Square root Crystals for G_n.

Tianyi Yu (UQÀM)

joint work with

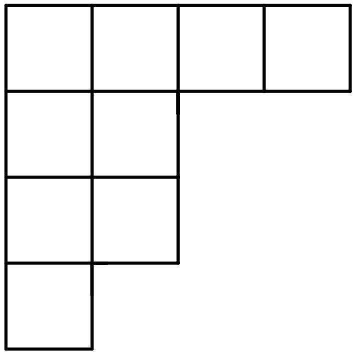
Eric Marberg, Kam-Hung Tong (HKUST)

Outline

- Expand $\left\{ \begin{matrix} s_{\nu/\mu} \\ s_\nu \cdot s_\mu \\ F_w \end{matrix} \right\}$ into s_λ via normal crystals. (Classical Story)

- Expand $\left\{ \begin{matrix} G_{\nu/\mu} \\ G_\nu \cdot G_\mu \\ G_w \end{matrix} \right\}$ into G_λ via normal $\sqrt{\ }$-crystals. (new!)

SSYT



Partition 42210

$\le$

$<$

1	1	2	4
2	3		
4	5		
5			

Semi-Standard Young Tableau.
of shape 42210

SSYT of shape 210.

1	1
2	

1	2
2	

1	1
3	

1	3
3	

$$x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 \quad x_1 x_3^2$$

2	2
3	

2	3
3	

1	2
3	

1	3
2	

$$+ x_2^2 x_3 + x_2 x_3^2 + x_1 x_2 x_3 + x_1 x_2 x_3$$

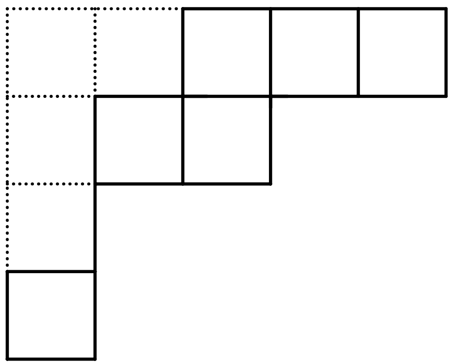
$$= S_{210}(x_1, x_2, x_3).$$

Schur Expansions

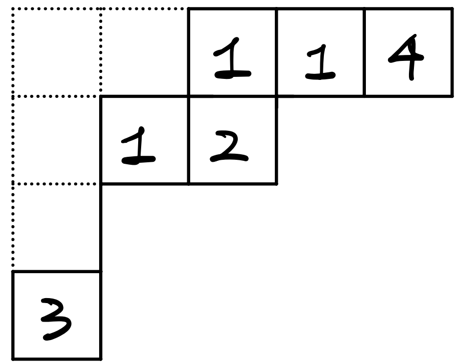
Problem : Expand ? ? ? into the Schur basis.

Skew Schur polynomial $s_{\lambda/\mu}$

Shape of $5331/2110$



SSYT of shape $5331/2110$



$$s_{\lambda/\mu} = \sum_{T} x^{wt(T)}$$ where T is a SSYT of shape λ/μ.

$$s_{321/110} = s_{211} + s_{220} + s_{310}$$

Schur Expansions

Problem : Expand ? ? ? into the Schur basis.

Products of Schurs $s_\nu \times s_\mu$

- $s_{210} \times s_{210} = s_{420} + s_{411} + s_{330} + 2s_{321} + s_{222}$

- Decompose tensor products of finite dimensional representations of GL_n.

- Give structure constants in the cohomology of Grassmannians, describing how Schubert classes intersect.

Schur Expansions

Problem : Expand ? ? ? into the Schur basis.

Stanley symmetric polynomial F_w

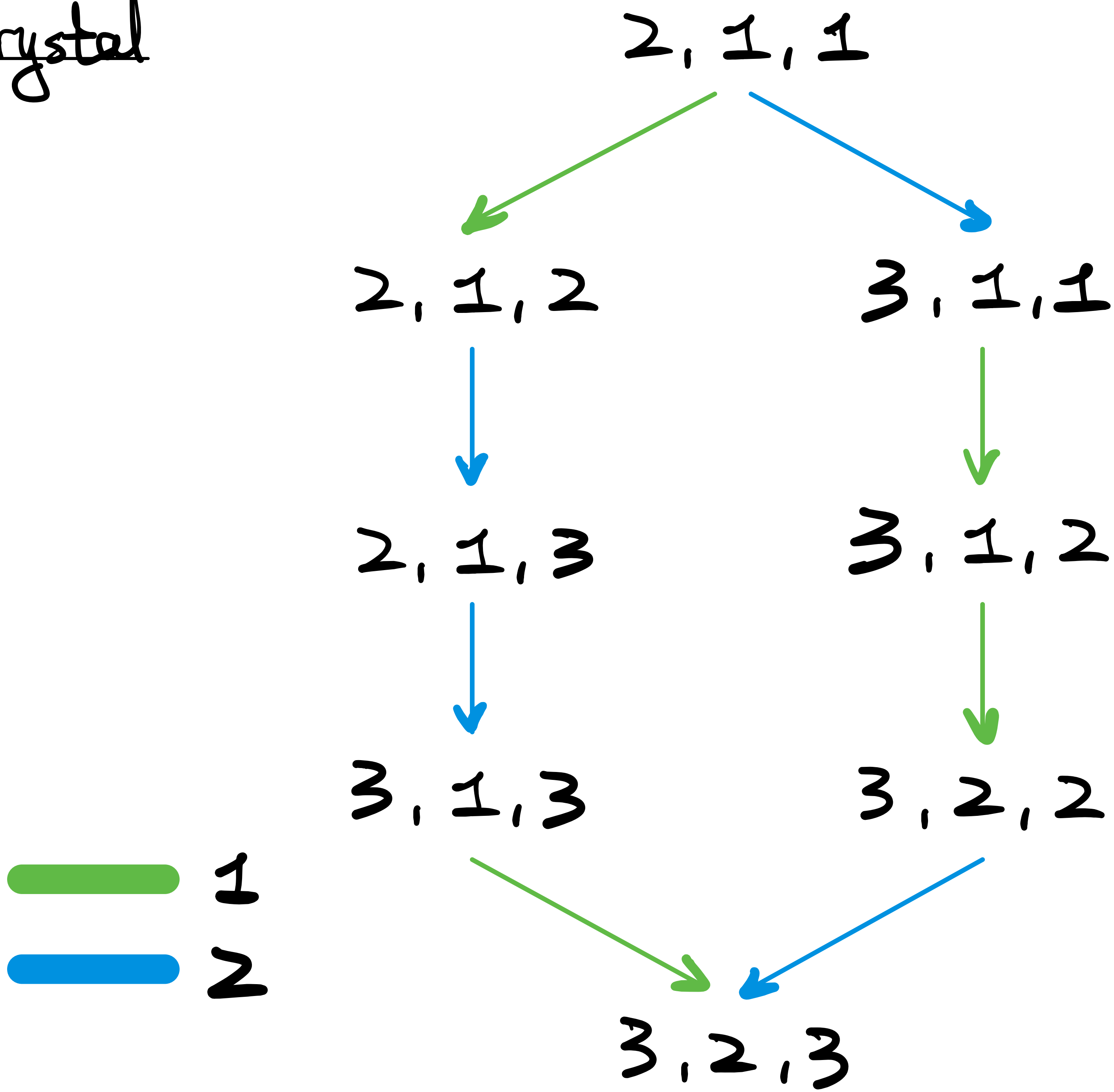
Decreasing Factorization of reduced words of permutations:

- $w = 321$: $(s_2 s_1)(\ )(s_2)$, $(s_2)(s_1)(s_2)$, $(s_1)(s_2 s_1)(\ )$ ...
  $x_1^2 x_3$ $\quad$ $x_1 x_2 x_3$ $\quad$ $x_1 x_2^2$

- $F_{321} = S_{210}$, $\quad F_{21543} = S_{310} + S_{220} + S_{211}$, $\quad F_{31524} = S_{220} + S_{310}$

- Decompose the generalized Specht module of a Rothe diagram into the irreducibles.

Crystal



Notations and terms

- $f_i(A) = B$ if $A \xrightarrow{i} B$

- i-string :

$$A_0 \xrightarrow{i} A_1 \cdots A_3 \xrightarrow{i} A_4$$

- Analogy : Seattle Light Rail

Lynnwood $\xrightarrow{1}$ Mountlake $\xrightarrow{1}$ $\cdots$ $\xrightarrow{1}$ Federal Way

$$f_1(\text{U district}) = \ ?$$

$$\begin{tikzcd}
 & 2,1,1 \arrow[dl] \arrow[dr] & \\
2,1,2 \arrow[d] & & 3,1,1 \arrow[d] \\
2,1,3 \arrow[d] & & 3,1,2 \arrow[d] \\
3,1,3 \arrow[dr] & & 3,2,2 \arrow[dl] \\
 & 3,2,3 &
\end{tikzcd}$$

Single box crystal

$n=3$:

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}$$

Generating Function : $S_{100} = x_1 + x_2 + x_3$

$n=4$:

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \boxed{4}$$

Tensor product of crystal

C_1, C_2 are two crystals.

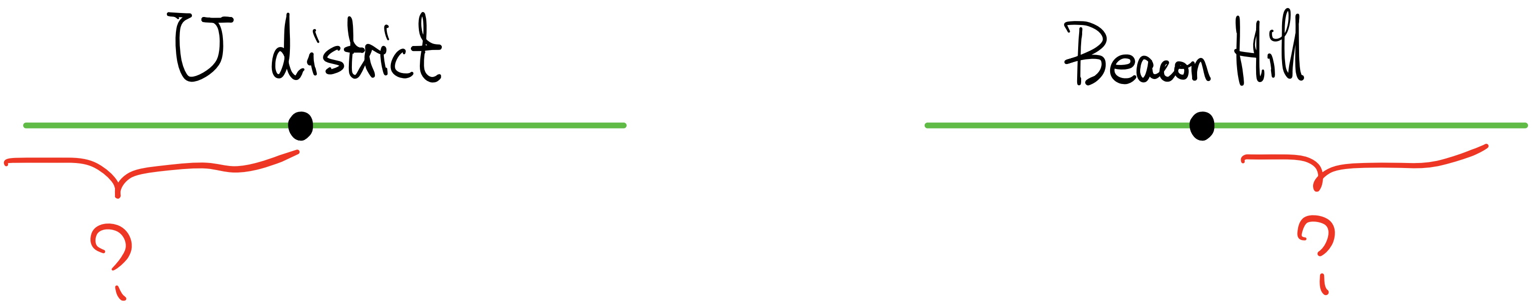
$C_1 \otimes C_2$ have vertices (v_1, v_2) where v_1 is a vertex in C_1 and v_2 is a vertex in C_2

$$f_i(v_1, v_2) = \begin{cases} (f_i(v_1), v_2) & \text{if } a \geq b \\ (v_1, f_i(v_2)) & \text{if } a < b \end{cases}$$

Tensor product example

$$f_i(v_1, v_2) = \begin{cases} (f_i(v_1),\ v_2) & \text{if } a \geq b \\ (v_1,\ f_i(v_2)) & \text{if } a < b \end{cases}$$

$$f_1(\text{U district}, \text{Beacon Hill}) = (\ ?\ ,\ ?\ )$$



Tensor product example

$$f_i(v_1, v_2) = \begin{cases} (f_i(v_1), v_2) & \text{if } a \geq b \\ (v_1, f_i(v_2)) & \text{if } a < b \end{cases}$$

f_1(U district, Beacon Hill) = (U district, Mount Baker)

Crystal on words

A word is a sequence of numbers in $\{1, 2, \dots, n\}$.

$n=3$, length $=2$. we have words:

1, 1	2, 1	3, 1
1, 2	2, 2	3, 2
1, 3	2, 3	3, 3

Crystal on words

We may view it as tensor product of "single box crystal".

$$\begin{tikzcd}
1,1 \arrow[d] & 2,1 \arrow[r] & 3,1 \arrow[d] \\
1,2 \arrow[r] \arrow[d] & 2,2 \arrow[d] & 3,2 \\
1,3 \arrow[r] & 2,3 \arrow[r] & 3,3
\end{tikzcd}$$

Normal Crystal

Decompose it into connected components.

Each component is called a normal crystal.

$$\begin{tikzcd}
1,1 \arrow[d] & 2,1 \arrow[r] & 3,1 \arrow[d] \\
1,2 \arrow[r] \arrow[d] & 2,2 \arrow[d] & 3,2 \\
1,3 \arrow[r] & 2,3 \arrow[r] & 3,3
\end{tikzcd}$$

Summary

- Define one box crystal

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}$$

- Tensor one box crystals together

$$\begin{tikzcd}
1,1 \arrow[d] & 2,1 \arrow[r] & 3,1 \arrow[d] \\
1,2 \arrow[r] \arrow[d] & 2,2 \arrow[d] & 3,2 \\
1,3 \arrow[r] & 2,3 \arrow[r] & 3,3
\end{tikzcd}$$

Crystal on words.

- Break into connected components.

Properties

"Lynnwood"

- Unique source, called "highest-weight element". Its wt is a partition λ.

- Generating function of wt is just s_λ.

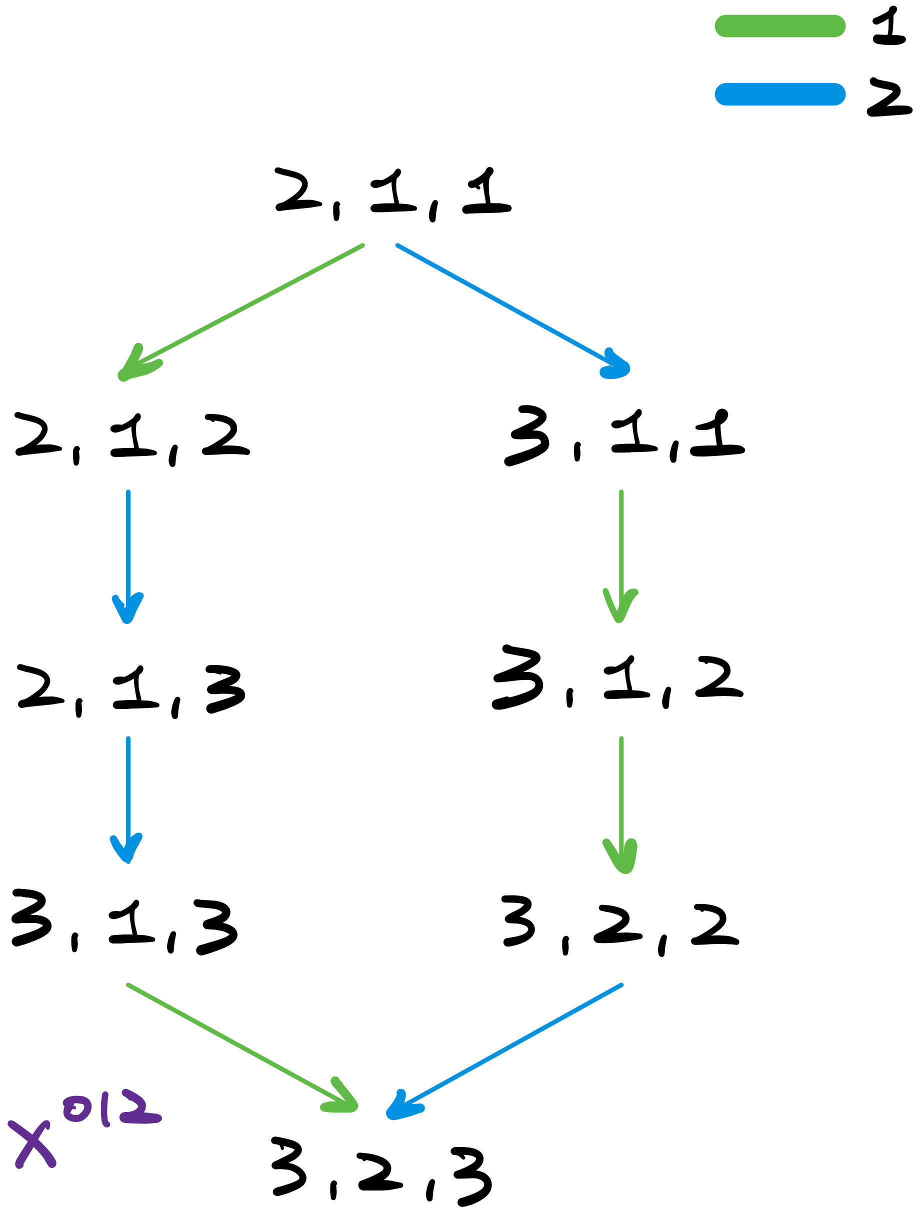
$$
\begin{tikzcd}
& \boxed{2,1,1} \arrow[dl, "1"'] \arrow[dr, "2"] & \\
2,1,2 \arrow[d, "2"] & & 3,1,1 \arrow[d, "1"] \\
2,1,3 \arrow[d, "2"] & & 3,1,2 \arrow[d, "1"] \\
3,1,3 \arrow[dr, "1"'] & & 3,2,2 \arrow[dl, "2"] \\
& 3,2,3 &
\end{tikzcd}
$$

Properties

- Unique source, called "highest-weight element". Its wt is a partition λ.

- Generating function of wt is just s_λ.

$$s_{210} = x^{210} + x^{120} + x^{201} + x^{111} + x^{111} + x^{102} + x^{021} + x^{012}$$



Lattice Word

- A word is a lattice word if you have seen weakly more i than $i+1$ at any time when you read from right to left.

$3, 1, 2, 2, 1, 1$ ✓ — Lattice Word!

$2, 1, 2, 3, 1, 1$! — Not Lattice!

Fact: Lattice words are the highest-weight elements.

Tableaux crystal

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad\qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

$$\downarrow \qquad\qquad\qquad\qquad \uparrow$$

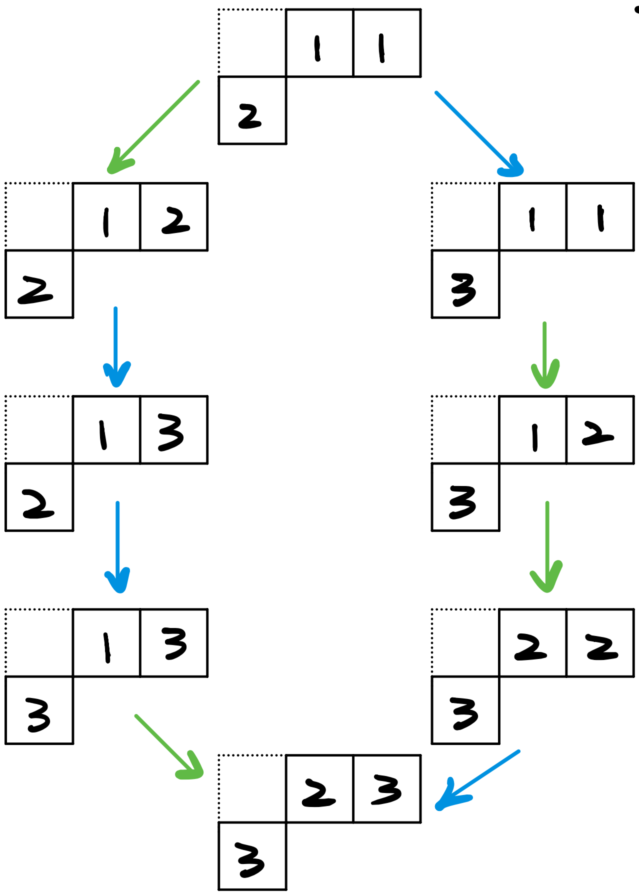
$$4,2,1,4,1,3 \xrightarrow{\;1\;} 4,2,1,4,2,3$$

(left to right ;
bottom to top.)

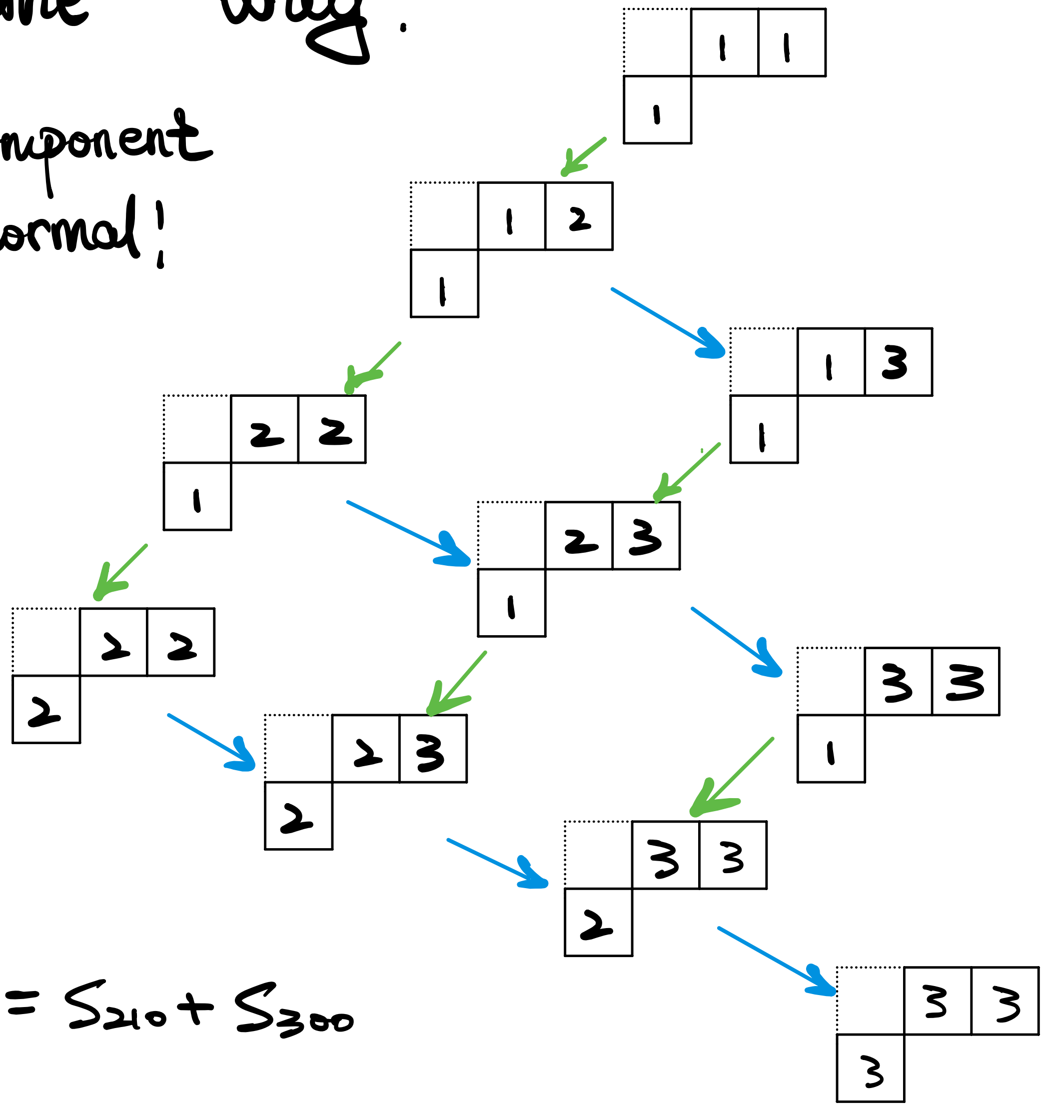
- Gives a normal crystal structure on SSYTs.

Skew Schur to Schur

We can define a crystal structure on skew SSYTs in the same way.

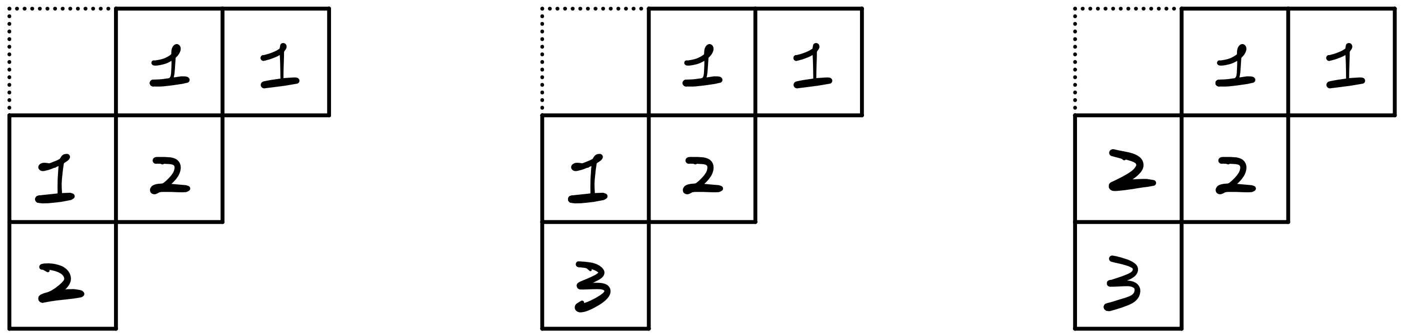


Each component is normal!



$$s_{310/100} = s_{210} + s_{300}$$

Skew Schur to Schur



Reading Words : 21211   31211   32211

lattice

$$s_{321/100} = s_{320} + s_{311} + s_{221}.$$

Product of Schurs to Schur

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\ ,\ \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \xrightarrow{\ 2\ } \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\ ,\ \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

213 322 313 322

Pairs with lattice reading words:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\ ,\ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\ ,\ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\ ,\ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$$

21121 31121 21321

$$S_{210} \cdot S_{110} = S_{320} + S_{311} + S_{221}.$$

Stanley Symmetric Polynomial to Schur

Thm [Edelman - Greene 1987 ;
(Crystal approach) Morse - Schilling 2015]

$$
\begin{tikzcd}[column sep=tiny]
 & (s_3s_1)(s_4s_2)(\ ) \arrow[d] & \\
 & (s_3s_1)(s_4)(s_2) \arrow[dl] \arrow[dr] & \\
(s_3)(s_4s_1)(s_2) \arrow[dr] & & (s_3s_1)(\ )(s_4s_2) \arrow[dl] \\
 & (s_3)(s_1)(s_4s_2) \arrow[d] & \\
 & (\ )(s_3s_1)(s_4s_2) &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
(s_3s_1)(s_2)(s_4) \arrow[d] \\
(s_1)(s_3s_2)(s_4) \arrow[d] \\
(s_1)(s_3)(s_4s_2)
\end{tikzcd}
$$

$$F_{s_3s_1s_4s_2}(x_1, x_2, x_3) = s_{220} + s_{221}$$

Symmetric Grothendieck Polynomial G_λ

$$S_{100} = x_1 + x_2 + x_3$$

$$G_{100} = x_1 + x_2 + x_3 + x_1x_2 + x_1x_3 + x_2x_3 + x_1x_2x_3.$$

- Also form a basis of symmetric polynomials.
- Not homogeneous!
- Form a basis for the K-theory of the Grassmannians.

SVT

Definition [Buch]

Each cell filled by a non-empty subset of $[n]$.

Weakly increasing $\rightarrow$

Strictly increasing $\downarrow$

1	12	234
2	4	
34		

$\longrightarrow \quad wt(T) = (2,3,2,3)$

Thm [Buch, 2002] $G_\lambda = \sum_{T \in SVT(\lambda)} x^{wt(T)}$

G_λ - expansion Problems

Problem : Expand ? ? ? into the G_λ - basis.

- $G_{\nu/\mu}$: Sum of "Skew SVT" [Buch 2002]
- $G_\mu \cdot G_\nu$: a product of two symmetric Grothendieck polynomials. [Buch 2002]
- G_w : K-theoretic analogue of F_w. [Buch-Kresch-Shimozono-Tamvakis-Yong 2008]

Classical Normal Crystal

- Define one box crystal

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}$$

- Tensor one box crystals together

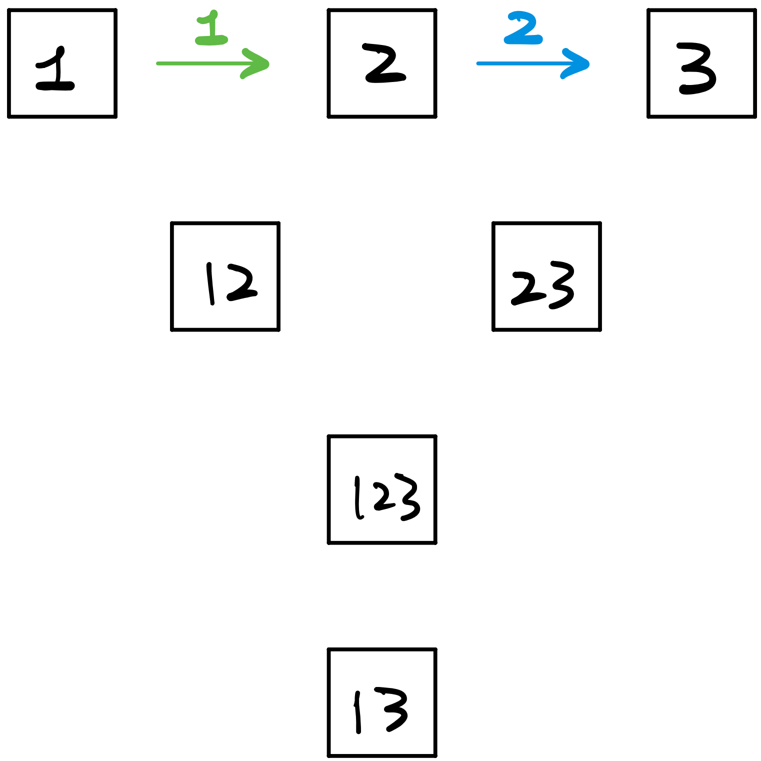
$$\begin{tikzcd}
1,1 \arrow[d] & 2,1 \arrow[r] & 3,1 \arrow[d] \\
1,2 \arrow[r] \arrow[d] & 2,2 \arrow[d] & 3,2 \\
1,3 \arrow[r] & 2,3 \arrow[r] & 3,3
\end{tikzcd}$$

Crystal on words.

- Break into connected components.

~~Classical~~ New Normal Crystal

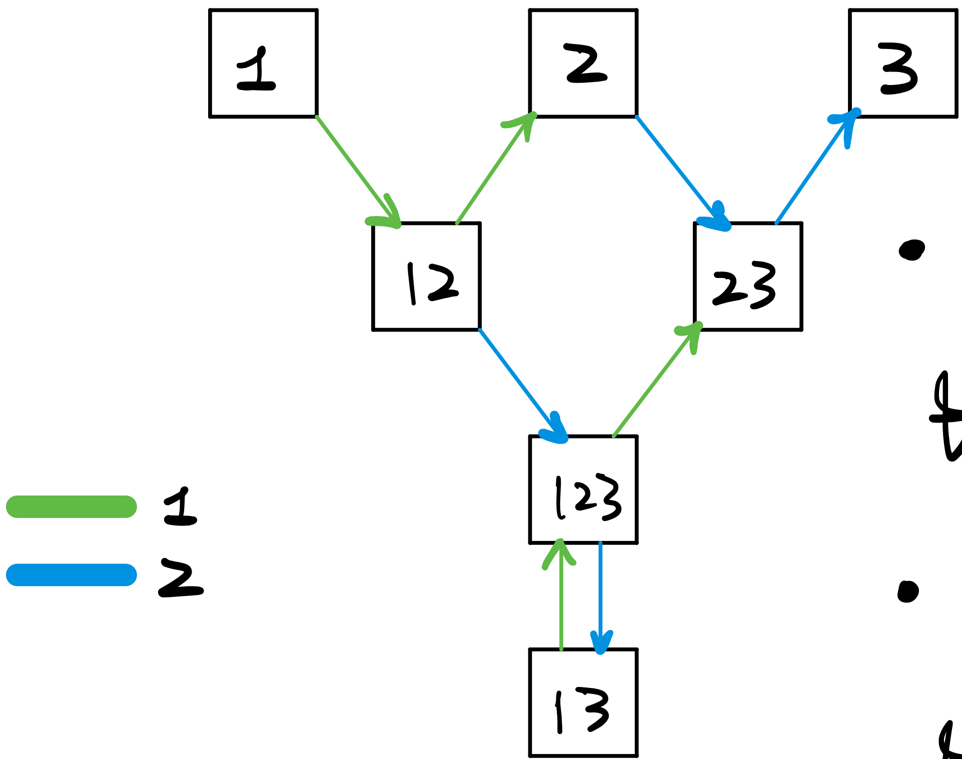
- Define one box crystal



Our new SVTs!

New one box Crystal

- Define one box crystal [Y, 2021]

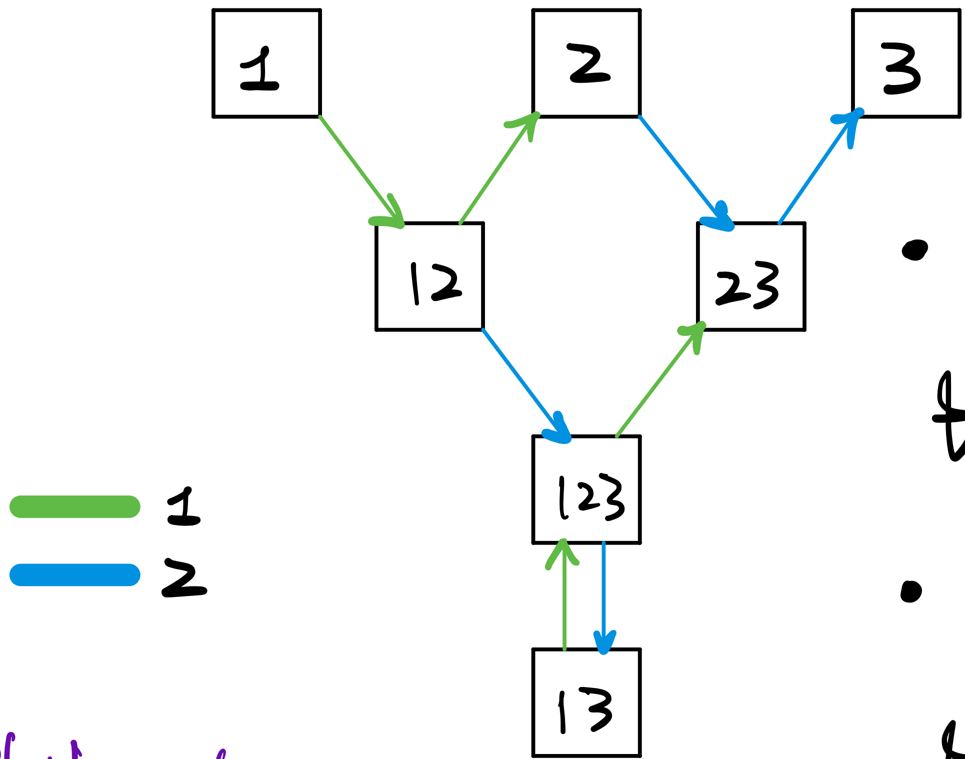


Rule :

- f_i adds $i+1$ if the box has only i
- f_i removes i if the box has i & $i+1$.

New one box Crystal

- Define one box crystal [Y. 2021]



Rule:

- f_i adds $i+1$ if the box has only i
- f_i removes i if the box has i & $i+1$.

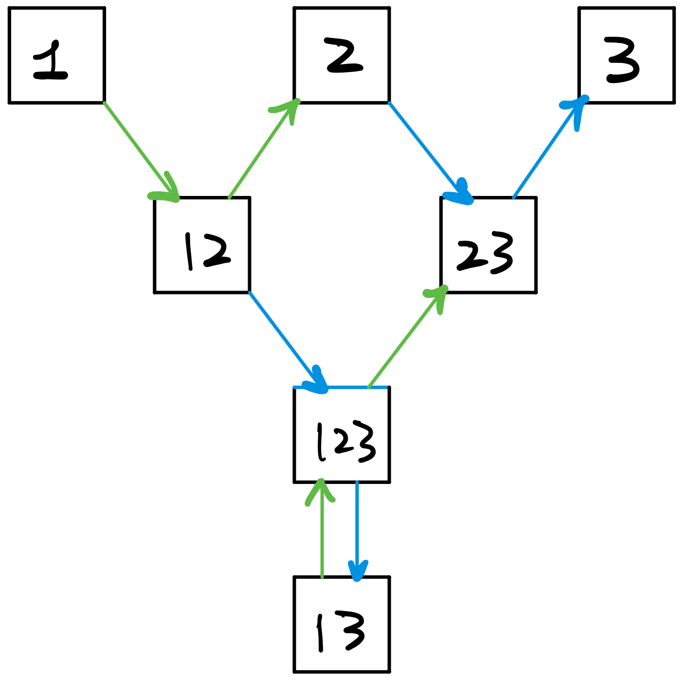
Philosophy

P.h.D. life : Student → Student & researcher → researcher

~~Classical~~ New Normal Crystal

- Define one box crystal

Done!



Next!

- Tensor one box crystals together
- Break into connected components.

Tensor product of crystal

C_1, C_2 are two crystals.

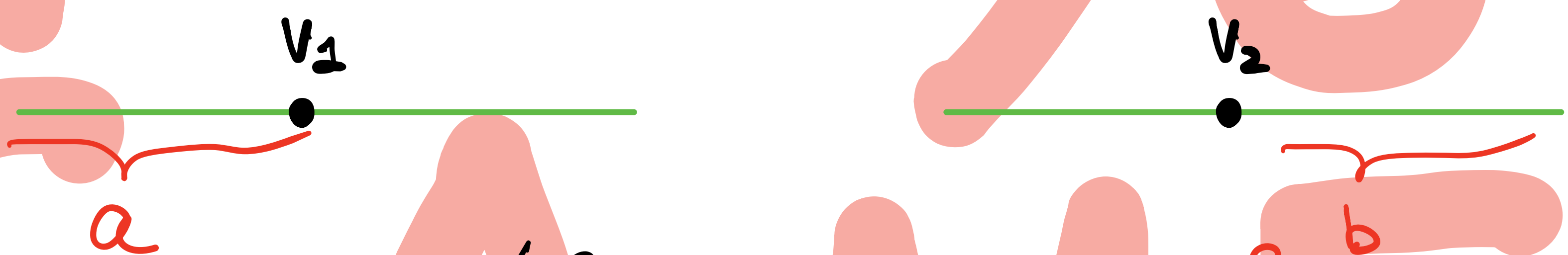
$C_1 \otimes C_2$ have vertices (v_1, v_2) where v_1 is a vertex in C_1 and v_2 is a vertex in C_2

$$f_i(v_1, v_2) = \begin{cases} (f_i(v_1), v_2) & \text{if } a \geq b \\ (v_1, f_i(v_2)) & \text{if } a < b \end{cases}$$

Tensor product of crystal

C_1, C_2 are two crystals.

$C_1 \otimes C_2$ have vertices (v_1, v_2) where v_1 is a vertex in C_1 and v_2 is a vertex in C_2



$$f_i(v_1, v_2) = \begin{cases} (f_i(v_1), v_2) & \text{if } a \geq b \\ (v_1, f_i(v_2)) & \text{if } a < b \end{cases}$$

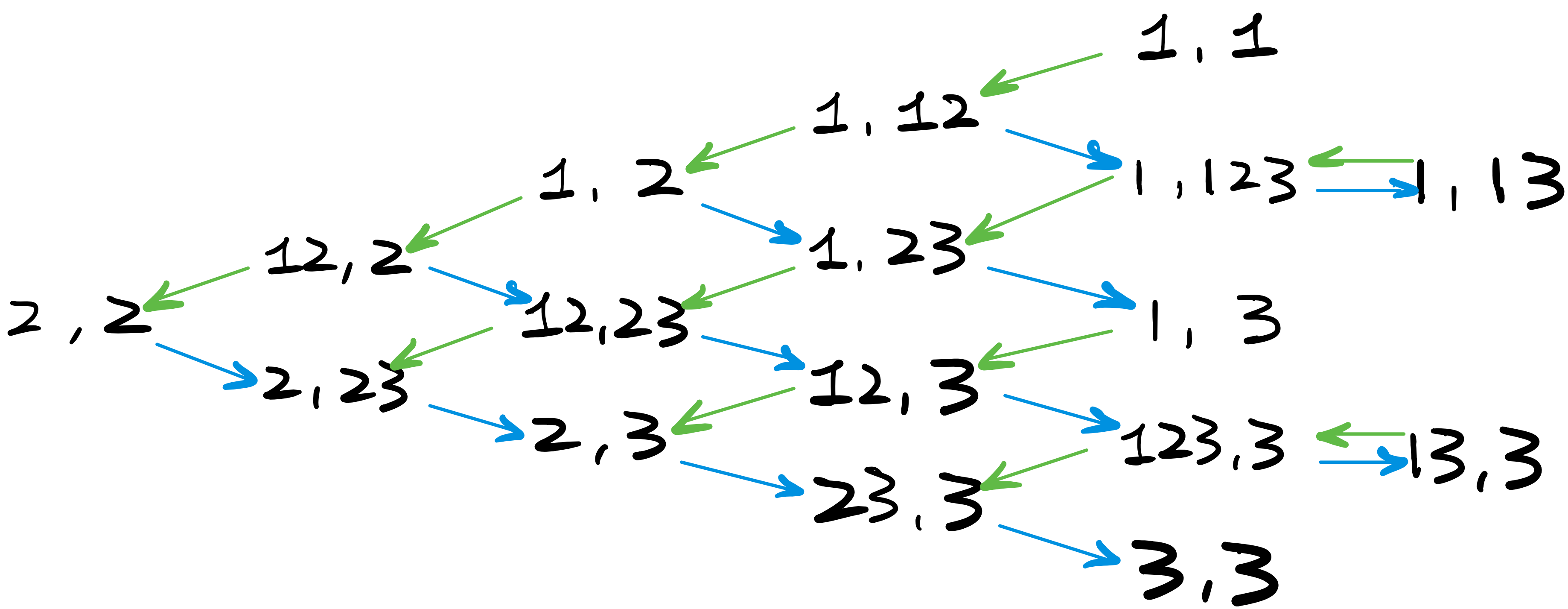
$\sqrt{\ }$-crystal on set-valued words

- Set-valued words:

  A sequence of subsets of $\{1, 2, \ldots, n\}$.

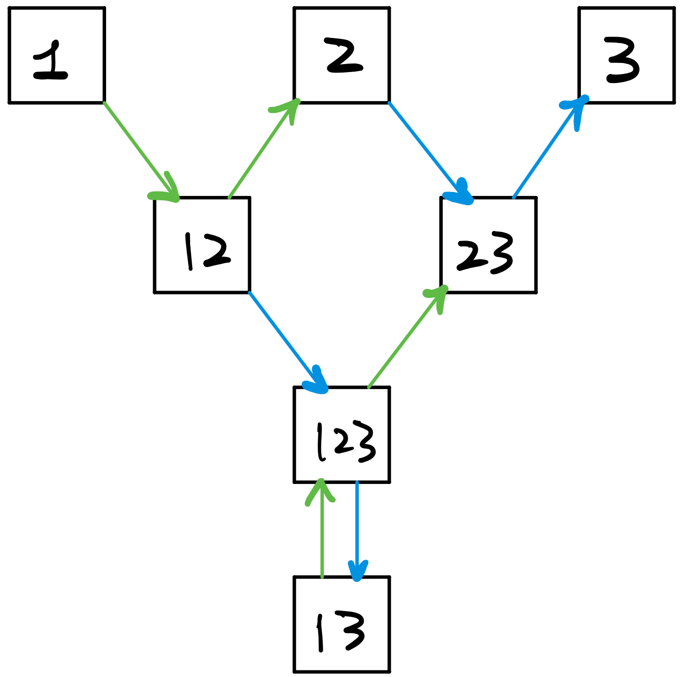
  e.g. $12, 13, 1, 123$.

View it as a tensor of one-box-$\sqrt{\ }$-crystal.



~~Classical~~ New Normal $\sqrt{\ }$- Crystal

- Define one box crystal



Same as before!

- Tensor one box crystals together
- Break into connected components.

Highest weight elements

Highest-weight elements: Set-valued words that are lattice words if you read each set in increasing order.

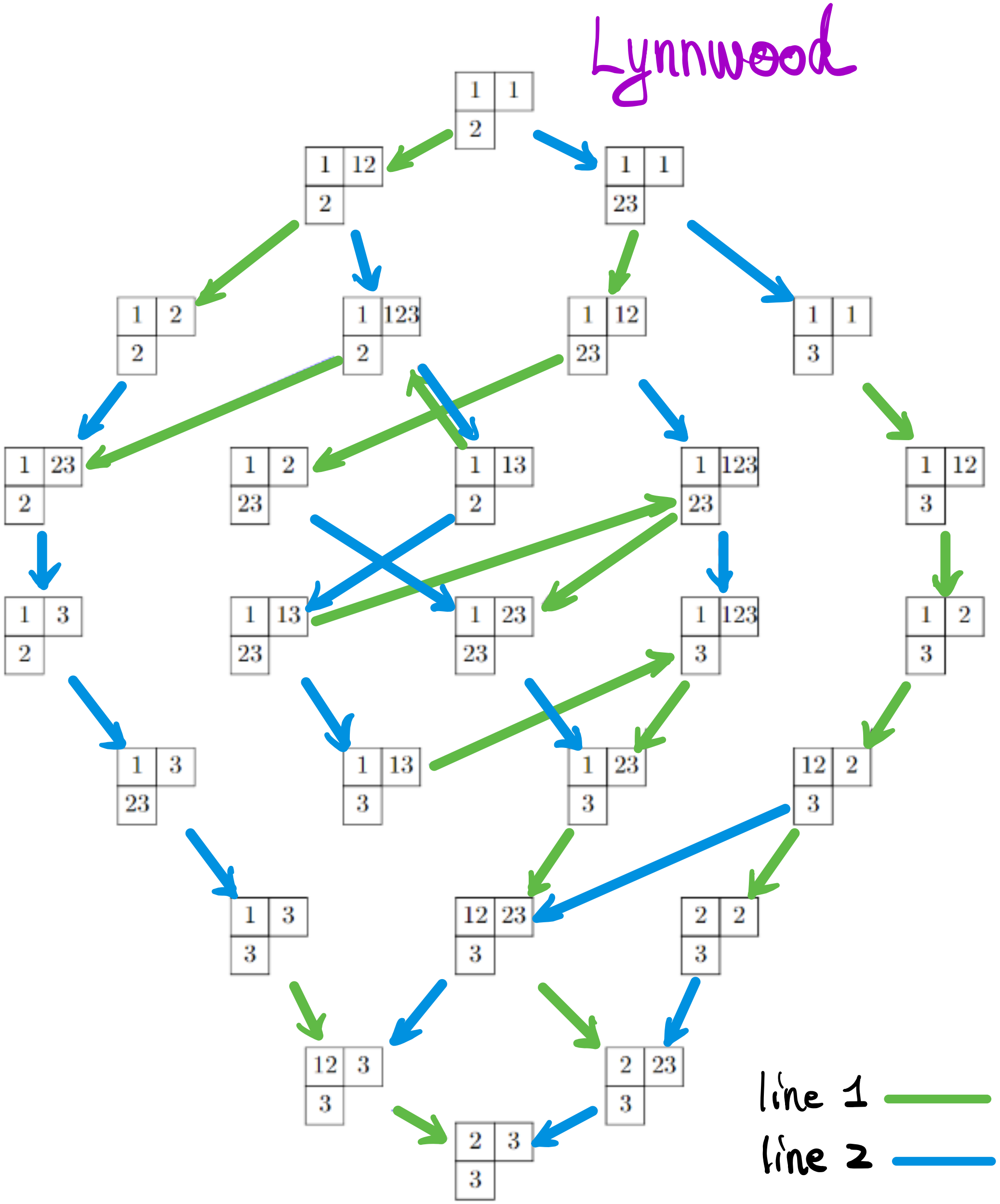
2, 13, 12, 123, 2, 1, 1

2131212321 1

lattice!

Highest weight!
(Source in the graph).

Normal $\mathfrak{J}$-crystal on SVT [Y, 2021]



12	24	4
34		

Reading Word

34, 12, 24, 4

Read the sets in the same order as before.

Dreams

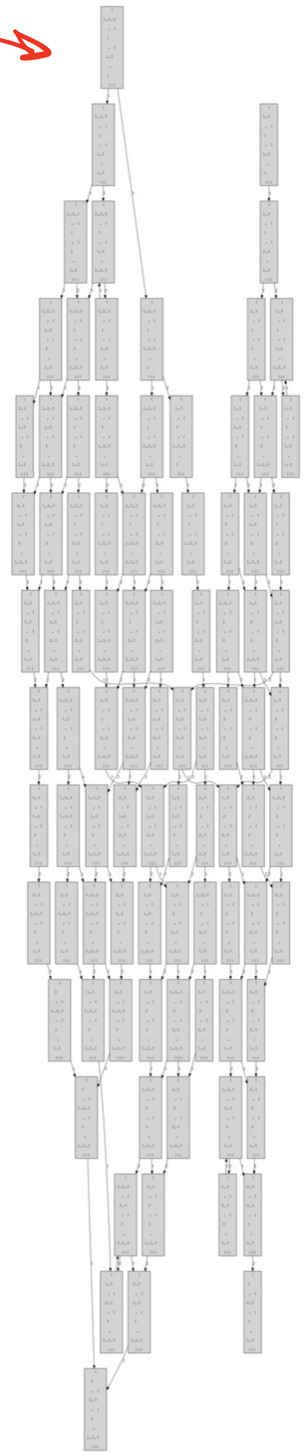
[Marberg – Tong , 2024]

- Has a unique highest-weight element
- Generating function is a G_λ.

123, 1, 12, 1

13, 1, 12, 1

Two highest weight elements



Generating function is

$G_{421} + G_{411}$

Normal $\sqrt{}$-crystal properties

~~Dreams~~ Nightmares

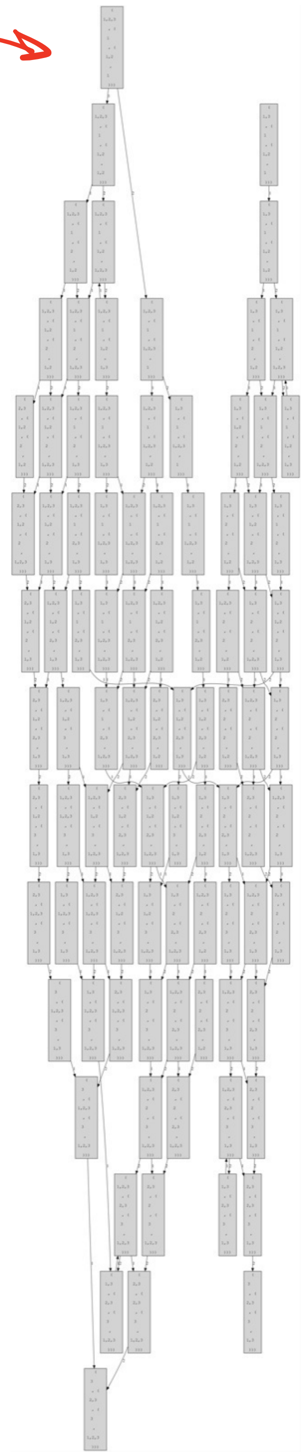
[Marberg – Tong , 2024]

- ~~Has a unique highest-weight element~~
- ~~Generating function is a G_λ.~~

123, 1, 12, 1

wt = (4,2,1)

Two highest weight elements



13, 1, 12, 1

wt = (4,1,1)

Generating function is

$G_{421} + G_{411}$

Normal $\sqrt{\ }$-crystal properties

~~Dreams~~ ~~Nightmares~~ Conjecture

[Marberg – Tong , 2024]

- ~~Has a unique highest-weight element~~
- Generating function is a ~~G_λ~~. sum of G_λ, where λ ranges over wt of all highest-weight elements.

Normal $\sqrt{}$-crystal properties

~~Dreams~~ ~~Nightmares~~ ~~Conjecture~~ THEOREM

[Marberg – Tong – Y, 202~~4~~5]

- ~~Has a unique highest-weight element~~
- Generating function is a ~~G_λ~~. sum of G_λ, where λ ranges over wt of all highest-weight elements.

Expansion of $G_{\lambda/\mu}$

	1	1
1	2	
2		

	1	1
1	2	
3		

	1	1
2	2	
3		

	1	1
12	2	
3		

	1	1
1	2	
23		

21211 31211 32211 312211 231211

Lattice

$$G_{321/100} = G_{320} + G_{311} + G_{221} + G_{321} + G_{321}$$

(Recover Buch's formula from our normal $\sqrt{\ }$- crystal perspective)

Expansion of $G_\lambda G_\mu$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$$

21121   31121   21321

$$\begin{array}{|c|c|}\hline 1 & 13 \\ \hline 2 \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 23 \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 13 \\ \hline 2 \\ \cline{1-1}\end{array}\ ,\ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$$

211321   231121   211321

$$G_{210} \cdot G_{110} = G_{320} + G_{311} + G_{221} + 3\, G_{321}$$

(Recover Buch's formula from our normal $\sqrt{\ }$- crystal perspective)

And more ...

- $\sqrt{}$ - crystal on decreasing factorization of Hecke words.

  $\Rightarrow$ recovers G_w to G_λ

[Buch - Kresch - Shimozono - Tamvakis - Yong 2008]

- (new!) $\sqrt{}$ - crystal on set-valued decomposition tableaux.

$\Rightarrow$ Progress toward Cho--Ikeda Conjecture.

Why call it "square root"?

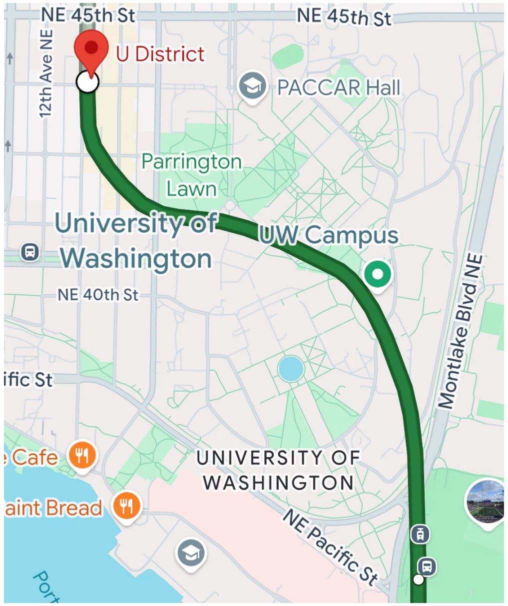
$$
\begin{tikzcd}
1,1 \arrow[dd, "f_1"] & & 1,1 \arrow[d, "\sqrt{f_1}"] \\
 & & 1,1\tfrac{1}{2} \arrow[d, "\sqrt{f_1}"] \\
1,2 \arrow[dd, "f_1"] \arrow[rr, Rightarrow] & & 1,2 \arrow[d, "\sqrt{f_1}"] \\
 & & 1\tfrac{1}{2},2 \arrow[d, "\sqrt{f_1}"] \\
2,2 & & 2 \quad 2
\end{tikzcd}
$$

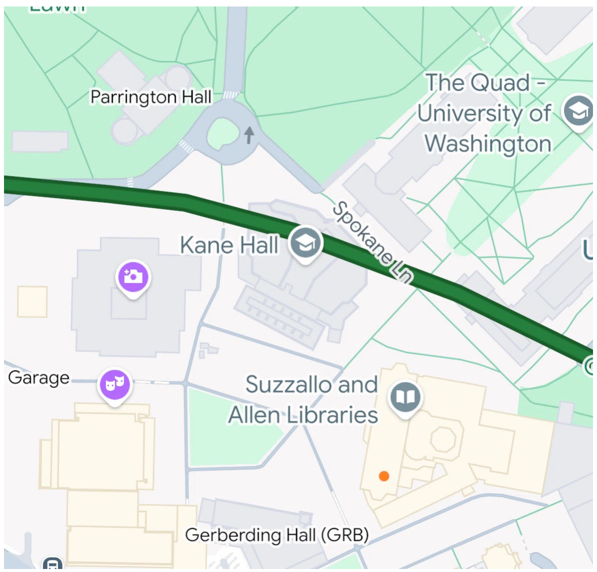
Why call it "square root"?

$$
\begin{tikzcd}
1,1 \arrow[d, "f_1"] \\
1,2 \arrow[d, "f_1"] \\
2,2
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
1,1 \arrow[d, "\sqrt{f_1}"] \\
1,1\tfrac{1}{2} \arrow[d, "\sqrt{f_1}"] \\
1,2 \arrow[d, "\sqrt{f_1}"] \\
1\tfrac{1}{2},2 \arrow[d, "\sqrt{f_1}"] \\
2\ \ 2
\end{tikzcd}
$$

$$
\begin{tikzcd}
{} \arrow[d, "f_1"] \\
\text{U district} \arrow[d, "f_1"] \\
\text{Univ of Wash} \arrow[d, "f_1"] \\
{}
\end{tikzcd}
\quad \Rightarrow \quad ?
$$

Why call it "square root"?





$\downarrow \sqrt{f_i}$

U district

$\downarrow \sqrt{f_i}$

Kane Hall!

$\downarrow \sqrt{f_i}$

Univ of Wash

$\downarrow \sqrt{f_i}$

Thank you!

Link
Kane hall station
Link
Kane
hall
station
Link
Marberg, Tong, and Yu.
Grothendieck positivity for
normal square root crystals.
Adv. Math. (2026)