Proof Homework 3 Solutions

1. Let \( \{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\} \) and \( \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\} \) be bases for \( \mathbb{R}^n \) and let \( A, B \) be the corresponding matrices with these vectors as columns, respectively. Find the matrix for the unique linear map \( T : \mathbb{R}^n \to \mathbb{R}^n \) satisfying
   \[
   T(\vec{a}_1) = \vec{b}_1, \quad T(\vec{a}_2) = \vec{b}_2, \quad \ldots, \quad T(\vec{a}_n) = \vec{b}_n
   \]
in terms of \( A, B \), and justify your answer. Hint: Use the standard basis vectors \( e_1, e_2, \ldots, e_n \).

Here, \( A \) is invertible as it columns form a basis for \( \mathbb{R}^n \) and thus are linearly independent. Notice that for all \( j \),
   \[
   A e_j = \vec{a}_j, \quad B e_j = \vec{b}_j
   \]
Then, \( e_j = A^{-1} \vec{a}_j \), so
   \[
   (BA^{-1}) \vec{a}_j = B(A^{-1} \vec{a}_j) = B(e_j) = \vec{b}_j = T(\vec{a}_j).
   \]
Thus, the matrix for \( T \) is \( BA^{-1} \).

Method 2: Alternatively, when we multiply matrices, we see that
   \[
   \begin{bmatrix} T \end{bmatrix} \begin{bmatrix} \vec{a}_1 & \ldots & \vec{a}_n \end{bmatrix} = \begin{bmatrix} T(\vec{a}_1) & \ldots & T(\vec{a}_n) \end{bmatrix} = \begin{bmatrix} \vec{b}_1 & \ldots & \vec{b}_n \end{bmatrix},
   \]
which means \( [T]A = B \) as matrices, so \( [T] = BA^{-1} \).

2. Let \( S_1, S_2 \) be subspaces of \( \mathbb{R}^n \). Define
   \[
   S_1 + S_2 := \{ \vec{x} + \vec{y} \mid \vec{x} \in S_1, \vec{y} \in S_2 \}
   \]
That is, \( S_1 + S_2 \) is the set of all vectors that are some vector in \( S_1 \) plus some vector in \( S_2 \). For example, if \( S_1 = \text{span}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\} \) and \( S_2 = \text{span}\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\} \), then \( S_1 + S_2 = \mathbb{R}^2 \).

(a) Show that \( S_1 + S_2 \) is always a subspace of \( \mathbb{R}^n \).

(b) Show that if the only vector in both \( S_1 \) and \( S_2 \) is \( \vec{0} \), then
   \[
   \dim(S_1 + S_2) = \dim(S_1) + \dim(S_2)
   \]
Hint: Think about how you can get a basis for \( S_1 + S_2 \) from a basis for \( S_1 \) and a basis for \( S_2 \). Remember you must justify that it is a basis. More generally, letting \( S_1 \cap S_2 \) is the set of all vectors that lie in both \( S_1 \) and \( S_2 \),
   \[
   \dim(S_1 + S_2) = \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2)
   \]
This last statement is not part of the proof assignment, since the proof is a bit too involved for the level of Math 308. (But you can try it for fun if you want...)

(a) First, \( \vec{0} \in S_1 \) and \( \vec{0} \in S_2 \), so \( \vec{0} = \vec{0} + \vec{0} \in S_1 + S_2 \). Secondly, consider any \( \vec{u}, \vec{v} \in S_1 + S_2 \). This means we can write
   \[
   \vec{u} = \vec{x}_1 + \vec{y}_1, \quad \vec{v} = \vec{x}_2 + \vec{y}_2
   \]
for some \(x_1', x_2' \in S_1\) and \(y_1', y_2' \in S_2\). Hence,
\[
\vec{u} + \vec{v} = (x_1' + y_1') + (x_2' + y_2') = (x_1' + x_2') + (y_1' + y_2')
\]
where \(x_1' + x_2' \in S_1\) and \(y_1' + y_2' \in S_2\). Thus, \(\vec{u} + \vec{v}\) is some vector in \(S_1\) plus some vector in \(S_2\), so \(\vec{u} + \vec{v} \in S_1 + S_2\). This means \(S_1 + S_2\) is closed under addition. Also, for any scalar \(c\), we have
\[
c\vec{u} = c(x_1' + y_1') = cx_1' + cy_1'
\]
where \(cx_1' \in S_1\), and \(cx_2' \in S_2\). Thus, \(c\vec{u} \in S_1 + S_2\), and \(S_1 + S_2\) is closed under scalar multiplication.

(b) Many students misinterpreted the statement “the only vector in both \(S_1\) and \(S_2\) is \(0\)”, as \(S_1 = \{0\}\) and \(\{0\}\). I do not foresee this confusion, or I would have tried to clarify it. I meant that \(0\) is the only vector that lies in \(S_1\) AND \(S_2\). Many students confused the AND for OR. For example, \(S_1 = \text{span}\{1\} \) and \(S_2 = \text{span}\{0\}\) satisfies this condition. Equivalently, one could say the intersection of \(S_1\) and \(S_2\) is zero.

Let \(u_1', \ldots, u_k'\) be a basis for \(S_1\), and \(v_1', \ldots, v_m'\) be a basis for \(S_2\). We claim the combination of these bases \(B = \{u_1', \ldots, u_k', v_1', \ldots, v_m'\}\) forms a basis for \(S_1 + S_2\). First, for \(w \in S_1 + S_2\), we can write \(\vec{w} = \vec{x} + \vec{y}\), with \(\vec{x} \in S_1, \vec{y} \in S_2\). Then, \(\vec{x}\) is a linear combination of \(u_1', \ldots, u_k'\), and \(\vec{y}\) is a linear combination of \(v_1', \ldots, v_m'\), so
\[
\vec{w} = \vec{x} + \vec{y} = c_1u_1' + \cdots + c_ku_k' + d_1v_1' + \cdots + d_mv_m'
\]
is a linear combination of the vectors in \(B\). Therefore, \(B\) spans \(S_1 + S_2\). But we must also show \(B\) is linearly independent. Suppose
\[
c_1u_1' + \cdots + c_ku_k' + d_1v_1' + \cdots + d_mv_m' = \vec{0}
\]
We can rewrite this as
\[
c_1u_1' + \cdots + c_ku_k' = -(d_1v_1' + \cdots + d_mv_m')
\]
Now, the left hand side lies in \(S_1\), and the right hand side lies in \(S_2\), which means both sides lie in \(S_1\) and in \(S_2\). By the hypothesis that \(0\) is the only vector in \(S_1\) and in \(S_2\), we conclude both sides must be \(0\). Then,
\[
c_1u_1' + \cdots + c_ku_k' = \vec{0} \Rightarrow c_1, \ldots, c_k = 0,
\]
because \(u_1', \ldots, u_k'\) are linearly independent, and
\[
d_1v_1' + \cdots + d_mv_m' = \vec{0} \Rightarrow d_1, \ldots, d_m = 0,
\]
because \(v_1', \ldots, v_m'\) are linearly independent. Thus, all the coefficients are 0, which tells us \(u_1', \ldots, u_k', v_1', \ldots, v_m'\) is linearly independent.

3.

(a) Suppose that \(A, B\) are matrices that satisfy \(AB = BA\). Show that if \((\lambda, \vec{v})\) is an eigenpair of \(A\), then \((\lambda, B\vec{v})\) is also an eigenpair of \(A\).

(b) Suppose that \((\lambda_1, \vec{v}_1')\) and \((\lambda_2, \vec{v}_2')\) are eigenpairs of \(A\), with \(\lambda_1 \neq \lambda_2\), and \(\vec{v}_1', \vec{v}_2' \neq \vec{0}\). Show that \(\vec{v}_1', \vec{v}_2'\) are linearly independent.

(a) Since \((\lambda, \vec{v})\) is an eigenpair of \(A\),
\[
A\vec{v} = \lambda\vec{v} \Rightarrow B(A\vec{v}) = B(\lambda\vec{v}) \Rightarrow (BA)\vec{v} = \lambda B\vec{v}
\]
So, since \(AB = BA\), we conclude that
\[
AB\vec{v} = \lambda B\vec{v} \Rightarrow A(B\vec{v}) = \lambda (B\vec{v})
\]
so \((\lambda, B\vec{v})\) is also an eigenpair of \(A\).
(b) Suppose that \((\lambda_1, \vec{v}_1)\) and \((\lambda_2, \vec{v}_2)\) are eigenpairs of \(A\), with \(\lambda_1 \neq \lambda_2\), and \(\vec{v}_1, \vec{v}_2 \neq \vec{0}\). Show that \(\vec{v}_1, \vec{v}_2\) are linearly independent.

(c) Since \((\lambda_1, \vec{v}_1)\) and \((\lambda_2, \vec{v}_2)\) are eigenpairs of \(A\),

\[
A\vec{v}_1 = \lambda_1 \vec{v}_1, \quad A\vec{v}_2 = \lambda_2 \vec{v}_2.
\]

Now, suppose that

\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0} \implies
\]

Applying matrix multiplication by \(A\) to both sides, we get, by linearity,

\[
c_1 A\vec{v}_1 + c_2 A\vec{v}_2 = \vec{0} \implies c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 = \vec{0}.
\]

Mutliplying the first equation by \(\lambda_2\) gives

\[
c_1 \lambda_2 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 = \vec{0}.
\]

Subtracting the above equation then yields

\[
c_1 (\lambda_1 - \lambda_2) \vec{v}_1 = \vec{0}.
\]

So, because \(\vec{v}_1 \neq \vec{0}\), we must have \(c_1 (\lambda_2 - \lambda_1) = 0\), so \(c_1 = 0\) because \(\lambda_1 \neq \lambda_2\). Then, plugging this in means \(c_2 \vec{v}_2 = \vec{0}\), so \(c_2 = 0\) as well, since \(\vec{v}_2 \neq \vec{0}\). Therefore, \(\vec{v}_1, \vec{v}_2\) are linearly independent.