Eigenvalues and resonances of highly oscillatory potentials

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Columbia University,
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Waves and resonances in odd dimension d

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$$(\partial_t^2 - \Delta_{\mathbb{R}_x^d} + V)u = 0, \quad (u|_{t=0}, \partial_t u|_{t=0}) = (f_0, f_1). \tag{1}$$

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This is reflected in the fact that the spectrum of $-\Delta_{\mathbb{R}_x^d} + V$ on $L^2(\mathbb{R}^d)$ is the union of a **discrete set** (eigenvalues) with the **continuous spectrum** $[0, \infty)$.

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Eigenvalues μ are poles of $(-\Delta_{\mathbb{R}^d} + V - \mu)^{-1}$, hence (squares of) resonances. **Conversely, resonances inducing eigenvalues are the one lying on the complex half-line $i[0,\infty)$.**

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The expansion (2) comes from a **contour deformation** in the representation of u given by the spectral theorem:

$$u = \int_{\mathbb{R}} e^{-it\lambda} \frac{R_V(\lambda) - R_V(-\lambda)}{2\pi} f_1 d\lambda - \int_{\mathbb{R}} \lambda e^{-it\lambda} \frac{R_V(\lambda) - R_V(-\lambda)}{2\pi} f_0 d\lambda.$$

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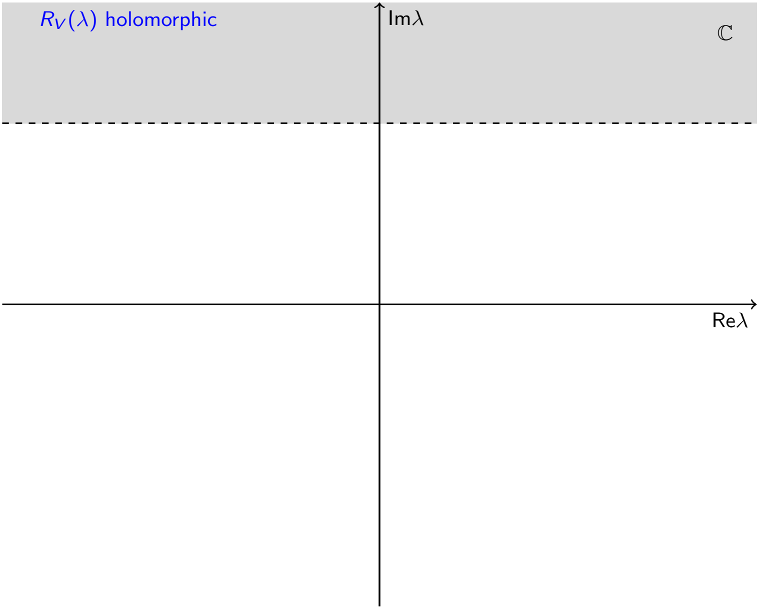
The poles λ_j of $R_V(\lambda)$ generate **residues** $u_j(x)e^{-i\lambda_j t}$ in (2). In particular, if $R_V(\lambda)$ has no poles above $\mathrm{Im}\lambda \geq -A$ – **resonance-free strip** – waves scattered by V decay locally like e^{-At}.

Resonances as poles of $R_V(\lambda) = (-\Delta + V - \lambda^2)^{-1}$



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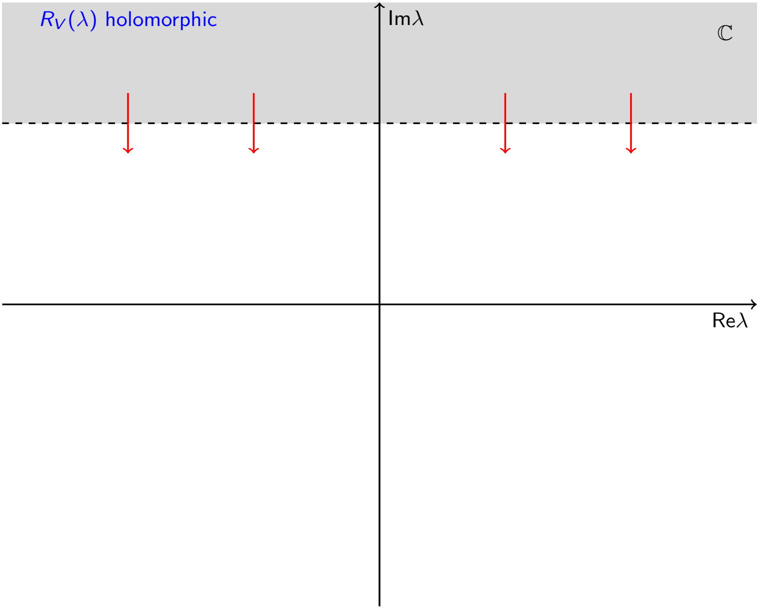

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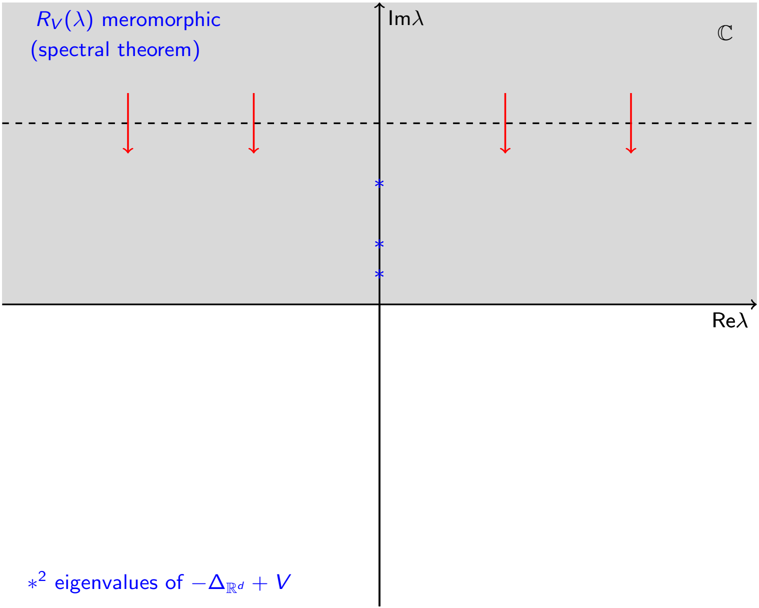

$*^2$ eigenvalues of $-\Delta_{\mathbb{R}^d} + V$

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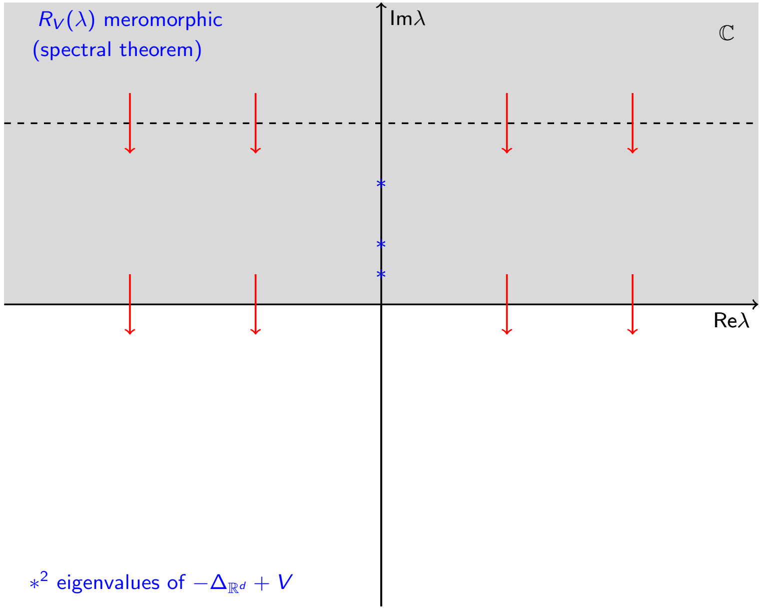

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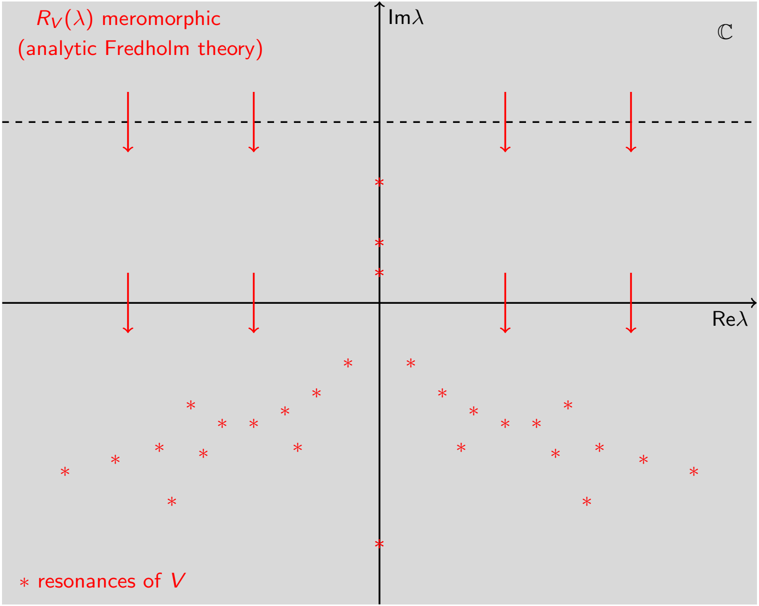

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The first model is an idealized version of the second one: **perfectly alternated oscillations** play the role of **randomness.**

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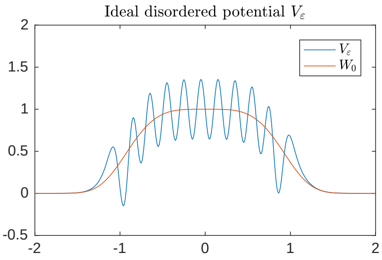

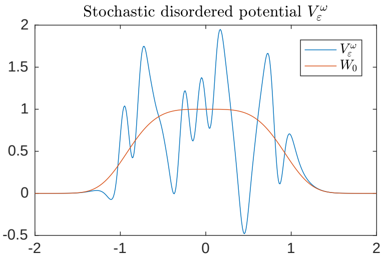

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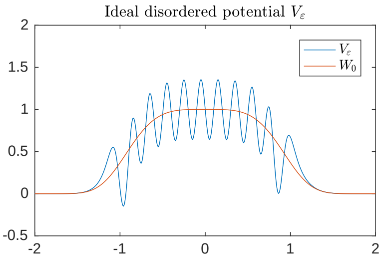

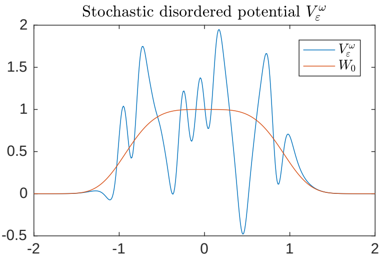

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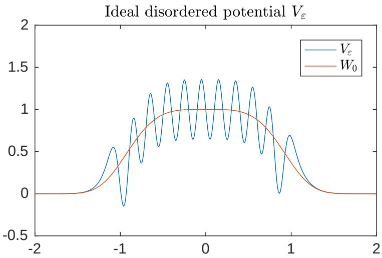

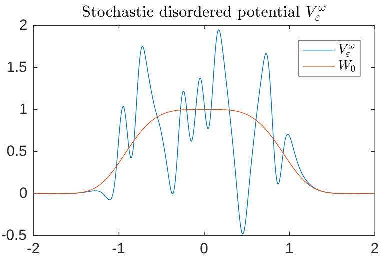

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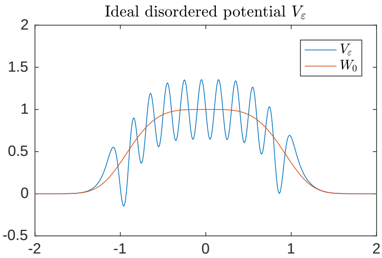

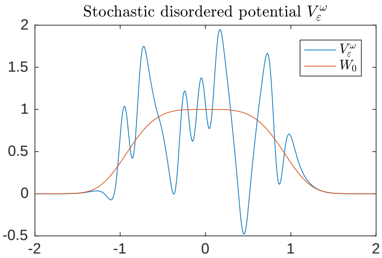

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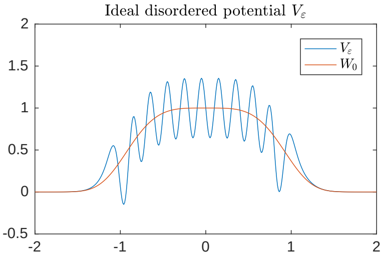

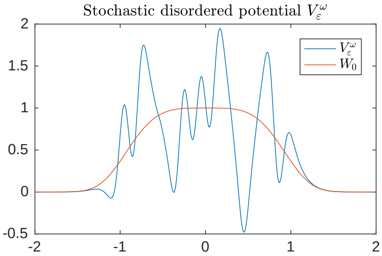

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$$= \varepsilon^d \sum_{|j|\le\varepsilon^{-1}} u_j(\omega)\varphi(\varepsilon j)\cdot \int q(x)dx + O(\varepsilon^{d+1}) \sum_{|j|\le\varepsilon^{-1}} |u_j(\omega)| \xrightarrow{\mathbb{P}\text{-a.s.}} 0 \quad \textbf{(K.S.L.L.N).}$$

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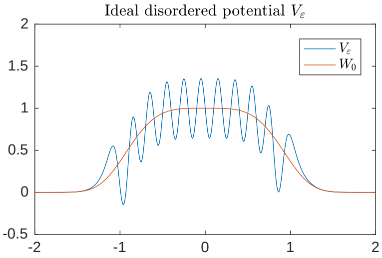

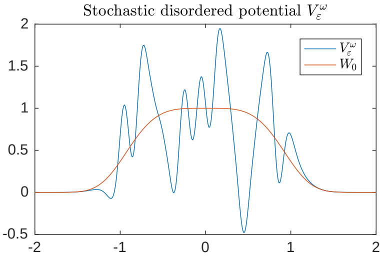

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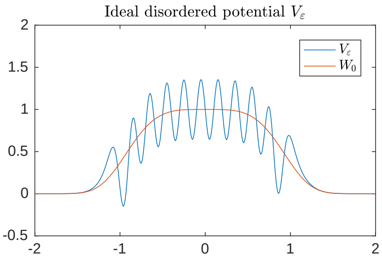

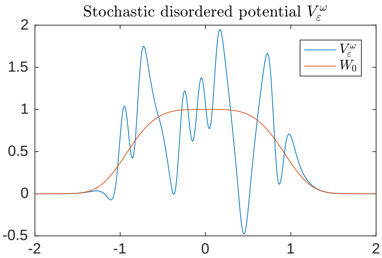

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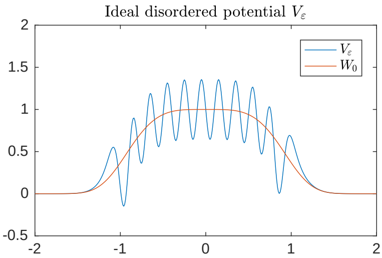

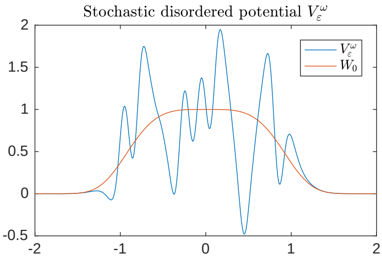

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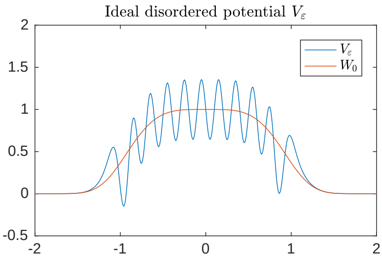
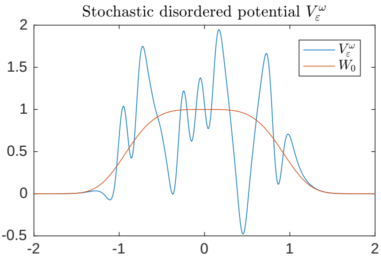
- Localize **resonance-free strips** of V_ε, V_ε^ω.

Waves in heterogeneous media





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Objectives:

- Localize **resonance-free strips** of V_ε, V_ε^ω.
- **Construct effective potentials** approximating eigenvalues/resonances of $-\Delta + V_\varepsilon$, $-\Delta + V_\varepsilon^\omega$.

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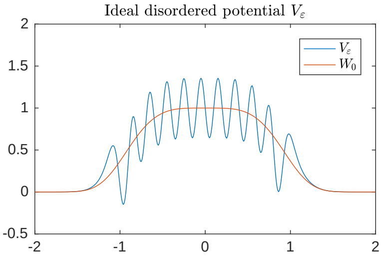

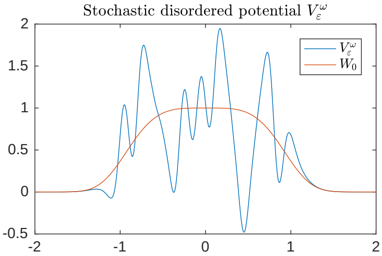

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- Localize **resonance-free strips** of $V_\varepsilon, V_\varepsilon^\omega$.
- **Construct effective potentials** approximating eigenvalues/resonances of $-\Delta + V_\varepsilon$, $-\Delta + V_\varepsilon^\omega$.
- Study **convergence** of eigenvalues/resonances of $V_\varepsilon, V_\varepsilon^\omega$ to the resonances of their **average/weak limit** W_0 as $\varepsilon \to 0$.

Existing results for V_ε

$$V_\varepsilon(x) = W_0(x) + \sum_{k\in\mathbb{Z}\setminus 0} W_k(x) e^{ikx/\varepsilon} \quad \textbf{(d=1)}.$$

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- [B'06], [BG'07]: for $W_0 = 0$, $-\partial_x^2 + V_\varepsilon$ has an **eigenvalue** given by

$$E_\varepsilon = -\frac{\varepsilon^4}{4}\int_{\mathbb{R}} \Lambda_0(x)dx + O(\varepsilon^5), \quad \Lambda_0(x) = \sum_k \frac{|W_k(x)|^2}{k^2}. \qquad (3)$$

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$$i\partial_t u - \partial_x^2 u + V_\varepsilon u = 0,$$

uniform as $\varepsilon \to 0$ despite the **presence of an eigenvalue near** 0.

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- if $d = 3$, $-\Delta + V_\varepsilon$ has **no eigenvalue** – proved in [Dr'15], in a more detailed version described here.
- if $d = 2$, $-\Delta + V_\varepsilon$ has a **unique eigenvalue** E_ε**,** that is exponentially close to 0:

$$E_\varepsilon = -\exp\left(-\frac{4\pi}{\varepsilon^2 \int_{\mathbb{R}^2} \Lambda_0(x)dx + o(\varepsilon^2)}\right), \quad \Lambda_0(x) = \sum_k \frac{|W_k(x)|^2}{|k|^2}.$$

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Results: resonance-free strips when $W_0 = 0$

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- The result **generalizes to stochastic potentials.**

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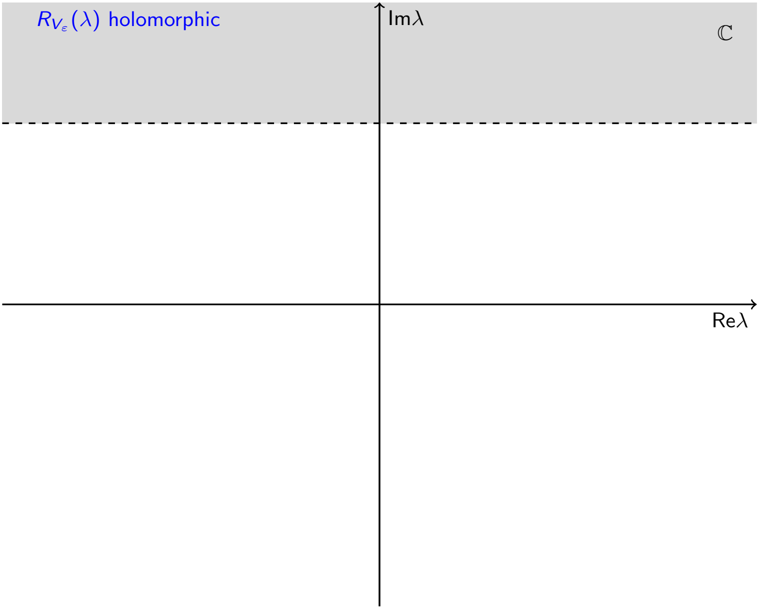

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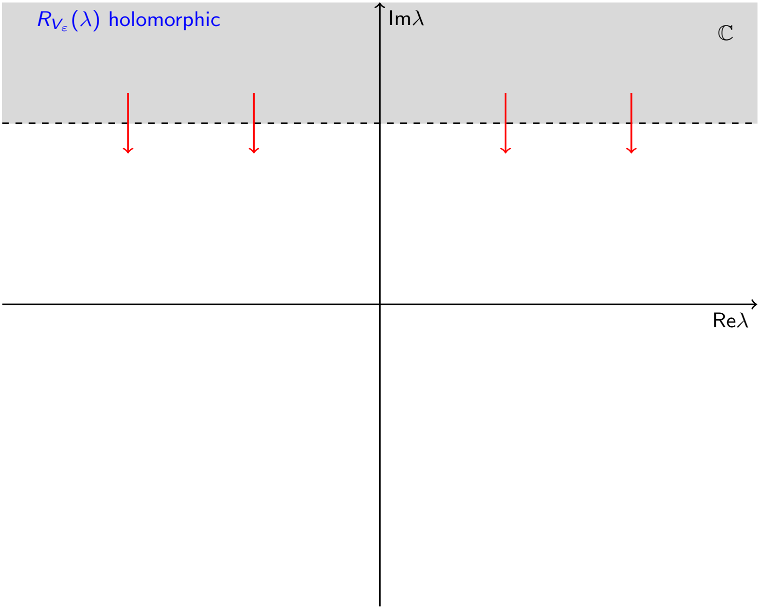

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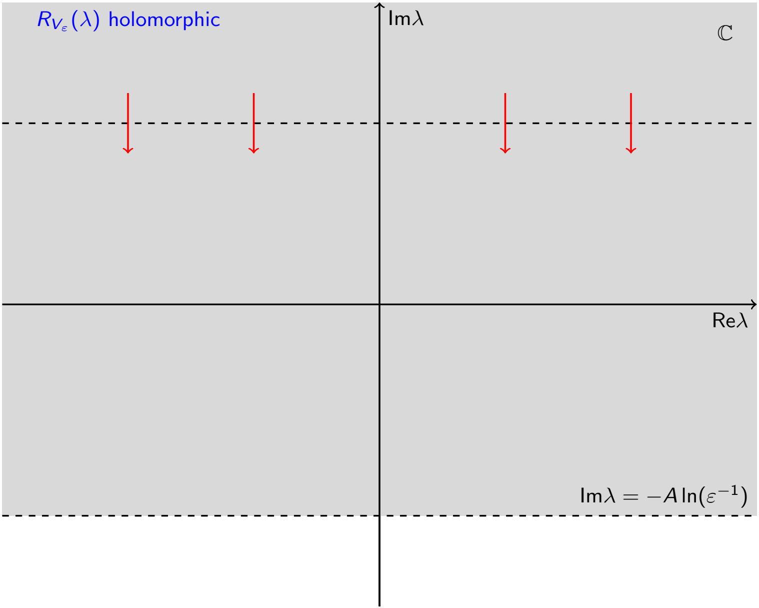

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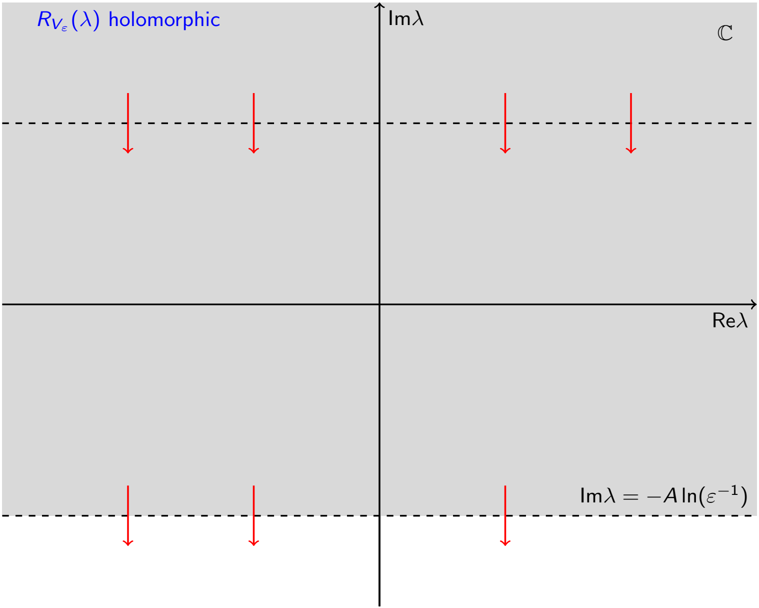

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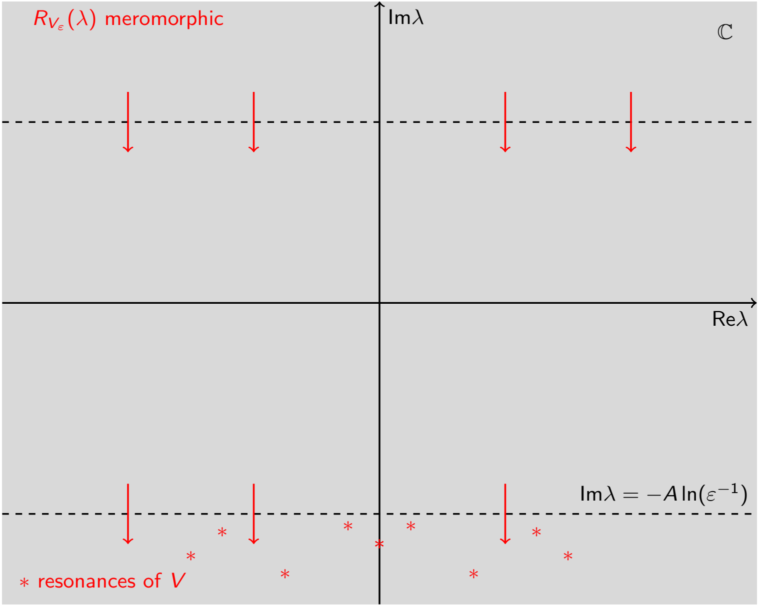

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This would show that for such λ's, $\mathrm{Id} + V_\varepsilon R_0(\lambda)\rho$ **is invertible by a Neumann series and conclude the proof.**

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A computation shows that $|V_\varepsilon|_{H^{-2}} = O(\varepsilon^2)$.

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Fact: $|\rho R_0(\lambda)\rho|_{L^2 \to H^2} = O(e^{C(\mathrm{Im}\lambda)_-})$. Hence

$$|(V_\varepsilon R_0(\lambda)\rho)^2|_{L^2 \to L^2} = O(e^{2C(\mathrm{Im}\lambda)_-})|V_\varepsilon|_{H^2 \to H^{-2}}. \tag{4}$$

The term $|V_\varepsilon|_{H^2 \to H^{-2}}$ is small: H^2 is an algebra when $d = 3$:

$$\forall f, g, \ |fg|_{H^2} \leq |f|_{H^2}|g|_{H^2}, \qquad \textbf{dual inequality: } |fg|_{H^{-2}} \leq |f|_{H^{-2}}|g|_{H^2}$$

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A computation shows that $|V_\varepsilon|_{H^{-2}} = O(\varepsilon^2)$. Combine with (4) to get

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Proof of the theorem when $d = 3$

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Important remark: When $d = 3$ the proof works for any family $\{V_\varepsilon\}$ that satisfies $|V_\varepsilon|_{H^{-2}} \leq \varepsilon^\delta$ for some $\delta > 0$.

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Recall that $q \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$, u_j are i.i.d, with $\mathbb{E}[u_j] = 0$ and compactly supported distributions, and

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For all $\delta > 0$, there exists C, c, A, ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$, **with probability** $1 - Ce^{-c/\varepsilon^{3-\delta}}$**,** V_ε^ω *has no resonances above the line* $\mathrm{Im}\lambda = -A\ln(\varepsilon^{-1})$.

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The theorem does not hold with probability higher than $1 - 2^{-1/\varepsilon^3}$**.**

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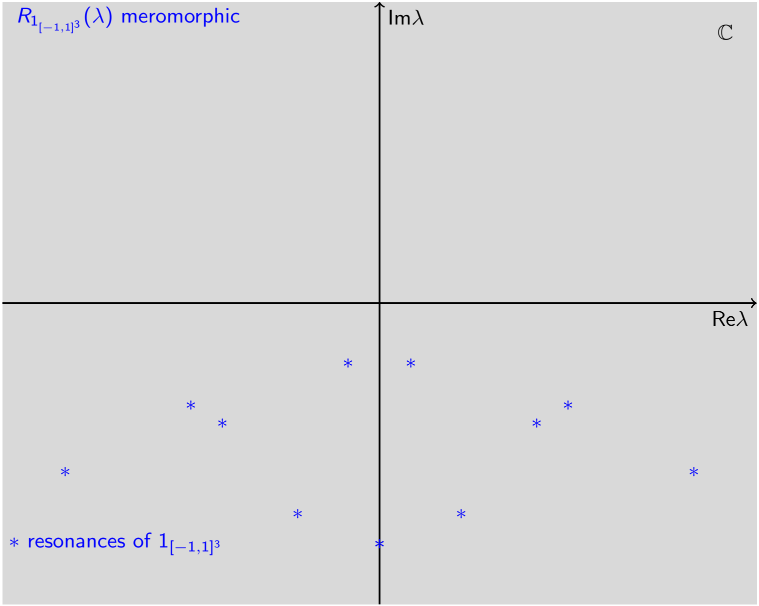

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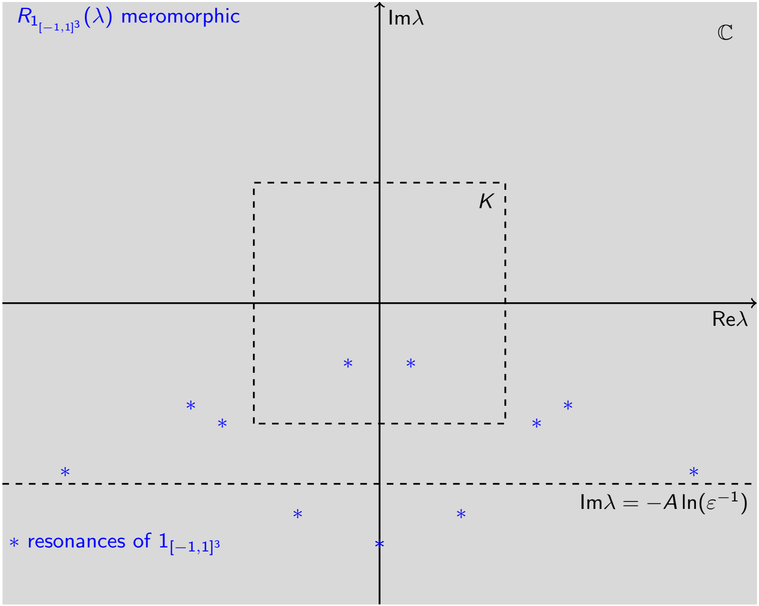

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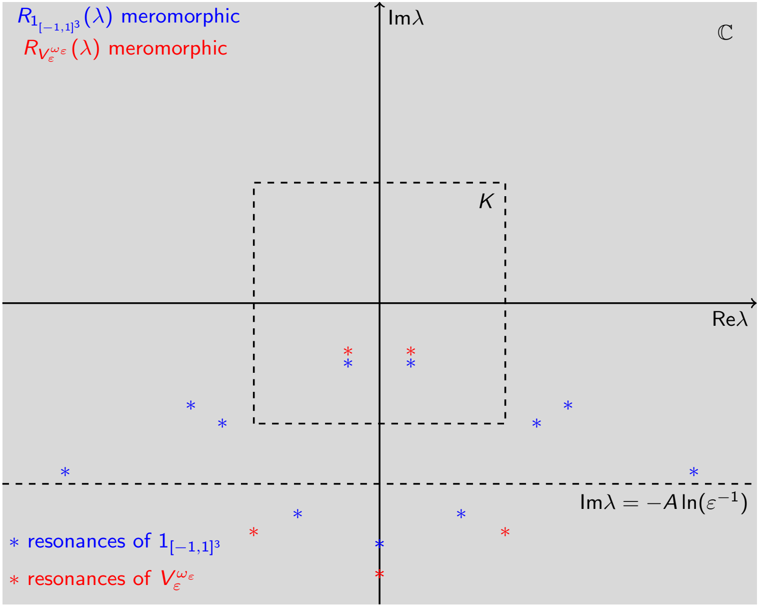

The case $W_0 \neq 0$, V_ε highly oscillatory

We go back to the **periodic case,** with $W_0 \neq 0$ this time:

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- Simple resonances of V_ε in compact sets depend **smoothly** on ε, despite the **singular** dependence of V_ε.
- The coefficients a_4, a_5, ... are computable.
- More complicated statements hold for non-simple resonances.

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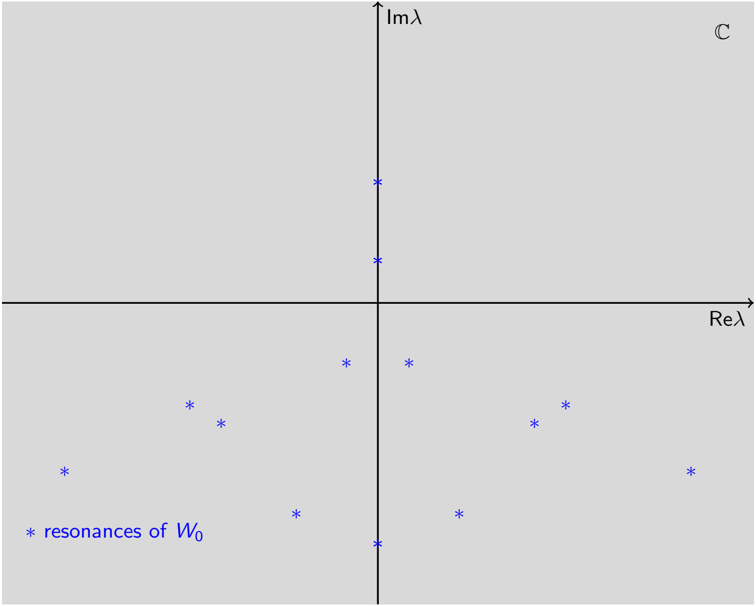
Comments:

- This theorem refines the **effective potential** $W_0 - \varepsilon^2\Lambda_0$ of Duchêne–Vukićević–Weinstein and generalizes it to **all odd dimensions.** Further refinements are possible with **non-linear resonances** and an effective potential depending on λ.

Convergence of resonances



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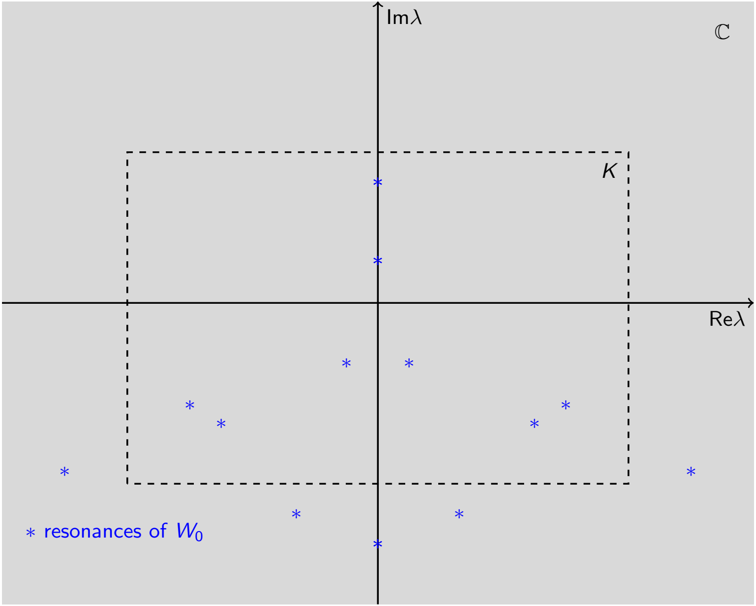

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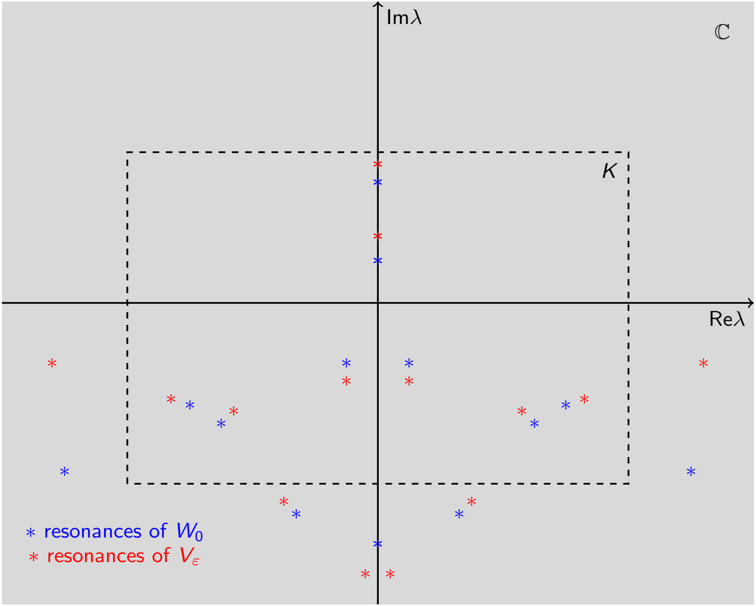

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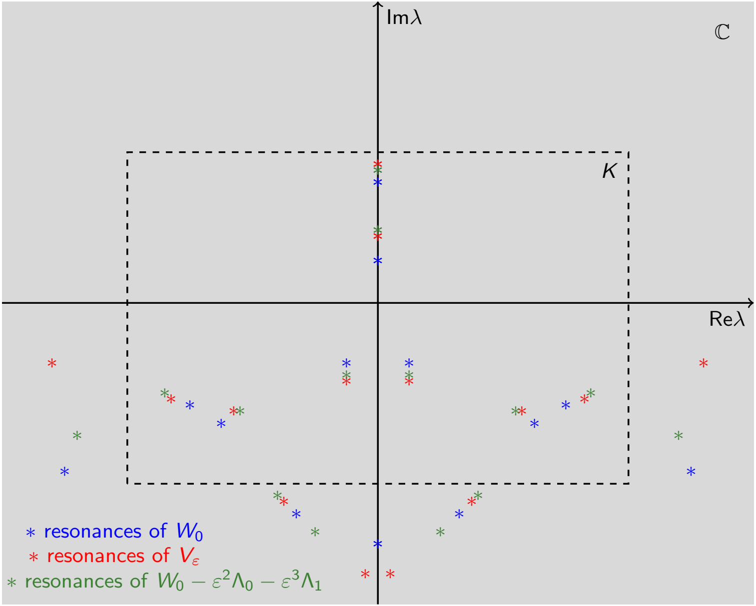

Numerical results (Thanks to D–V–W)

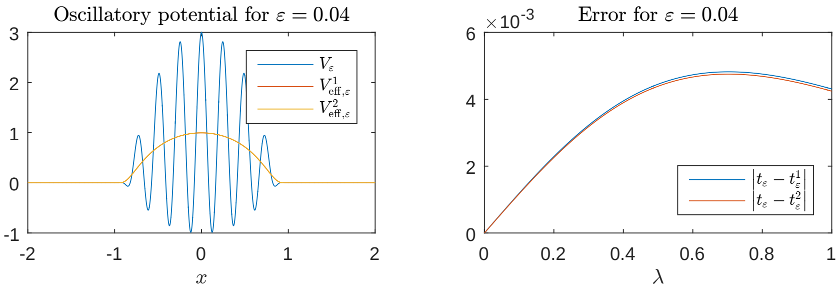

Figure: Oscillatory potential and **errors in approximating the transmission coefficient** of V_ε by the transmission coefficient of the Duchêne–Vukićević–Weinstein **effective potential** $W_0 - \varepsilon^2 \Lambda_0$ and by the refined one $W_0 - \varepsilon^2 - \varepsilon^3 \Lambda_1$. Here $\varepsilon = 1/25$.

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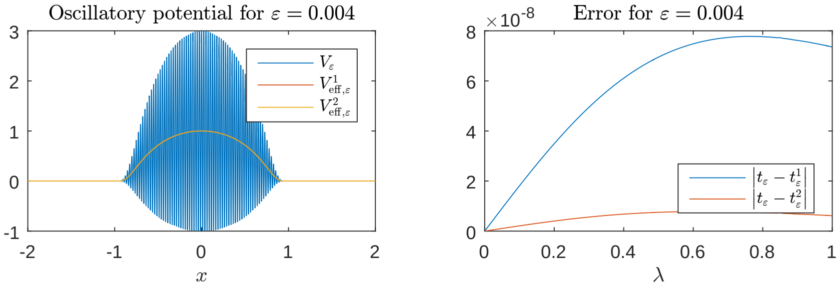

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Assume $d = 1$, $V_\varepsilon(x) \stackrel{\text{def}}{=} \sum_{k\neq 0} W_k(x) e^{ikx/\varepsilon}$, $D_{V_\varepsilon}(\lambda) \stackrel{\text{def}}{=} \det(\mathrm{Id} + \rho R_0(\lambda) V_\varepsilon)$.

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The operator $((D + \ell/\varepsilon)^2 - \lambda^2)^{-1}$ formally expands in powers of ε:

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This formally explains how the **effective potential** $-\varepsilon^2 \Lambda_0 - \varepsilon^3 \Lambda_1$ appears: it is produced by the **constructive interference of oscillatory terms.** The **actual proof of the theorem contains similar ideas,** but is more complicated. It uses expansion of modified Fredholm determinants, combinatorics, oscillatory integrals, operator-valued expansions,...

Resonances for random oscillatory potentials

Recall that $q \in C_0^\infty(\mathbb{R}^3)$, u_j are i.i.d, with $\mathbb{E}[u_j] = 0$ and compactly supported distributions, and

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$$\varepsilon^{-1/2}(\lambda_\varepsilon - \lambda_0) \xrightarrow{\text{law}} \mathcal{N}(0, \sigma^2), \ \ \sigma^2 = \mathbb{E}\left(u_j^2\right) \left(\int_{\mathbb{R}} q(x)dx\right)^2 \int_{-1}^{1} u(x)^2 v(x)^2 dx,$$

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$$\lambda_\varepsilon = \lambda_0 + i\varepsilon^2 \int_{\mathbb{R}^3} \frac{|\hat{q}(\xi)|^2}{|\xi|^2} \int_{[-1,1]^3} u(x)v(x)dx + o(\varepsilon^2),$$

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- We can show similar results for any value of d. When $d \geq 5$, the convergence of resonance is always **almost sure.** When $d = 1$ or 3, it is almost sure if $\hat{q}(\xi)$ **vanishes at sufficiently high order at** $\xi = 0$**.**

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- It is also expected that the convergence rate of λ_ε to λ_0 depend on d. The number of sites ε^{-d} grows with d, which makes **large deviation effects smaller and less likely,** and the **homogenization effect** highlighted in the case of highly oscillatory potentials **takes over.**

Open questions/current projects

- Study scattering resonances of **large** oscillatory potentials: $\varepsilon^{-\beta} V_\varepsilon$, $\varepsilon^{-\beta} V_\varepsilon^\omega$, $\beta \in (0, 2]$ – interesting work of Duchêne–Raymond, Dimassi, Dimassi–Duong in this direction.

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- Approximate the **dynamics** of $\partial_t^2 - \Delta_{\mathbb{R}^d} + V_\varepsilon$ by the one of $\partial_t^2 - \Delta_{\mathbb{R}^d} + V_\varepsilon^{\mathrm{eff}}$, away from the discrete spectrum.

Thanks for your attention!