Pollicott–Ruelle resonances via kinetic Brownian motion.

Alexis Drouot

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  is tangent to $S^*\mathbb{M}$ the cosphere bundle of $\mathbb{M}$. Its integral curves project to geodesics on $\mathbb{M}$. **It is called the generator of the geodesic flow.**

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- When $\varepsilon \to 0$, the projection of $z(t)$ to $\mathbb{M}$ **converges to the geodesic** starting at $z(0)$.
- When $\varepsilon \to \infty$, the projection of $z(\varepsilon^2 t)$ to $\mathbb{M}$ **converges in law to a Brownian motion** on $\mathbb{M}$.

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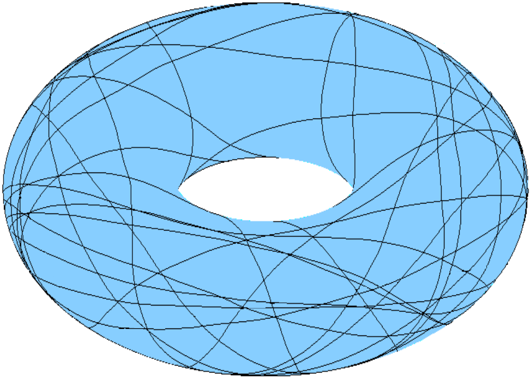

Figure: Projection of the kinetic Brownian motion on a 2-torus with $\varepsilon = 1/10$. The trajectories are locally close to geodesics – but not globally. **Simulation from Angst–Bailleul–Tardif.**

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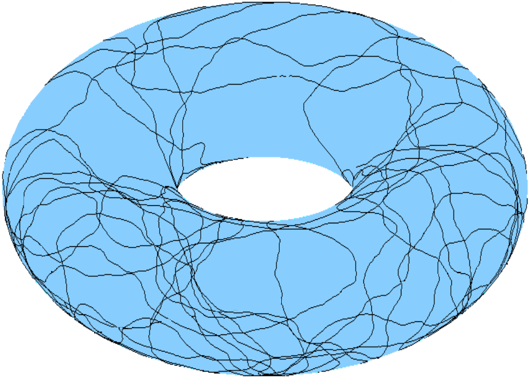

Figure: Projection of the kinetic Brownian motion on a 2-torus with $\varepsilon = 1$. The trajectories become random. **Simulation from Angst–Bailleul–Tardif.**

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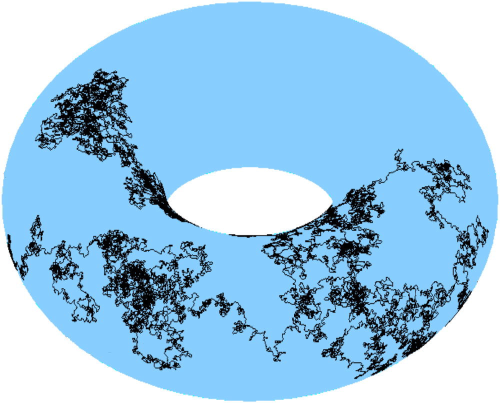

Figure: Projection of the kinetic Brownian motion on a 2-torus with $\varepsilon = 10$. The trajectories look completely random. **Simulation from Angst–Bailleul–Tardif.**

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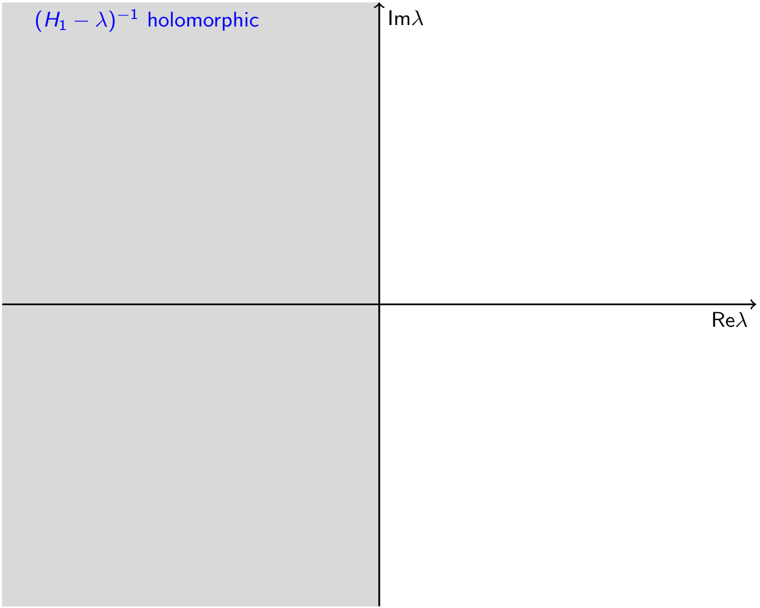

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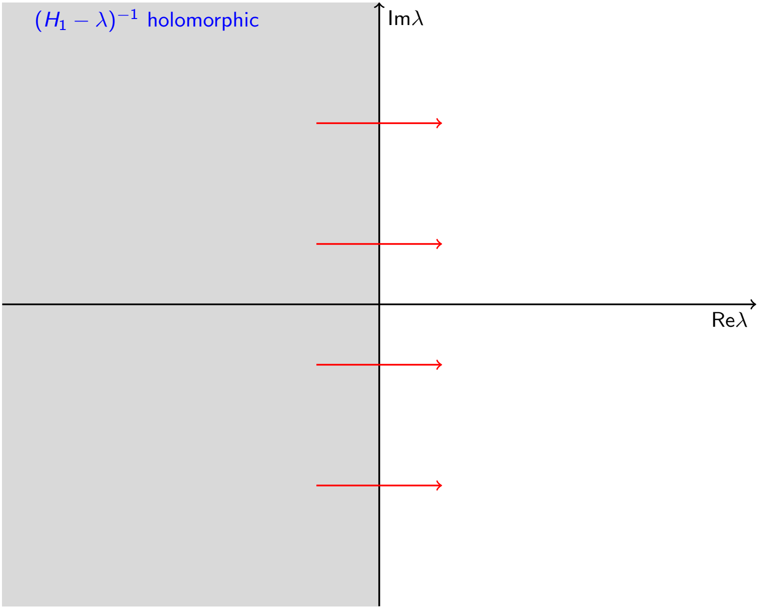

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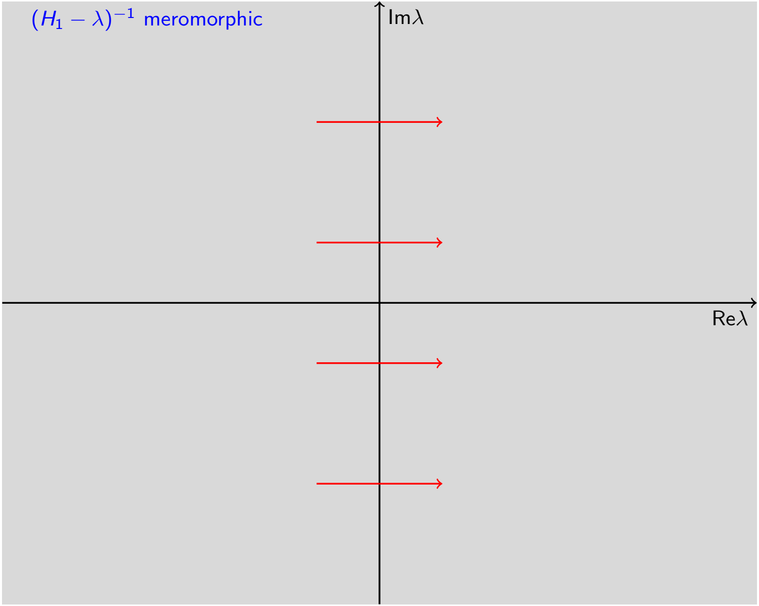

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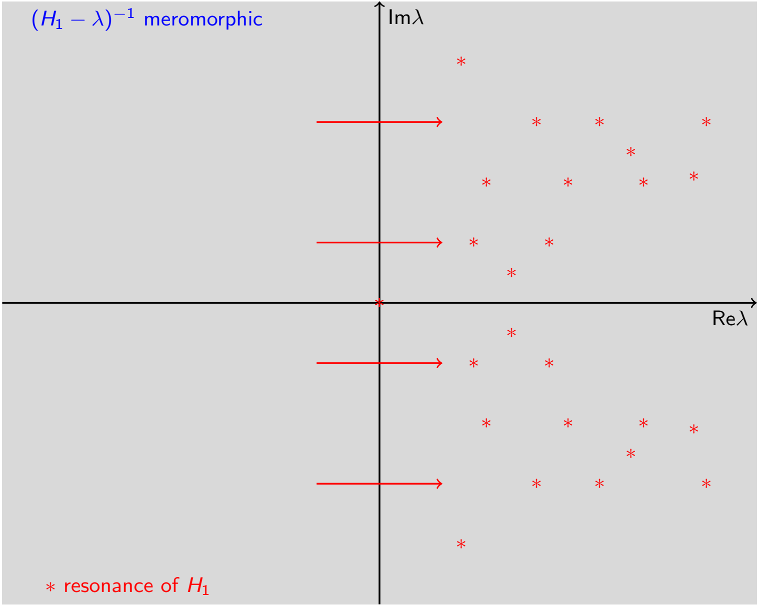

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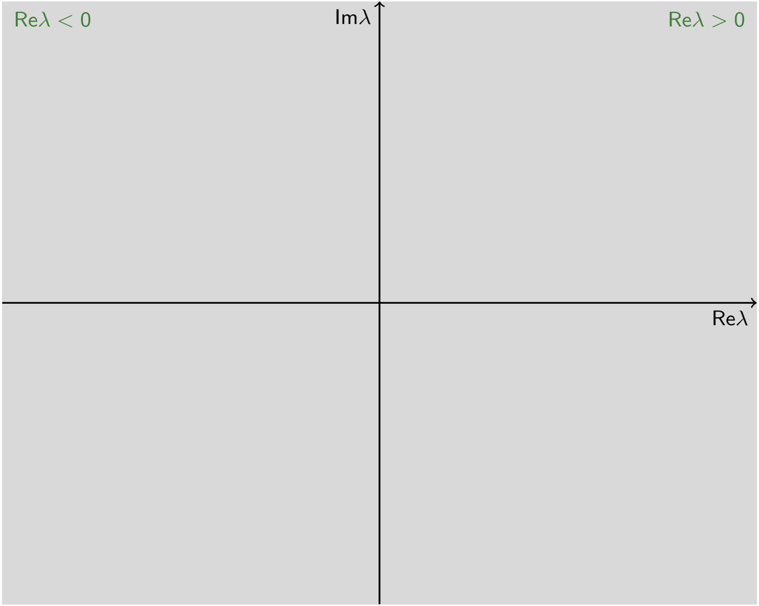

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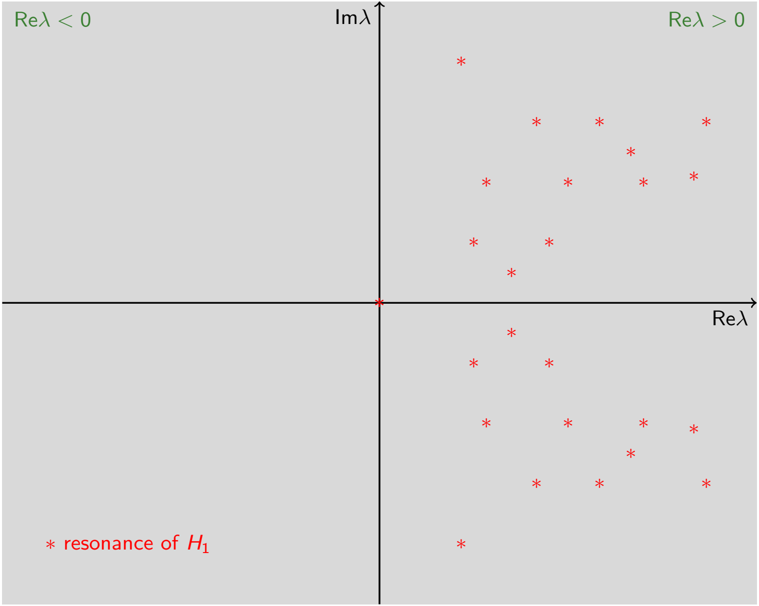

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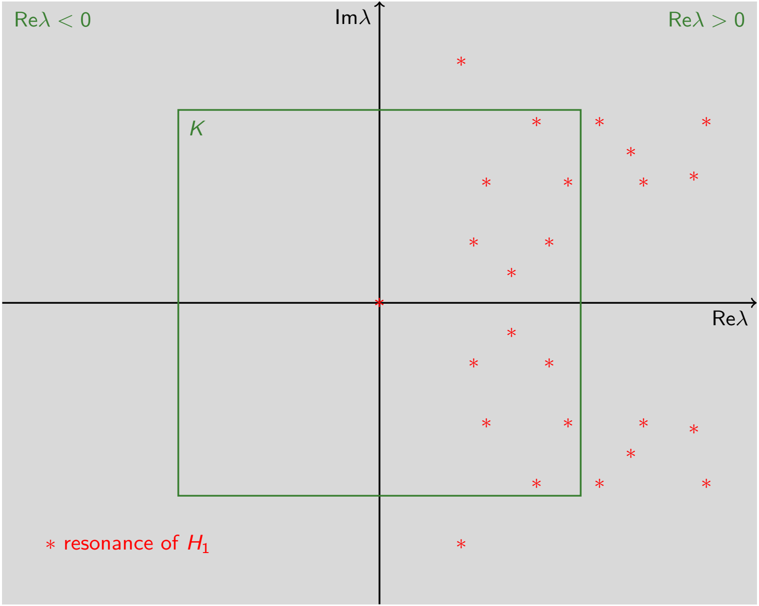

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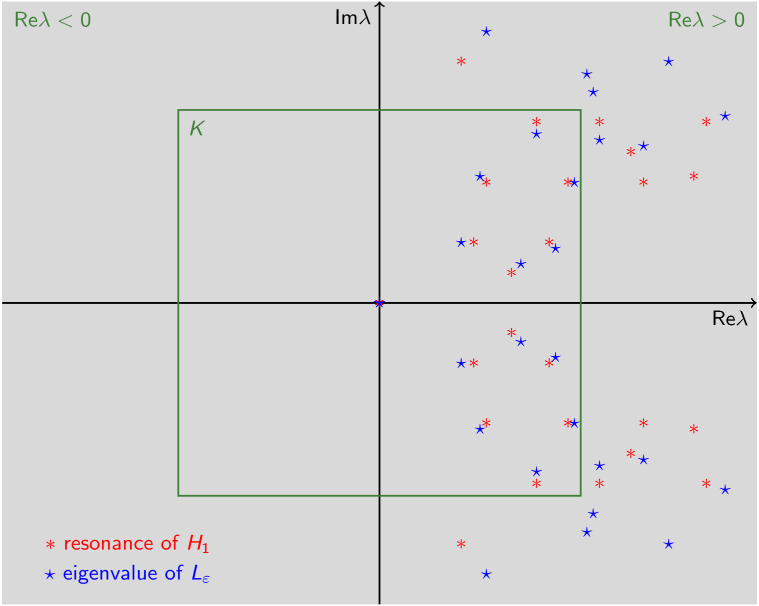

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Conclusion: the term εV^2 cannot be too big compared to L_ε.

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Below we show estimates for $\widehat{\mathcal{P}}$.

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Study of $\widehat{\mathcal{P}} = \varepsilon\partial_{x_1} - (x_1\xi_2)^2$ at fixed ξ_2 – after Lebeau

- **Control for small values of x_1:** take φ such that $(\varphi^2)'$ is a bump function with value 1 on $[-t, t]$.

$$\mathrm{Re}\langle\widehat{\mathcal{P}}v, \varphi^2 v\rangle = |\varphi x_1\xi_2 v|_{L^2}^2 + \mathrm{Re}\langle\varphi\varepsilon\partial_{x_1}v, \varphi v\rangle \geq \varepsilon\int(\varphi^2)'|v|^2;$$

$$\varphi^2 = O(t) \Rightarrow \int_{|x_1|\leq t}|v|^2 \leq C\varepsilon^{-1}t|\widehat{\mathcal{P}}v|_{L^2}|v|_{L^2}.$$

- **Control for large values of x_1:**

$$\int_{|x_1|\geq t}|v|^2 \leq \int_{|x_1|\geq t}\frac{x_1^2}{t^2}|v|^2 = \frac{1}{(t\xi_2)^2}\int_{|x_1|\geq t}(x_1\xi_2)^2|v|^2 \leq \frac{1}{(t\xi_2)^2}|\widehat{\mathcal{P}}v|_{L^2}|v|_{L^2}.$$

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Conclusion: $\varepsilon^{2/3}\||\varepsilon H_2|^{2/3}u|_{L^2} \leq C|Pu|_{L^2}$, which implies the **optimal subelliptic estimate** $\varepsilon^{2/3}|u|_{H_\varepsilon^{2/3}} \leq C|\varepsilon L_\varepsilon u|_{L^2} + \ldots$.

Maximal hypoellipticity for $L_\varepsilon = H_1 - \varepsilon V^2$

The **subelliptic estimate** $\varepsilon^{2/3}|u|_{H_\varepsilon^{2/3}} \leq C|Pu|_{L^2} + \ldots$ and standard manipulations yields the **hypoelliptic estimate**

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It remains to show that $(L_\varepsilon - \lambda)^{-1}$ **continues meromorphically on the same spaces as $(H_1 - \lambda)^{-1}$.**

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Based on the splitting (2), Faure–Sjöstrand and Dyatlov–Zworski constructed **semiclassical weighted Sobolev spaces** $\mathcal{H}$ such that if $0 \le Q$ is a suitable absorbing potential near the zero section, $|\lambda| \le R$,

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This and an adjoint inequality implies that $(H_1 - \lambda)^{-1} : \mathcal{H} \to \mathcal{H}$**, holomorphic and well defined for** $\operatorname{Re}\lambda < 0$**, extends meromorphically to** $\{|\lambda| \le R\}$**.**

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Goal: Fredhom estimate: if $0 \leq Q$ is a suitable absorbing potential near the zero section, $|\lambda| \leq R$ then

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This shows (3). The adjoint estimate shows that $L_\varepsilon + Q - \lambda$ is invertible, **hence** $(L_\varepsilon - \lambda)^{-1} : \mathcal{H} \to \mathcal{H}$ **continues meromorphically.**

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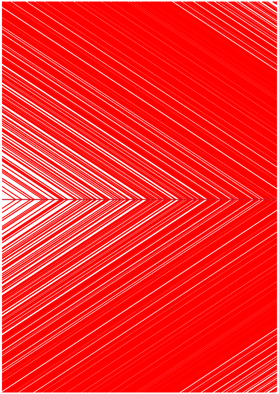

Limit set in $\{\operatorname{Re}\lambda \geq 0\}$ of the spectrum of $H_1 - \varepsilon\Delta$ on $S^*\mathbb{T}^2$.

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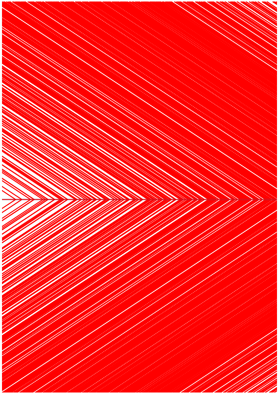

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Thanks for your attention!