Eigenvalues for highly disordered potentials

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Waves and resonances

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This is reflected in the spectrum of $-\Delta_{\mathbb{R}^3} + V$ on $L^2(\mathbb{R}^3)$: it is the union of a **discrete set** (eigenvalues) with the **continuous spectrum** $[0, \infty)$.

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Eigenvalues μ are poles of $(-\Delta_{\mathbb{R}^3} + V - \mu)^{-1}$, hence (squares of) resonances. **Conversely, resonances inducing eigenvalues are the one lying on the complex half-line** $i[0, \infty)$**.**

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The expansion (2) comes from a **contour deformation** in the representation of u given by the spectral theorem:

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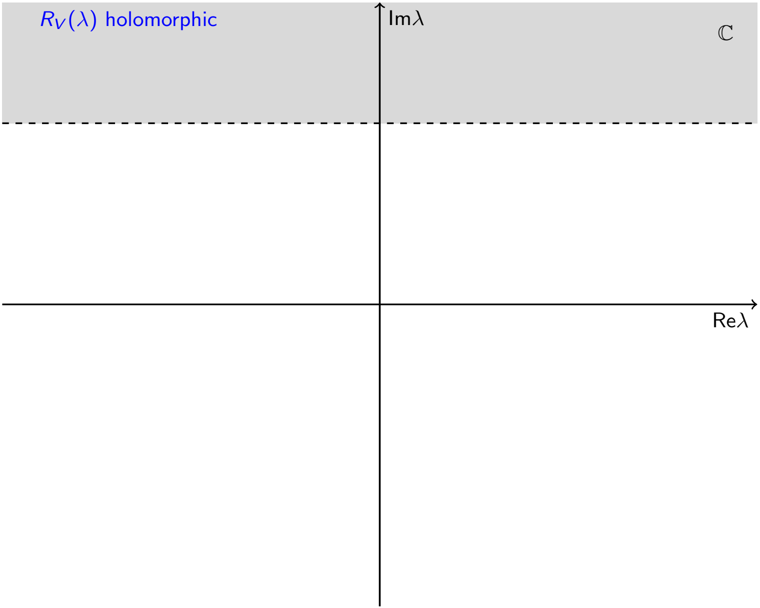
The poles λ_j of $R_V(\lambda)$ generate **residues** $u_j(x)e^{-i\lambda_j t}$ in (2). In particular, if $R_V(\lambda)$ has no poles above $\mathrm{Im}\lambda \geq -A$ – **resonance-free strip** – waves scattered by V decay locally like e^{-At}.

Resonances as poles of $R_V(\lambda) = (-\Delta + V - \lambda^2)^{-1}$



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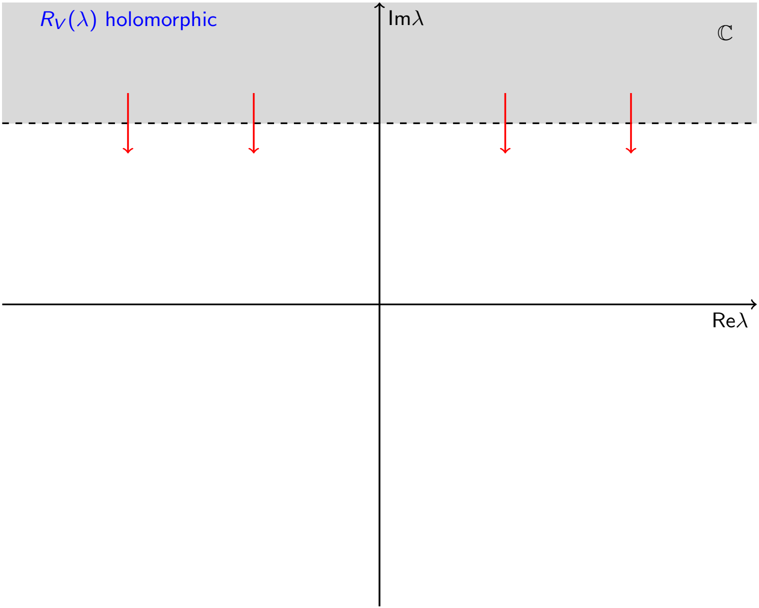

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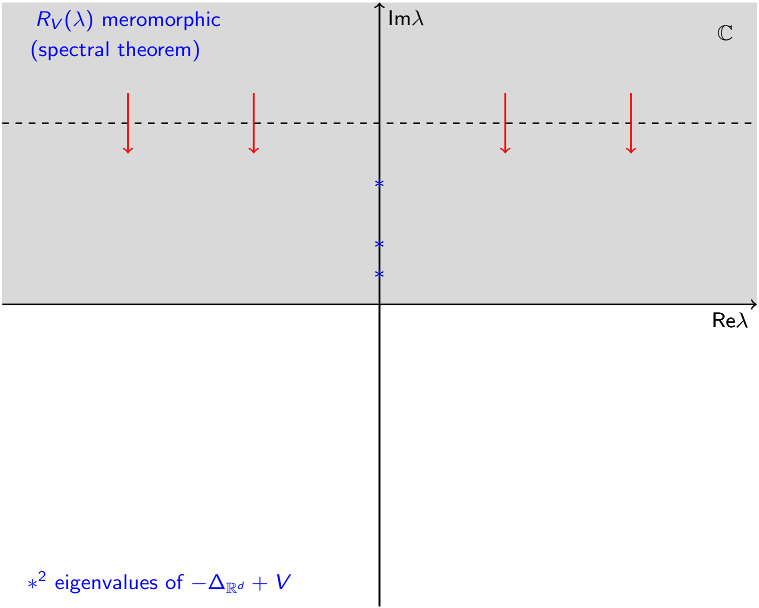

$*^2$ eigenvalues of $-\Delta_{\mathbb{R}^d} + V$

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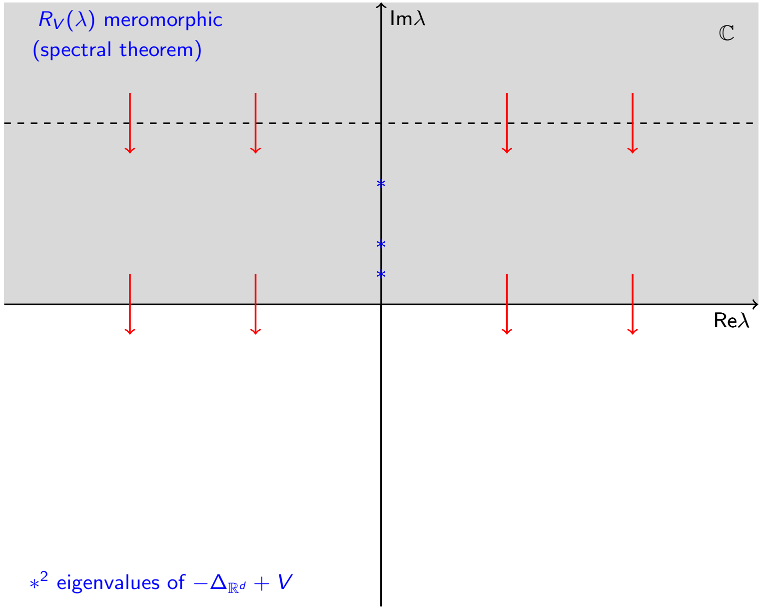

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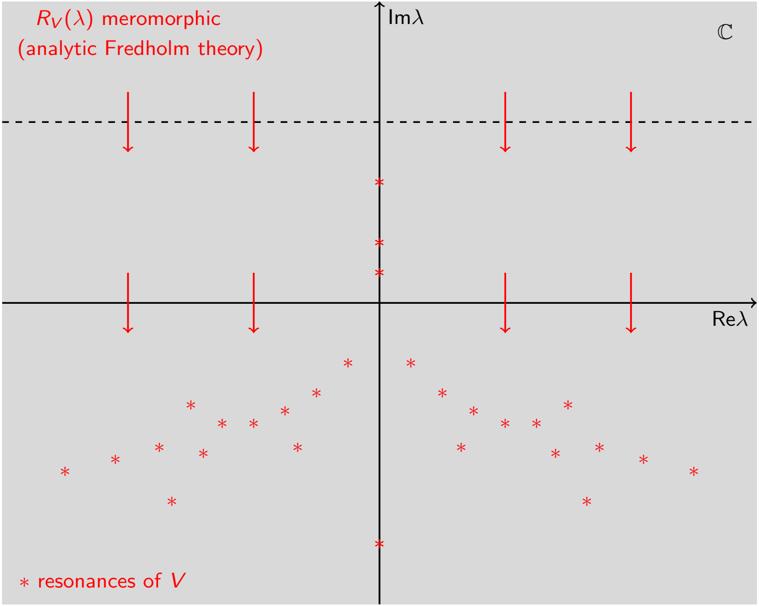

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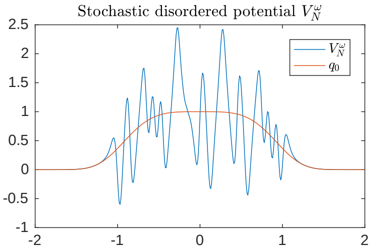

Example of potential V_N with $N = 20$ in blue, with q_0 in red.

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We observe a weak averaging effect on V_N.

Does this transfer to resonances of V_N, i.e. are resonances of V_N well approximated by resonances of q_0?

Result I: convergence of resonances

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In fact, after removing a set of probability $O(e^{-cN^{3/2}})$, for $q_0 \equiv 0$ resonances of V_N lie **below the logarithmic line $\Im\lambda = -A\ln(N)$; and waves scattered by V_N decay like N^{-At}.**

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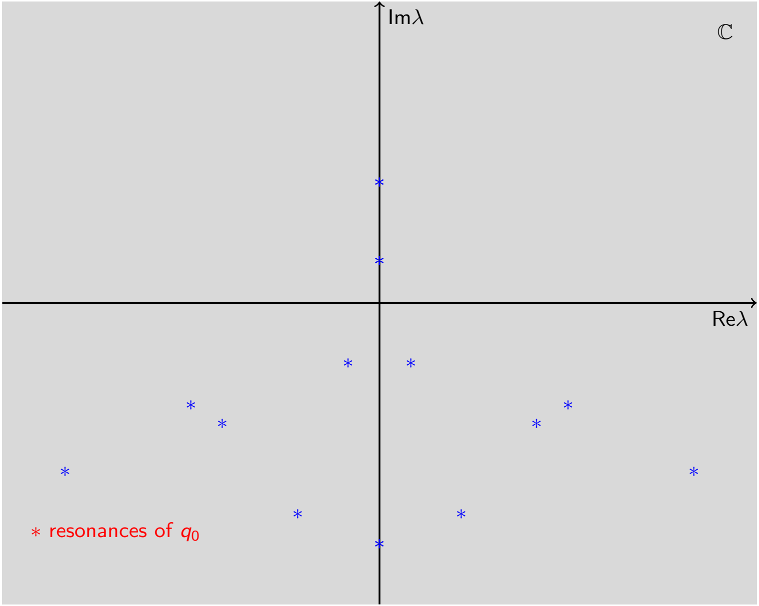

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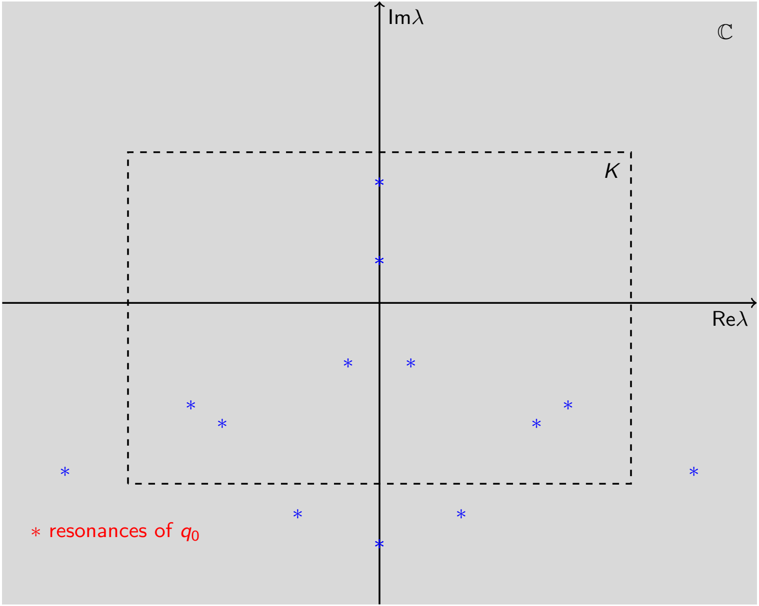

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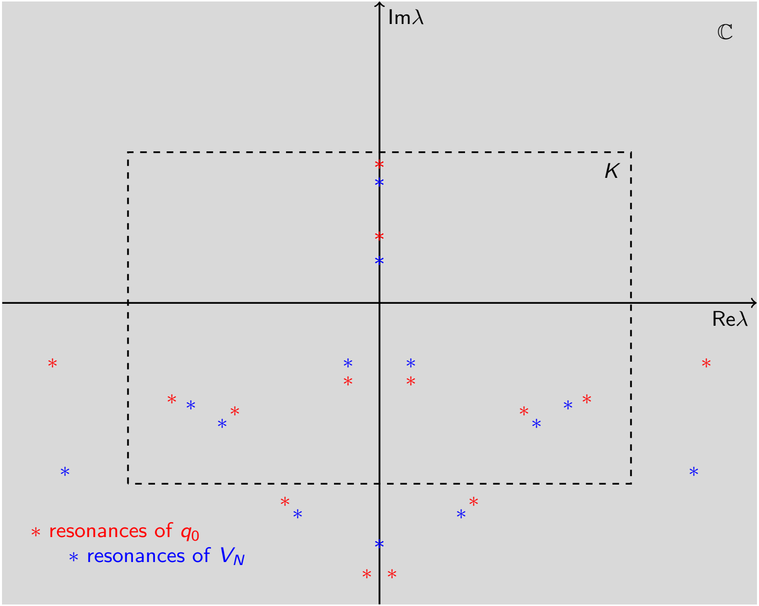

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Thanks for your attention!