• This exam consists of 3 problems on 5 pages, including this cover sheet.

• Show all work for full credit.

• You may use a scientific calculator during this exam, but it is by no means required. Graphing calculators and other electronic devices are not allowed.

• You may write on the back of a page, but indicate that you have done so.

• You may use 1 double-sided 8.5" by 11" sheet of handwritten notes.

• You have 50 minutes to complete the exam.
1.) Short Answer Questions - 5 points each

(a) Suppose that $S, T : \mathbb{R}^2 \to \mathbb{R}^2$ are linear maps given by the following formulas:

\[
S \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 3x_1 + 2x_2 \\ x_1 - 5x_2 \end{bmatrix}, \quad T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 - 4x_2 \\ -x_1 + 3x_2 \end{bmatrix}
\]

Find a formula for $(S \circ T) \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$. (Note: The correct matrix for $S \circ T$ is worth 4 points.)

(b) Suppose the matrix $A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$ has reduced row echelon form

\[
B = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \end{bmatrix}
\]

Express $a_3$ as a linear combination of $a_1$ and $a_2$. 

(c) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a linear map that satisfies $\begin{bmatrix} 3 \\ -5 \end{bmatrix} \in \ker(T)$. Find all possible matrices for $T$. (Your answer should involve a free variable.)

(d) Let $A, B, C$ be matrices. Solve for the matrix $X$ in the equation

$$AX(X + B)^{-1} = C$$

Assume matrices are invertible as needed.
2.) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map that satisfies

\[ T \left( \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 6 \end{bmatrix}, \quad T \left( \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}. \]

Note: The following can be done with or without finding a matrix for $T$.

(a) (3 points) Find $T \left( 5 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right)$.

(b) (6 points) Find $T \left( \begin{bmatrix} 7 \\ -12 \end{bmatrix} \right)$.

(c) (6 points) Find a vector $\vec{x} \in \mathbb{R}^2$ so that $T (\vec{x}) = \begin{bmatrix} -5 \\ 21 \end{bmatrix}$. 
3. (a) (2 points) Let \( S = \text{span}\{ \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \} \subset \mathbb{R}^3 \). Find a linear map \( U : \mathbb{R}^n \rightarrow \mathbb{R}^3 \) (for some \( n \)) that has \( \text{range}(U) = S \). (The matrix for \( U \) is worth 1 point.)

(b) (4 points) Using same \( S \) as in (a), find a linear map \( W : \mathbb{R}^3 \rightarrow \mathbb{R}^m \) (for some \( m \)) that has \( \text{ker}(W) = S \). (The matrix for \( W \) is worth 3 points.)

(c) (3 points) Find a basis for \( \mathbb{R}^3 \) containing \( \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \), and explain why it is a basis for \( \mathbb{R}^3 \).

(d) (6 points) Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear map. Show that

\[
\text{ker}(T) = \{ \vec{x} \in \mathbb{R}^n \mid T(\vec{x}) = \vec{0} \}
\]

is a subspace of \( \mathbb{R}^n \).