Structure constants of Peterson Schubert calculus

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Table of Contents

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- $C_\Delta := (\langle \alpha_i, \alpha_j \rangle)_{i,j\in\Delta} = (c_{ij})_{i,j\in\Delta}$ Cartan matrix associated with $\mathfrak{g}$,

  e.g., $C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ for $\mathfrak{sl}_4$.

Flag variety and Bruhat decompositions

Flag variety and Bruhat decompositions

Let G/B be the complete flag variety, e.g.,

$$SL_{n+1}(\mathbb{C})/B \cong \left\{F_* = (0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^{n+1}) \,\middle|\, \dim_{\mathbb{C}} F_i = i\right\}.$$

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Bruhat decomposition of G/B

$$B \curvearrowright G/B = \bigsqcup_{w \in W} BwB/B,$$

where each $BwB/B \cong \mathbb{C}^{l(w)}$ is the **Schubert cell** X_w°.

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Similarly,

$$B^- \curvearrowright G/B = \bigsqcup_{w \in W} B^- wB/B,$$

where $B^- wB/B \cong \mathbb{C}^{l(w_0)-l(w)}$ is the **opposite Schubert cell** Ω_w°.

Peterson variety

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Fix a regular nilpotent $e \in \mathfrak{b} = \mathrm{Lie}(B)$. The **Peterson variety** is

$$\mathrm{Pet}_G := \{gB \in G/B \mid \mathrm{Ad}_{g^{-1}}(e) \in \mathfrak{b} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{-\alpha}\},$$

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- Type A_{n-1} case:
  $\mathrm{Fl}_n := \left\{V_\bullet = (0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n) \mid \dim_{\mathbb{C}} V_i = i\right\},$

$$N := \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{C}^{n \times n},$$

$\mathrm{Pet}_n \subset \mathrm{Fl}_n$ has a concrete geometric description in terms of flags

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- There is a one-dimensional sub-torus $S \subset T$ acting on Pet_G.

Peterson & Schubert

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- $\mathrm{Pet}_G \cap X_w^\circ \neq \emptyset \iff \mathrm{Pet}_G \cap \Omega_w^\circ \neq \emptyset \iff w = w_J$ for some $J \subset \Delta$, where w_J is the longest element of the parabolic subgroup $W_J \subset W$.

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- For any $I \subset \Delta$, denote

$$\mathrm{Pet}_{G,I}^\circ := \mathrm{Pet}_G \cap X_{w_I}^\circ \cong \mathbb{C}^{|I|},$$

the **Peterson cell**.

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the **Peterson Schubert variety**.

Presentation of $H_S^*(\mathrm{Pet}_G, \mathbb{Q})$

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For any weight $\mu : T \to \mathbb{C}^\times$, consider the line bundle

$$L_\mu := G \times^B \mathbb{C}_{-\mu},$$

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Theorem (Harada–Horiguchi–Masuda, 2015)

There is an isomorphism

$$H_S^*(\mathrm{Pet}_G; \mathbb{Q}) \cong \frac{\mathbb{Q}[\varpi_1, \ldots, \varpi_n, t]}{\left(\sum_{j=1}^n \langle \alpha_i, \alpha_j \rangle \varpi_i \varpi_j - 2t\varpi_i \;\middle|\; 1 \le i \le n\right)},$$

where ϖ_i corresponds to the equivariant first Chern class $c_1(L_{\varpi_i})$.

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Theorem (Goldin–Mihalcea–Singh, 2024)

For $I \subset \Delta$, fix a Coxeter element v_I in W. Define the **Peterson Schubert class**

$$p_I := \frac{\iota^* \sigma_{v_I}}{m(v_I)} \in H_S^*(\mathrm{Pet}_G, \mathbb{Q}),$$

where $\iota : \mathrm{Pet}_G \hookrightarrow G/B$ and $m(v_I) \in \mathbb{Z}_{\geq 0}$ is a certain intersection multiplicity.

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2. *Define*

$$p_I \cdot p_J = \sum_K c_{I,J}^K p_K,$$

then the structure constants $c_{I,J}^K \in H_S^*(pt) \cong \mathbb{Z}[t]$ are in fact $\in \mathbb{Z}_{\geq 0}[t]$.

Table of Contents

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- For Lie type A, Abe–Horiguchi–Kuwata–Zeng (2024) found a different combinatorial model for non-equivariant structure constants.

Peterson Schubert monomials

For $I \subset \Delta$, let C_I be the Cartan sub-matrix, W_I denotes the parabolic subgroup. Combine the equivariant Giambelli formula of Drellich and the intersection multiplicity formula proved by Goldin–Singh, we can get

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Theorem (Folklore?)

For $I \subset \Delta$, the Peterson Schubert class p_I is represented by

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Now the problem of Peterson Schubert calculus becomes:
how can we expand the product of two (square-free) monomials in the above monomial basis modulo some **quadratic** relations?

Equivariant structure constants

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Theorem (Gui–Jia–Yu–Z.–Zhu, 2025)

Suppose $I = \{i_1, \ldots, i_p\} \subset \Delta$. *The equivariant structure constant* $c_{I,J}^K$ *is equal to*

$$\frac{\det(C_I)\det(C_J)\,|W_K|}{|W_I|\,|W_J|\det(C_K)}$$ *multiplied by the* (J,K)*-entry of the product* $D_{i_1}\cdots D_{i_p}$,

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- Transform structure constants into an infinite matrix series related to **Coxeter adjacency matrix** $A_K := 2E - C_K$.

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where each matrix D_i *has entries* $(d_{i,J}^K)_{J,K\subset\Delta}$ *defined by*

$$d_{i,J}^K = \begin{cases} [C_K^{-1}]_{i,s}/[C_K^{-1}]_{s,s}, & i \in K,\ |K| = |J|+1,\ K\setminus J = \{s\}; \\ 2t\sum_{k\in K}[C_K^{-1}]_{i,k}, & i\in K,\ K = J; \\ 0, & \textit{otherwise.} \end{cases}$$

- Transform structure constants into an infinite matrix series related to **Coxeter adjacency matrix** $A_K := 2E - C_K$.
- Use **Perron–Frobenius theorem** to prove convergency and relate to the inverse of a matrix.

Equivariant structure constants

Theorem (Gui–Jia–Yu–Z.–Zhu, 2025)

Suppose $I = \{i_1, \ldots, i_p\} \subset \Delta$. *The equivariant structure constant* $c_{I,J}^{K}$ *is equal to*

$$\frac{\det(C_I)\det(C_J)\,|W_K|}{|W_I|\,|W_J|\det(C_K)}$$ *multiplied by the* (J,K)*-entry of the product* $D_{i_1}\cdots D_{i_p}$*,*

where each matrix D_i *has entries* $(d_{i,J}^{K})_{J,K\subset\Delta}$ *defined by*

$$d_{i,J}^{K} = \begin{cases} [C_K^{-1}]_{i,s}/[C_K^{-1}]_{s,s}, & i \in K,\ |K| = |J|+1,\ K \setminus J = \{s\}; \\ 2t\sum_{k\in K}[C_K^{-1}]_{i,k}, & i \in K,\ K = J; \\ 0, & \textit{otherwise}. \end{cases}$$

- Transform structure constants into an infinite matrix series related to **Coxeter adjacency matrix** $A_K := 2E - C_K$.
- Use **Perron–Frobenius theorem** to prove convergency and relate to the inverse of a matrix.
- Use **Sherman–Morrison formula** to compute inverse.

Example of equivariant structure constants

Example (Type B_3, $I = \{2\}$)

For Lie type B_3, the structure constants matrix of $I = \{2\}$ is

$$d_{I,J}^{K} = \begin{array}{cc} & J \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2t & 0 & \frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2t & 0 & 0 & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2t & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2t \end{pmatrix} & \begin{matrix} \emptyset \\ \{1\} \\ \{2\} \\ \{3\} \\ \{1,2\} \\ \{1,3\} \\ \{2,3\} \\ \{1,2,3\} \end{matrix} \end{array}$$

Then $d_{\{2\},\{1,2\}}^{\{1,2\}} = 2t$, $d_{\{2\},\{1,2\}}^{\{1,2,3\}} = \frac{4}{3}$, hence $\varpi_2 \cdot \varpi_1\varpi_2 = 2t\varpi_1\varpi_2 + \frac{4}{3}\varpi_1\varpi_2\varpi_3$.

Non-equivariant structure constants

Since $H^*(\mathrm{Pet}_G;\mathbb{Z})$ vanishes in odd degrees, Pet_G is **equivariantly formal**. By letting $t=0$, we get a simple formula for all non-equivariant structure constants with respect to the basis $\{p_I\}_{I\subset\Delta}$ of $H^*(\mathrm{Pet}_G;\mathbb{Z})$.

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Theorem (Gui–Jia–Yu–Z.–Zhu, 2025)

The non-equivariant structure constant of Peterson Schubert calculus is equal to

$$\frac{\det(C_I)\det(C_J)\,|W_K|}{|W_I|\,|W_J|\det(C_K)} \text{ multiplied by the } (J,K)\text{-entry of the product } M_{i_1}\cdots M_{i_p},$$

where each matrix M_i has entries $(m_{i,J}^K)_{J,K\subset\Delta}$ defined by

$$m_{i,J}^K = \begin{cases} \dfrac{[C_K^{-1}]_{i,s}}{[C_K^{-1}]_{s,s}}, & i \in K,\ |K| = |J|+1,\ K \setminus J = \{s\}; \\ 0, & \textit{otherwise}. \end{cases}$$

Example of non-equivariant structure constants

Example (Type A_9, $I = \{3,6,8\}$)

For Lie Type A_9, $I = \{3,6,8\}$, $J = \{1,3,5,6,7\}$, from our formula and letting $t = 0$, one can deduce

$$\varpi_I \cdot \varpi_J = \frac{18}{35}\varpi_{\{1,2,3,4,5,6,7,8\}} + \frac{1}{5}\varpi_{\{1,2,3,5,6,7,8,9\}} + \frac{2}{7}\varpi_{\{1,3,4,5,6,7,8,9\}}.$$

Hence, we have

$$\begin{aligned} p_I \cdot p_J =& \frac{1}{1!\cdot 1!\cdot 1!}\cdot\frac{1}{1!\cdot 1!\cdot 3!}\varpi_I\varpi_J \\ =& \frac{1}{3!}\left(\frac{18}{35}\varpi_{\{1,2,3,4,5,6,7,8\}} + \frac{1}{5}\varpi_{\{1,2,3,5,6,7,8,9\}} + \frac{2}{7}\varpi_{\{1,3,4,5,6,7,8,9\}}\right) \\ =& \frac{8!}{3!}\frac{18}{35}p_{\{1,2,3,4,5,6,7,8\}} + \frac{3!\cdot 5!}{3!}\frac{1}{5}p_{\{1,2,3,5,6,7,8,9\}} + \frac{7!}{3!}\frac{2}{7}p_{\{1,3,4,5,6,7,8,9\}} \\ =& 3456p_{\{1,2,3,4,5,6,7,8\}} + 24p_{\{1,2,3,5,6,7,8,9\}} + 240p_{\{1,3,4,5,6,7,8,9\}}. \end{aligned}$$

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Corollary

The structure constant $c_{I,J}^{K} \neq 0$ (equivalently, $d_{I,J}^{K} \neq 0$) if and only if

- *$K \supset I \cup J$, and*
- *For each connected component K_k of K, we have $|K_k| \leq |K_k \cap I| + |K_k \cap J|$.*

Table of Contents

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Example (Type A_{n-1} weight polytope—Permutohedron)

Consider the natural action of S_n on $\mathbb{R}^n$. For $a_1, \ldots, a_n \in \mathbb{R}$, the **permutohedron** *$P_n(a_1, \ldots, a_n)$ is the convex hull of points in the S_n-orbit of $(a_1, \ldots, a_n)$*

$$P_n(a_1, \ldots, a_n) := \mathrm{ConvexHull}\left\{\left(a_{w(1)}, \ldots, a_{w(n)}\right) \in \mathbb{R}^n \mid w \in S_n\right\}.$$

Standard permutohedra

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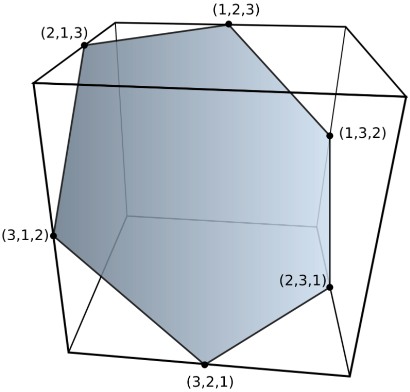

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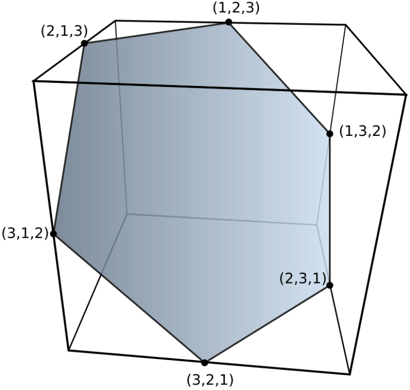

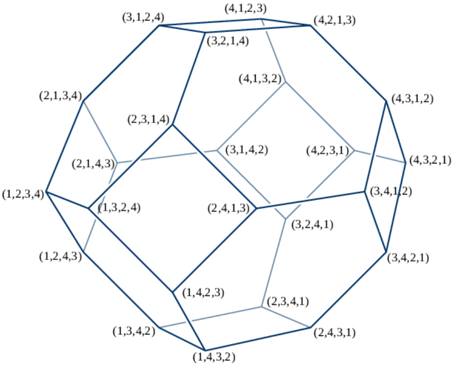

Mixed Φ-Eulerian numbers

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Definition (Postnikov, 2009)

Let $c_1, \ldots, c_n \geq 0$ with $\sum_{i=1}^{n} c_i = n$. The **mixed Φ-Eulerian number** $A^{\Phi}_{c_1,\ldots,c_n}$ is the coefficient of the volume polynomial

$$V_\Phi(u_1, \ldots, u_n) := \text{ volume of } P_\Phi\left(u_1\varpi_1 + \cdots + u_n\varpi_n\right) = \sum_{c_1,\ldots,c_n} A^{\Phi}_{c_1,\ldots,c_n} \frac{u_1^{c_1}}{c_1!} \cdots \frac{u_n^{c_n}}{c_n!}.$$

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- By definition, $A^{\Phi}_{c_1,\ldots,c_n}$ is exactly the **mixed volume** of c_1 copies of $P_\Phi\left(\varpi_1\right)$, c_2 copies of $P_\Phi\left(\varpi_2\right), \ldots$, and c_n copies of $P_\Phi\left(\varpi_n\right)$, multiplied by $n!$.

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- Here, the weight polytopes $P_\Phi\left(\varpi_1\right), P_\Phi\left(\varpi_2\right), \ldots, P_\Phi\left(\varpi_n\right)$ are called the **Φ-hypersimplices**.

Standard hypersimplices in $\mathbb{R}^3$

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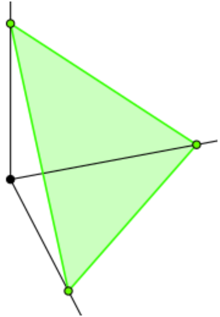
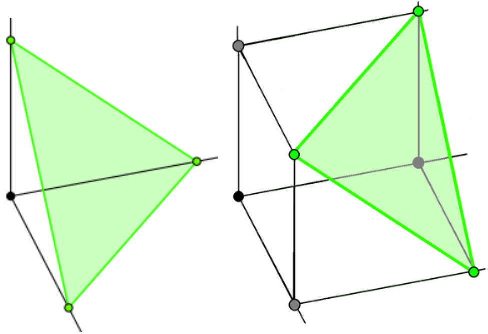


(a) $P_{A_2}(\varpi_1)$

Standard hypersimplices in $\mathbb{R}^3$



(a) $P_{A_2}(\varpi_1)$ (b) $P_{A_2}(\varpi_2)$

Mixed Eulerian numbers from Peterson Schubert calculus

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Theorem (Horiguchi, 2024)

Let Φ be an irreducible root system of rank n. Let $c_1, \ldots, c_n$ be non-negative integers with $c_1 + \cdots + c_n = n$. Then the mixed Φ-Eulerian number $A^{\Phi}_{c_1,\ldots,c_n}$ is equal to

$$A^{\Phi}_{c_1,\ldots,c_n} = \int_{\mathrm{Pet}_G} \varpi_1^{c_1} \varpi_2^{c_2} \cdots \varpi_n^{c_n}.$$

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- Hence, we have

$$\mathrm{Vol}\, P_{\Phi}\left(u_1\varpi_1 + \cdots + u_n\varpi_n\right) = \frac{1}{n!} \int_{\mathrm{Pet}_G} \left(u_1\varpi_1 + \cdots + u_n\varpi_n\right)^n,$$

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  which is exactly the volume polynomials of the nef divisors $\varpi_1, \ldots, \varpi_n$ on the Peterson variety Pet_G.
- This gives a different proof that $\mathrm{Vol}\, P_{\Phi}\left(u_1\varpi_1 + \cdots + u_n\varpi_n\right)$ is **Lorentzian**.

Formula for mixed Φ-Eulerian numbers

Formula for mixed Φ-Eulerian numbers

Theorem (Gui–Jia–Yu–Z.–Zhu, 2025)

Let Φ be an irreducible root system. Let $c_1, \ldots, c_n$ be non-negative integers with $c_1 + \cdots + c_n = n$. Then $A^{\Phi}_{c_1,\ldots,c_n}$ can be computed using the following formula

$$A^{\Phi}_{c_1,\ldots,c_n} = \frac{|W_\Phi|}{\det(C_\Phi)} [M_1^{c_1} \cdots M_n^{c_n}]_{\emptyset,\Delta},$$

where M_i's are the matrices defined as before

$$m^K_{i,J} = \begin{cases} \frac{[C_K^{-1}]_{i,s}}{[C_K^{-1}]_{s,s}}, & i \in K,\ |K| = |J| + 1,\ K \setminus J = \{s\}; \\ 0, & \textit{otherwise}. \end{cases}$$

Here $[\quad]_{\emptyset,\Delta}$ denotes the entry in the row indexed by $\emptyset$ and column indexed by Δ.

Examples of mixed Φ-Eulerian numbers

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Example (Type A_8)

For Lie type A_8 and $(c_1, \ldots, c_8) = (1, 0, 2, 3, 0, 0, 1, 1)$. Then from our formula

$$\begin{aligned} A^{\Phi}_{c_1,\ldots,c_8} &= 8! \left[(m^K_{1,J})(m^K_{3,J})^2(m^K_{4,J})^3(m^K_{7,J})(m^K_{8,J})\right]_{\emptyset,\{1,2,3,4,5,6,7,8\}} \\ &= 8! \cdot \frac{41}{70} \\ &= 23616. \end{aligned}$$

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Example (Type E_6)

For Lie type E_6 and $(c_1, \ldots, c_6) = (0, 1, 0, 2, 3, 0)$. Then from our formula

$$\begin{aligned} A^{\Phi}_{c_1,\ldots,c_6} &= \frac{|W_{E_6}|}{\det(C_{E_6})} \left[(m^K_{2,J})(m^K_{4,J})^2(m^K_{5,J})^3 \right]_{\emptyset,\{1,2,3,4,5,6\}} \\ &= 2^7 \cdot 3^3 \cdot 5 \cdot \frac{81}{40} \\ &= 34992. \end{aligned}$$

Thank you

Acknowledgment
This work was conducted in the **PACE** (PKU Algebraic Combinatorics Experience) at **Peking University**, we thank Yibo Gao who organized this wonderful REU.

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