1. Let \( S = \text{span}\{ \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m \}\).

(a) Show that if \( \vec{v} \in S \), and \( c \) is a scalar, then \( c\vec{v} \in S \).

(b) Show that if \( \vec{v} \in S \) and \( \vec{w} \in S \), then \( \vec{v} + \vec{w} \in S \).

(a) First, suppose that \( \vec{v} \in S \), and \( c \) is a scalar. Thus, for some scalars \( x_1, x_2, \ldots, x_m \),

\[
\vec{v} = x_1 \vec{u}_1 + x_2 \vec{u}_2 + \cdots + x_m \vec{u}_m,
\]

which means \( c\vec{v} \in \text{span}\{ \vec{u}_1, \ldots, \vec{u}_m \} = S \).

(b) Next, suppose that \( \vec{v}, \vec{w} \in S \). Thus, for some scalars \( x_1, x_2, \ldots, x_m \), and \( y_1, y_2, \ldots, y_m \),

\[
\vec{v} = x_1 \vec{u}_1 + x_2 \vec{u}_2 + \cdots + x_m \vec{u}_m, \\
\vec{w} = y_1 \vec{u}_1 + y_2 \vec{u}_2 + \cdots + y_m \vec{u}_m, \\
\vec{v} + \vec{w} = (x_1 + y_1) \vec{u}_1 + (x_2 + y_2) \vec{u}_2 + \cdots + (x_m + y_m) \vec{u}_m,
\]

which means \( \vec{v} + \vec{w} \in \text{span}\{ \vec{u}_1, \ldots, \vec{u}_m \} = S \).

2. Consider sets of vectors \( X = \{ \vec{u}_1, \ldots, \vec{u}_k \} \) and \( Y = \{ \vec{u}_1, \ldots, \vec{u}_m \} \), where \( k \leq m \) so that \( X \subseteq Y \) (\( X \) is a subset of \( Y \)).

(a) Show that if \( X \) is linearly dependent, so is \( Y \).

(b) Show that if \( Y \) is linearly independent, so is \( X \).

Be sure to refer to the definition of linear dependence/linear independence in your proof.

(a) First, suppose that \( X \) is linearly dependent. This means there exists scalars \( c_1, c_2, \ldots, c_k \), not all zero, such that

\[
c_1 \vec{u}_1 + c_2 \vec{u}_2 + \cdots + c_k \vec{u}_k = \vec{0}.
\]

Then, we can include \( u_{k+1}^+, \ldots, u_m^+ \) with coefficients of 0 to get

\[
c_1 \vec{u}_1 + c_2 \vec{u}_2 + \cdots + c_k \vec{u}_k + 0u_{k+1}^+ + \cdots + 0u_m^+ = \vec{0}.
\]

Since not all of the \( c_j \)'s are zero, we have a nontrivial linear combination of \( Y \) that is \( \vec{0} \). Thus, \( Y \) is linear dependent.

(b) This part is equivalent to part (a). If we let \( A \) be the statement “\( X \) is linearly dependent”, and \( B \) as the statement “\( Y \) is linearly dependent”, then (a) says that \( A \) implies \( B \). Then, (b) is saying that (NOT \( B \)) implies (NOT \( A \)). As logical statements, (b) is called the contrapositive of (a), and it happens always that the contrapositive of a statement is equivalent to the statement.

So, why is the contrapositive statement equivalent? We prove this by contradiction, which starts by assuming the conclusion is false, and then proceeds to derive a contradiction to the hypotheses. Hence,
the conclusion must have been true. So, here we go. We assume $A$ implies $B$, and want to show that (NOT $B$) implies (NOT $A$)? Suppose (Not $B$). Suppose to the contrary that (NOT $A$) does NOT hold, which means $A$ holds. Then, by our assumption, $B$ holds as well, contradicting the hypothesis that NOT $B$ holds. Hence, NOT $A$ must have been true in the first place, so NOT $B$ implies NOT $A$. This shows that if $A$ implies $B$ is true, its contrapositive, NOT $B$ implies NOT $A$ is true as well. To go the other way, assuming (NOT $B$) implies (NOT $A$), then its contrapositive must be true, but the contrapositive is just $A$ implies $B$. Thus, $A$ implies $B$, and (NOT $B$) implies (NOT $A$) are equivalent statements.

NOTE: The statement $B$ implies $A$ is the converse of $A$ implies $B$, and is not equivalent to it. To distinguish these, consider the claim

If it is July, then it is sunny.

Its converse is

If it is sunny, then it is July.

This need not be true, for it could also be sunny in June. On, the other hand, the contrapositive is

If it if not sunny, then it is not July.

This must be true, for it were July, it would have been sunny, so it could not be July.

3. Suppose that \{\vec{u}_1, \ldots, \vec{u}_m\} is linearly independent in $\mathbb{R}^n$, and $\vec{v}$ in $\mathbb{R}^n$ does not lie in $\text{span}\{\vec{u}_1, \ldots, \vec{u}_m\}$. Prove that \{\vec{u}_1, \ldots, \vec{u}_m, \vec{v}\} is linearly independent. (Hint: Looking at the proof we did in class showing that a set is linearly dependent if and only if one of the vectors is in the span of the others may help.)

Consider the equation

$$ c_1 \vec{u}_1 + \ldots + c_m \vec{u}_m + c\vec{v} = \vec{0}. $$

We want to show that $c_1, \ldots, c_m, c$ are all 0. First, suppose $c \neq 0$, and then we can solve for $\vec{v}$ to get

$$ \vec{v} = -\frac{1}{c} (c_1 \vec{u}_1 + \ldots + c_m \vec{u}_m) = -\frac{c_1}{c} \vec{u}_1 - \ldots - \frac{c_m}{c} \vec{u}_m \in \text{span}\{\vec{u}_1, \ldots, \vec{u}_m\}, $$

which contradicts that $\vec{v}$ is not in $\text{span}\{\vec{u}_1, \ldots, \vec{u}_m\}$. Therefore, $c = 0$, and we have

$$ c_1 \vec{u}_1 + \ldots + c_m \vec{u}_m = \vec{0}. $$

Now, the fact that \{\vec{u}_1, \ldots, \vec{u}_m\} is linearly independent forces $c_1, \ldots, c_m$ to all be zero as well. Thus, $c_1, \ldots, c_m, c$ are all 0, and \{\vec{u}_1, \ldots, \vec{u}_m, \vec{v}\} is linearly independent.