• This exam consists of 3 problems on 4 pages, including this cover sheet.

• Show all work for full credit.

• You may use a scientific calculator during this exam, but it is by no means required. Graphing calculators and other electronic devices are not allowed.

• You may write on the back of a page, but indicate that you have done so.

• You may use 1 double-sided 8.5” by 11” sheet of handwritten notes.

• You have 50 minutes to complete the exam.
1.) True/False Questions. Determine whether the following statements are True(T) or False(F) (2 points each).

(a) T  F Any subspace of \( \mathbb{R}^n \) has a basis consisting of a subset of the standard basis vectors.

(b) T  F The map \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) given by

\[
T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 5x_1 + 3 \\ 7x_2 + 4 \end{bmatrix}
\]

is linear.

(c) T  F The set \( \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| 3x = 5y - 2z \right\} \) is a subspace of \( \mathbb{R}^3 \).

(d) T  F \((AB)^2 = A^2B^2\) for all \( n \times n \) matrices \( A, B \).

(e) T  F A \( 3 \times 5 \) matrix \( A \) can have nullity(\( A \)) = 1.

(f) T  F Let \( \{v_1, v_2, v_3\} \) span \( \mathbb{R}^3 \), and let \( \{e_1, e_2, e_3\} \) denote the standard basis vectors for \( \mathbb{R}^n \). Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be a linear map satisfying

\[
T(e_1) = v_1, \ T(e_2) = v_2, \ T(e_3) = v_3.
\]

Then, \( T \) is invertible.

(g) T  F Let \( T : \mathbb{R}^2 \to \mathbb{R} \) be a linear transformation that has \( \begin{bmatrix} 2 \\ 5 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ 3 \end{bmatrix} \) in \( \ker(T) \). Then, \( T(\vec{x}) = 0 \) for all \( \vec{x} \in \mathbb{R}^2 \).

(h) T  F If \( S \) is a subspace of \( \mathbb{R}^n \), and \( B \) is a set of vectors that spans \( S \), then vectors can be added to \( B \) to form a basis for \( \mathbb{R}^n \).

(i) T  F There is a 2 by 2 matrix \( A \) such that \( A \neq O \), but \( A^2 = O \), where \( O \) refers to the 2 by 2 matrix of all zeroes.
2.) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map given by

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_2 - x_3 \\ x_1 - x_3 \\ x_1 - x_2 \end{bmatrix}.$$ 

(a) (3 points) Find the matrix $A$ such that $T(x) = Ax$.

(b) (7 points) Find $\ker(T)$ and express your answer as the span of a set of vectors.

(c) (2 points) What is $\dim(\text{range}(T))$?

(d) (6 points) Find a basis $B$ for $\text{range}(T)$ that is a subset of the columns of $A$. Then, justify that whatever columns you do not use lie in the span of the vectors in $B$. (Hint: Use Part b)
3.) (a) (6 points) Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \), \( U : \mathbb{R}^2 \to \mathbb{R}^2 \) and \( S : \mathbb{R}^2 \to \mathbb{R}^3 \) be the linear maps given by

\[
T(\vec{x}) = \begin{bmatrix} 4 & 5 \\ -2 & 1 \end{bmatrix} \vec{x}, \quad U(\vec{x}) = \begin{bmatrix} -3 & -5 \\ 4 & -3 \end{bmatrix} \vec{x}, \quad S(\vec{x}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \vec{x}.
\]

Find the matrix \( D \) for the map \( S \circ (T + U) \) so that

\[
S(T(\vec{x}) + U(\vec{x})) = D\vec{x}.
\]

(b) (8 points) Let \( \{\vec{v}_1, \ldots, \vec{v}_n\} \) be a set of vectors which spans \( \mathbb{R}^n \). Let \( S : \mathbb{R}^n \to \mathbb{R}^m \) and \( T : \mathbb{R}^n \to \mathbb{R}^m \) be linear maps satisfying

\[
S(\vec{v}_1) = T(\vec{v}_1), \quad S(\vec{v}_2) = T(\vec{v}_2), \quad \ldots, \quad S(\vec{v}_n) = T(\vec{v}_n).
\]

That is, \( S, T \) have the same output for each of the inputs \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \). Show that \( S(\vec{x}) = T(\vec{x}) \) for all \( \vec{x} \in \mathbb{R}^n \).