2, \dots k$ to get T, of which there are f^λ as $\mathrm{sh}(Q) = \mathrm{sh}(T) = \lambda$. Hence,

$$\lim_{n\to\infty} \frac{N(n,T)}{t_n} = \sum_{\sigma \in Z_k} \frac{F_n(\sigma)}{t_n} = \frac{|Z(k)|}{k!} = \frac{f^\lambda}{k!}.$$

Corollary 6.3. *The probablity that a random Young tableau on n cells has entry k in cell (i,j) approaches*

$$\frac{1}{k!} \sum f^\lambda f^{\lambda-(i,j)}$$

where that sum is over all $\lambda \vdash k$ such that (i,j) is an inner corner of λ, as $n \to \infty$.

To see this, k will be in entry (i,j) if and only if the subtableau for entries $1, \dots, k$ has (i,j) as an inner corner filled with k, of which there are $f^{\lambda-(i,j)}$, and each has probability $\frac{f^\lambda}{k!}$.

7 Conclusion

We have seen the definition of the Robinson-Schensted-Knuth Corrspondence, a description of some of its properties, and how it has been used to study a variety of questions in combinatorics, algebra, and representation theory. Of course, neither our list of its properties or uses is exhaustive.But there is still lets more to lear about and from RSK.

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