CHANGING BASES WITH PIPE DREAM COMBINATORICS

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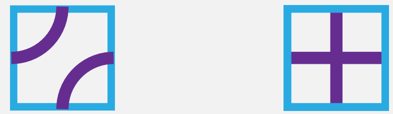
SCHUBERT AND GROTHENDIECK POLYNOMIALS

SCHUBERT POLYNOMIALS

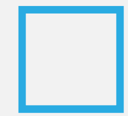
- The **complete flag variety** $\mathrm{GL}(n)/B$ has a special family of subvarieties $\{\mathcal{X}_w : w \in S_n\}$ called **Schubert varieties**.
- Each Schubert variety defines a **Schubert class** $\sigma_w \in H^*(\mathrm{GL}(n)/B) \approx \mathbb{Z}[x_1, \ldots, x_n]/I^{S_n}$.
- Lascoux and Schützenberger (1982) proposed a choice of coset representatives for the Schubert classes and called these polynomials $\mathfrak{S}_w$ **Schubert polynomials**.
- Subsequent work led to a rich combinatorial theory studying these polynomials and their properties.

PIPE DREAMS

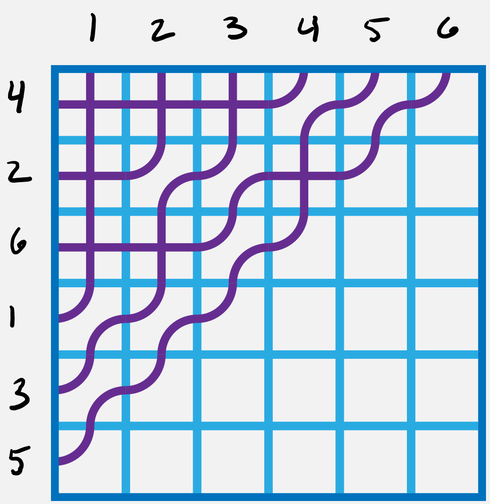
on main antidiagonal



above main antidiagonal



below main antidiagonal

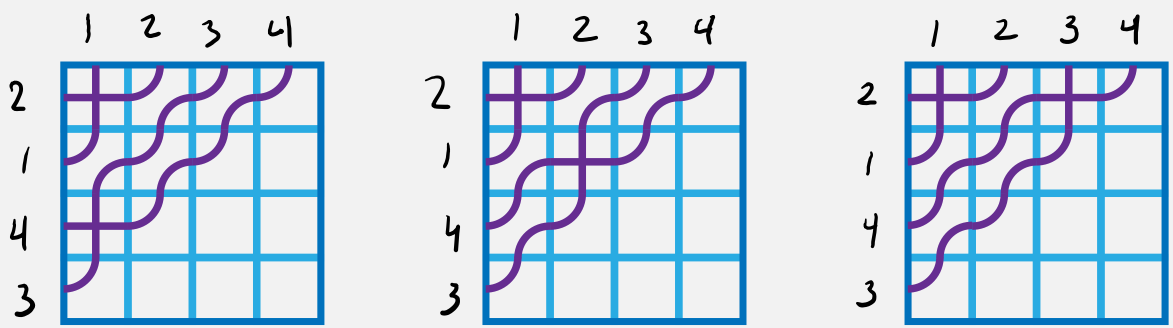


Fill $n \times n$ grid with n pipes that start at the top end at the left

$$wt(P) = x_1^3 x_2^2 x_3^2$$

THEOREM (FOMIN-KIRILLOV 1996, BERGERON-BILLEY 1993, …)

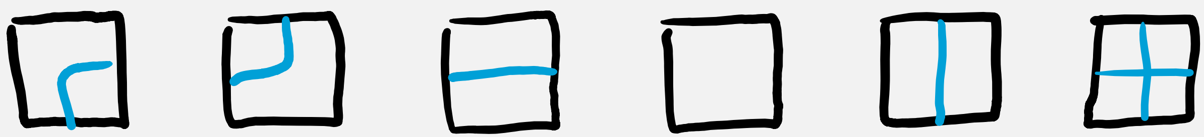
The Schubert polynomial $\mathfrak{S}_w(\boldsymbol{x})$ is a weighted sum over (reduced) pipe dreams for w.



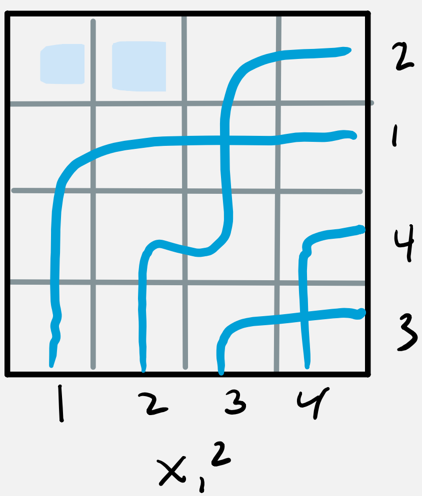
$$\mathfrak{S}_{2143}(x) = x_1 x_3 + x_1 x_2 + x_1^2 .$$

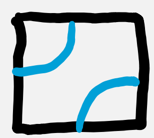
BUMPLESS PIPE DREAMS

A **bumpless pipe dream** (**BPD**) is a tiling of the $n \times n$ grid with



so that pipes start at the bottom of the grid and end at the right.

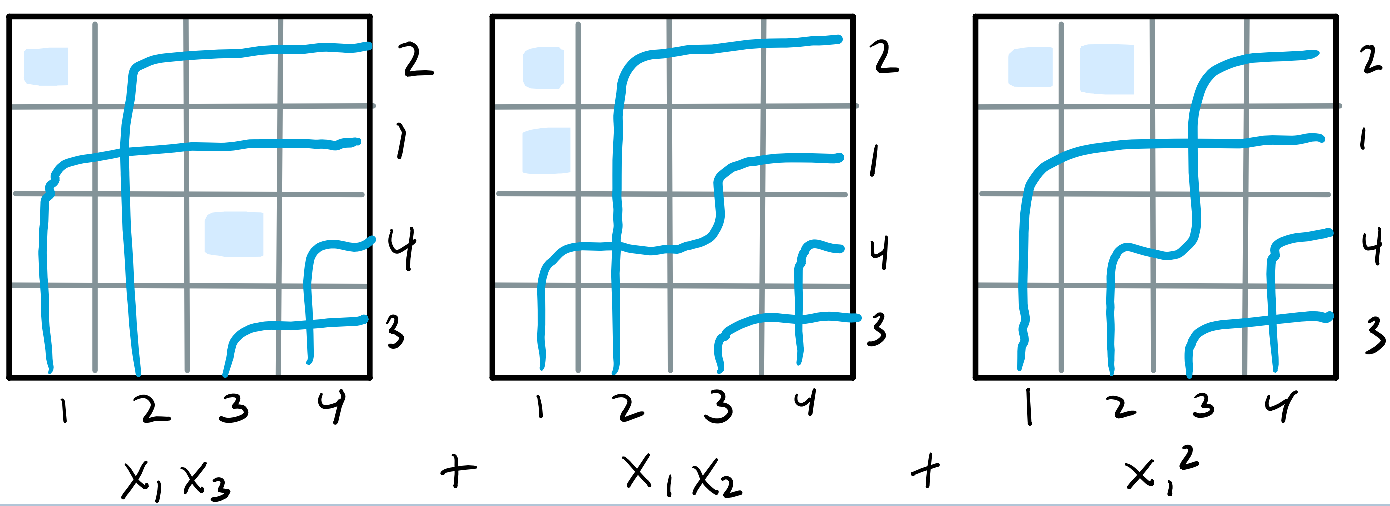




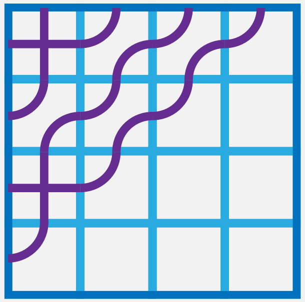
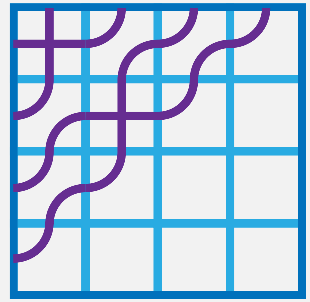
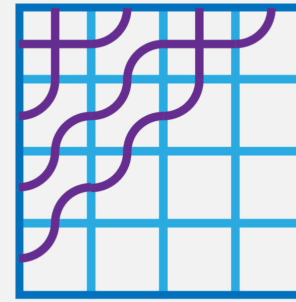
no bumps allowed!

THEOREM (LAM-LEE-SHIMOZONO 2021)

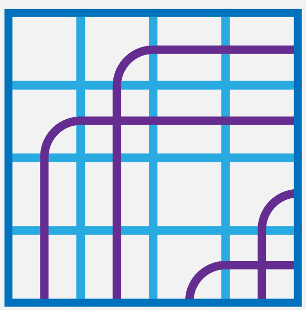
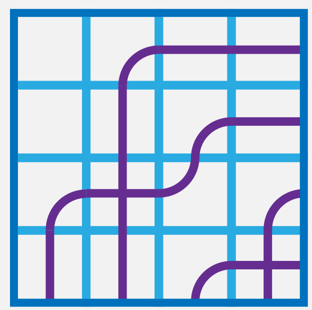
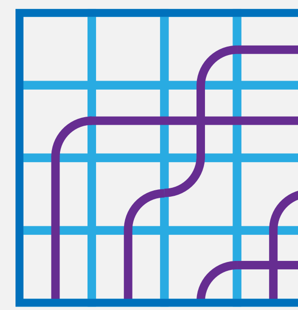
The Schubert polynomial $\mathfrak{S}_w$ is a weighted sum over reduced BPDs for w.



COMPARISON OF FORMULAS

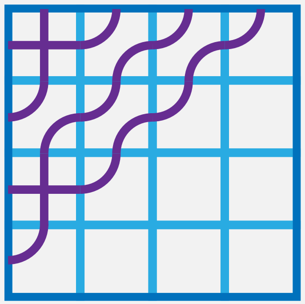
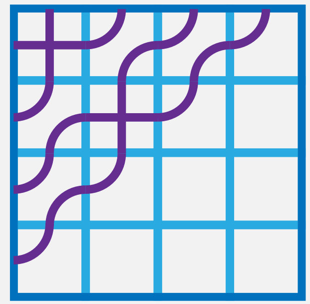
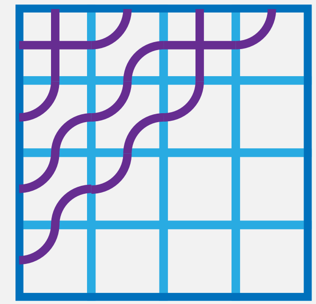




$$\mathfrak{S}_{2143}(x) = x_1 x_3 \quad + \quad x_1 x_2 \quad + \quad x_1^2$$

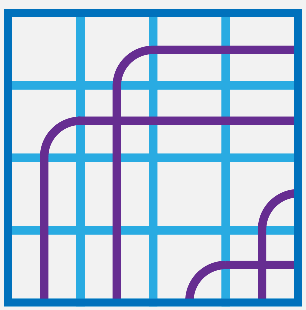
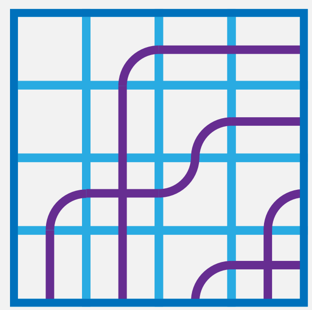
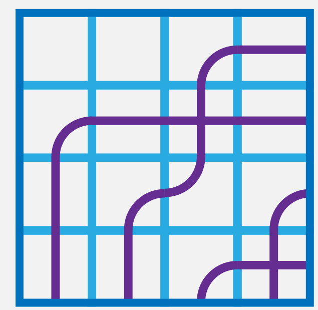




Weight preserving bijection due to Gao – Huang '21.

$$\mathfrak{S}_{2143}(x;y) = (x_1-y_1)(x_3-y_1) \quad + \quad (x_1-y_1)(x_2-y_2) \quad + \quad (x_1-y_1)(x_1-y_3)$$

$$\mathfrak{S}_{2143}(x) = x_1x_3 \quad + \quad x_1x_2 \quad + \quad x_1^2$$

$$\mathfrak{S}_{2143}(x;y) = (x_1-y_1)(x_3-y_3) \quad + \quad (x_1-y_1)(x_2-y_1) \quad + \quad (x_1-y_1)(x_1-y_2)$$

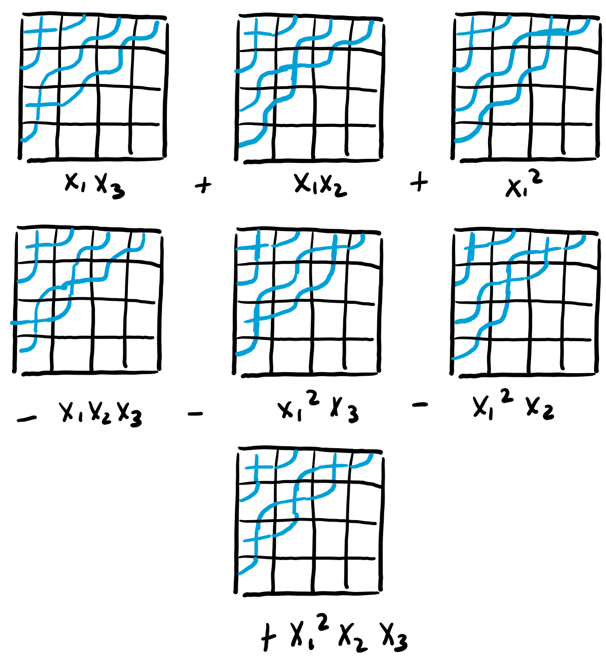
double Schubert polynomial

GROTHENDIECK POLYNOMIALS

- **Grothendieck polynomials** $\mathfrak{G}_w$ are K-theoretic analogues of Schubert polynomials
- Need not be homogeneous
- Lowest degree part of Grothendieck polynomial $\mathfrak{G}_w$ is the Schubert polynomial $\mathfrak{S}_w$

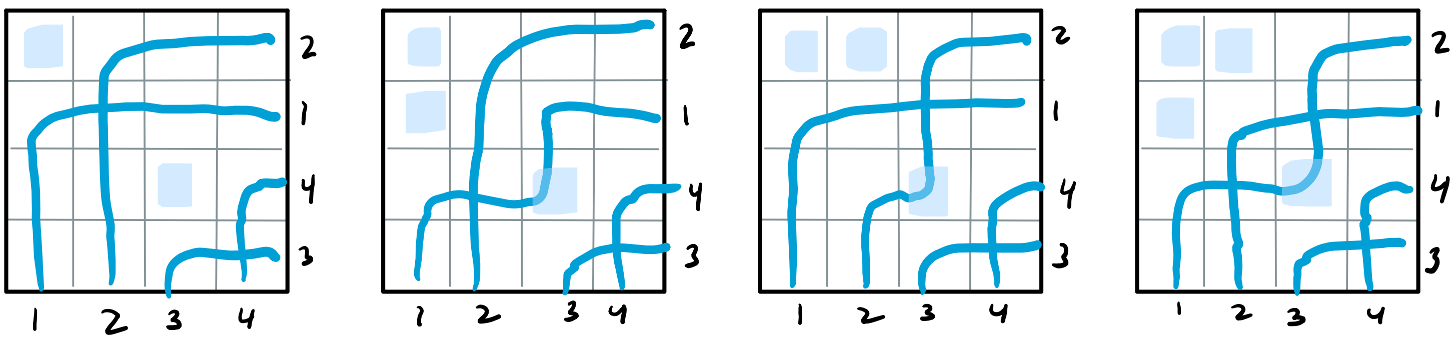
THEOREM (FOMIN-KIRILLOV 1994)

The Grothendieck polynomial $\mathfrak{G}_w$ is a weighted sum over pipe dreams for w.



THEOREM (WEIGANDT 2021*)

The Grothendieck polynomial $\mathfrak{G}_w$ is a weighted sum over BPDs for w.



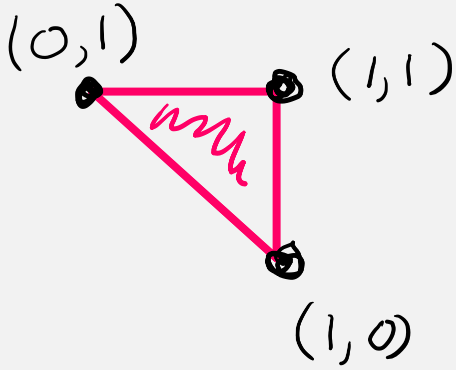
$$x_1 x_3 + x_1 x_2 (1-x_3) + x_1^2 (1-x_3) - x_1^2 x_2 (1-x_3)$$

* Lascoux 2002 ⇝ ASM formula

A FEW QUESTIONS

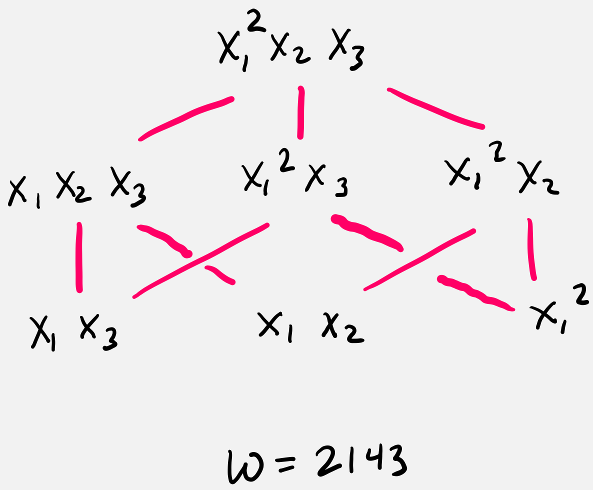
NEWTON POLYTOPES

- Monical, Tokcan, and Yong (2019) conjectured that Schubert and Grothendieck polynomials have **saturated Newton polytopes**
- Single Schubert polynomials (Fink- Mészáros-St. Dizier, 2018)
- Symmetric Grothendieck polynomials (Escobar-Yong, 2017)
- And more…
- $\mathfrak{G}_{132} = x_1 + x_2 - x_1 x_2$



MONOMIAL SUPPORT POSETS

- Mészáros, Setiabrata, and St. Dizier (2022) made a series of conjectures about the posets of monomials which appear in the support of Grothendieck polynomials



CHANGING BASES

- Both Schubert polynomials and Grothendieck polynomials form a basis of $\mathbb{Z}[x_1, x_2, \ldots]$.
- **Strategy:** Let's consider the expansion $\mathfrak{G}_w = \sum_u (-1)^{\ell(u)-\ell(w)} a_{w,u} \mathfrak{S}_u$.

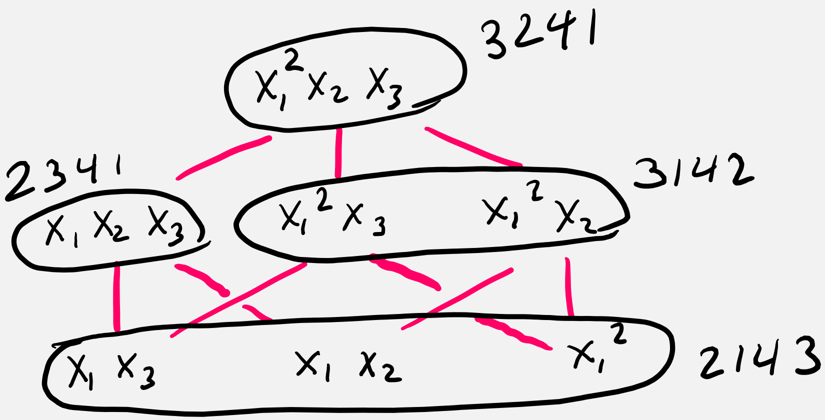
positive integers!

The **Schubert support poset** of $\mathfrak{G}_w$ is $\{u: a_{w,u} \neq 0\}$ ordered by **left weak order**.

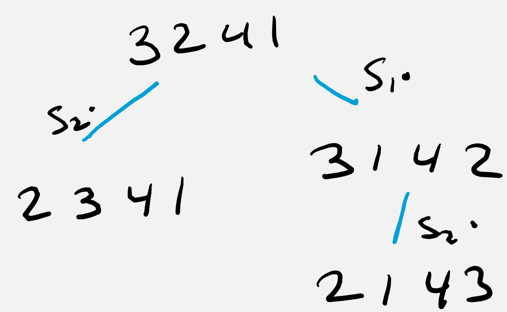
Studying Schubert support poset with Judy Chiang

SCHUBERT SUPPORT POSETS

$w = 2143$



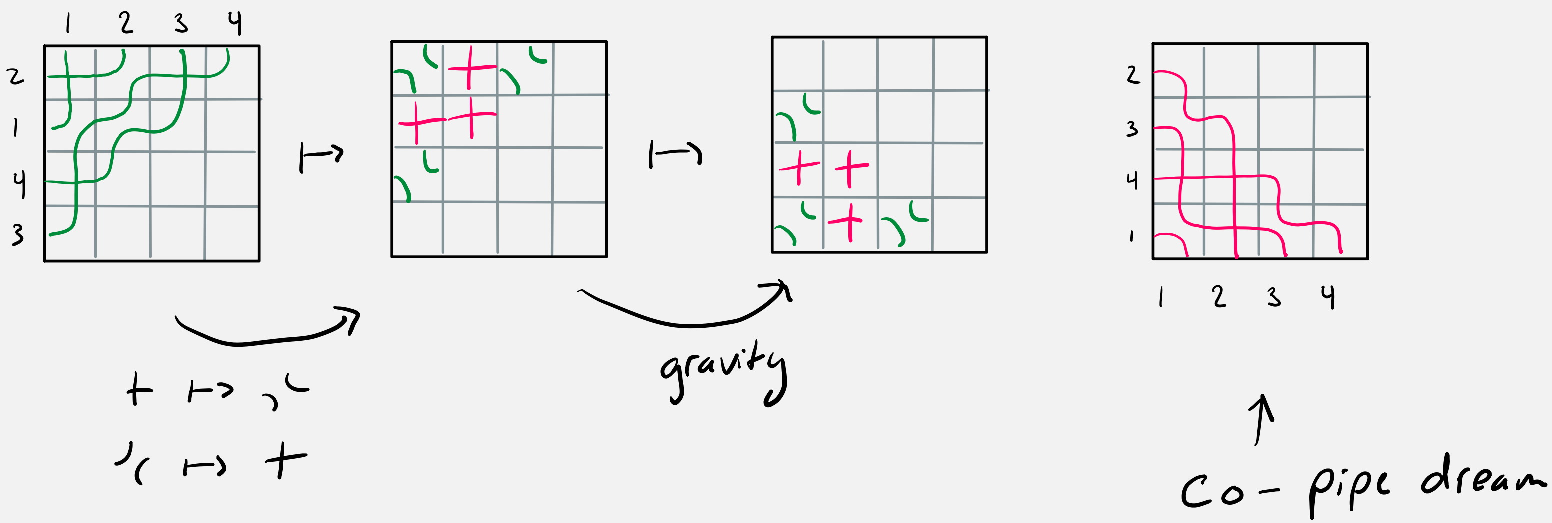
monomial support



schubert support

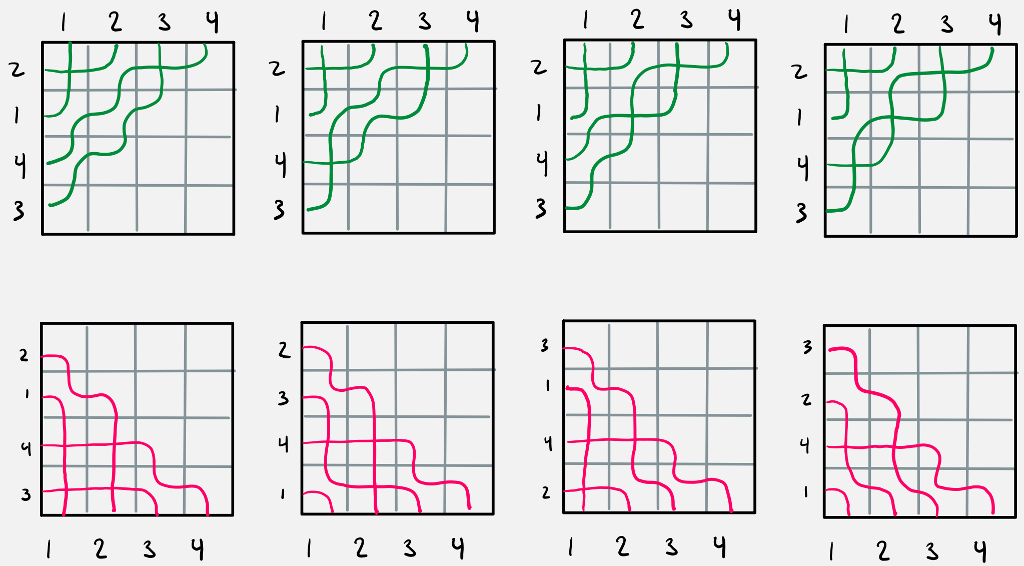
CHANGING BASES

CO-PIPE DREAMS



THEOREM (LENART, 1999)

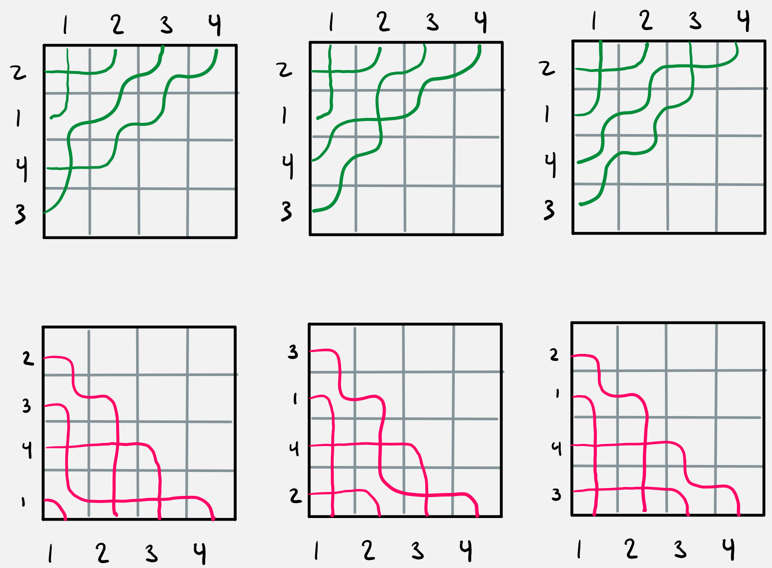
The Grothendieck polynomial is $\mathfrak{G}_w = \sum_v (-1)^{\ell(v)-\ell(w)} a_{w,v}\, \mathfrak{S}_v$, where $a_{w,v}$ counts pipe dreams for w with co-pipe dreams that are reduced and trace out v.



$\mathfrak{G}_{2143} = \mathfrak{S}_{2143} - \mathfrak{S}_{2341} - \mathfrak{S}_{3142} + \mathfrak{S}_{3241}$

THEOREM (LASCOUX, 2004)

The Schubert polynomial is $\mathfrak{S}_w = \sum_v b_{w,v}\mathfrak{G}_v$, where $b_{w,v}$ counts reduced pipe dreams for w with co-pipe dreams that trace out v.

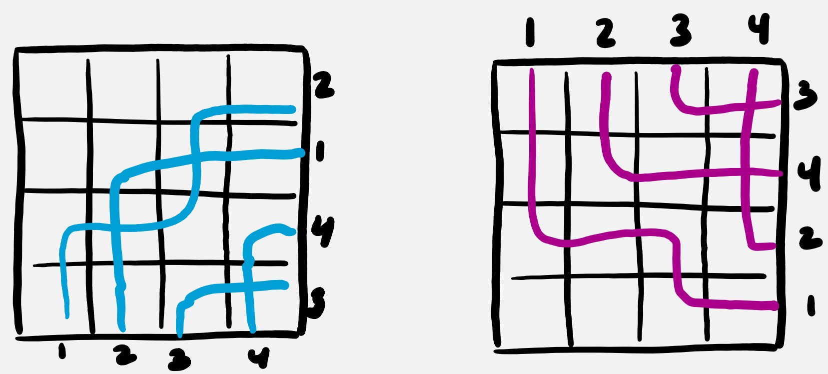


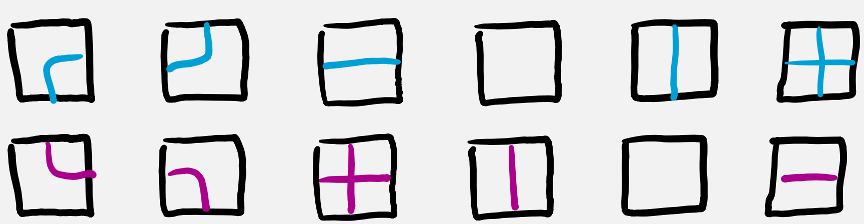
$\mathfrak{S}_{2143} = \mathfrak{G}_{2143} + \mathfrak{G}_{2341} + \mathfrak{G}_{3142}$

Cool upshot: The monomial and Grothendieck expansions of Schubert polynomials have the same number of terms!!!

WHAT ABOUT BPDS?

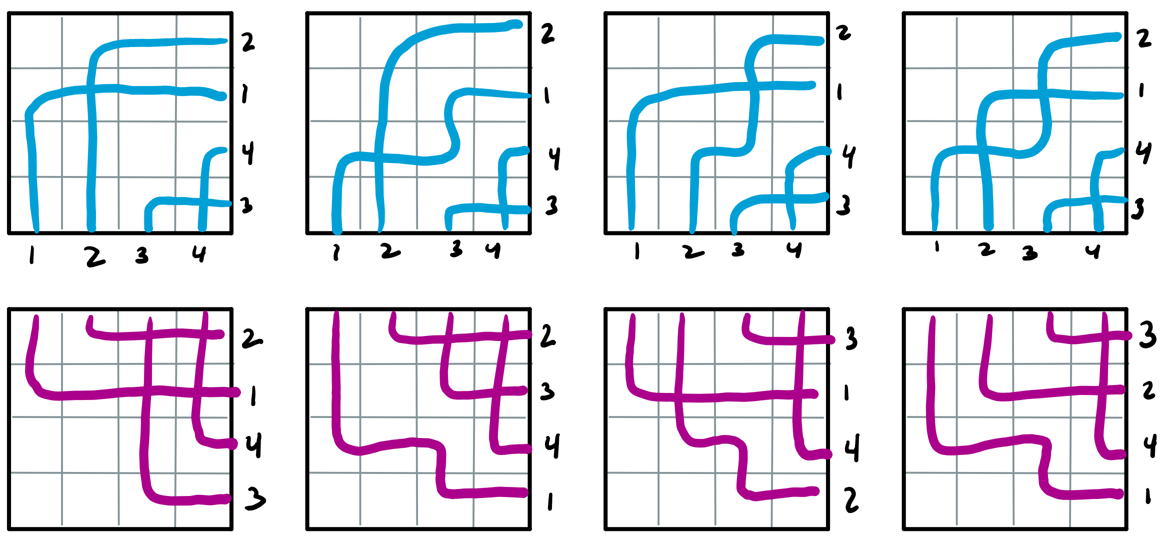
- New formulas for changing bases via BPDs!





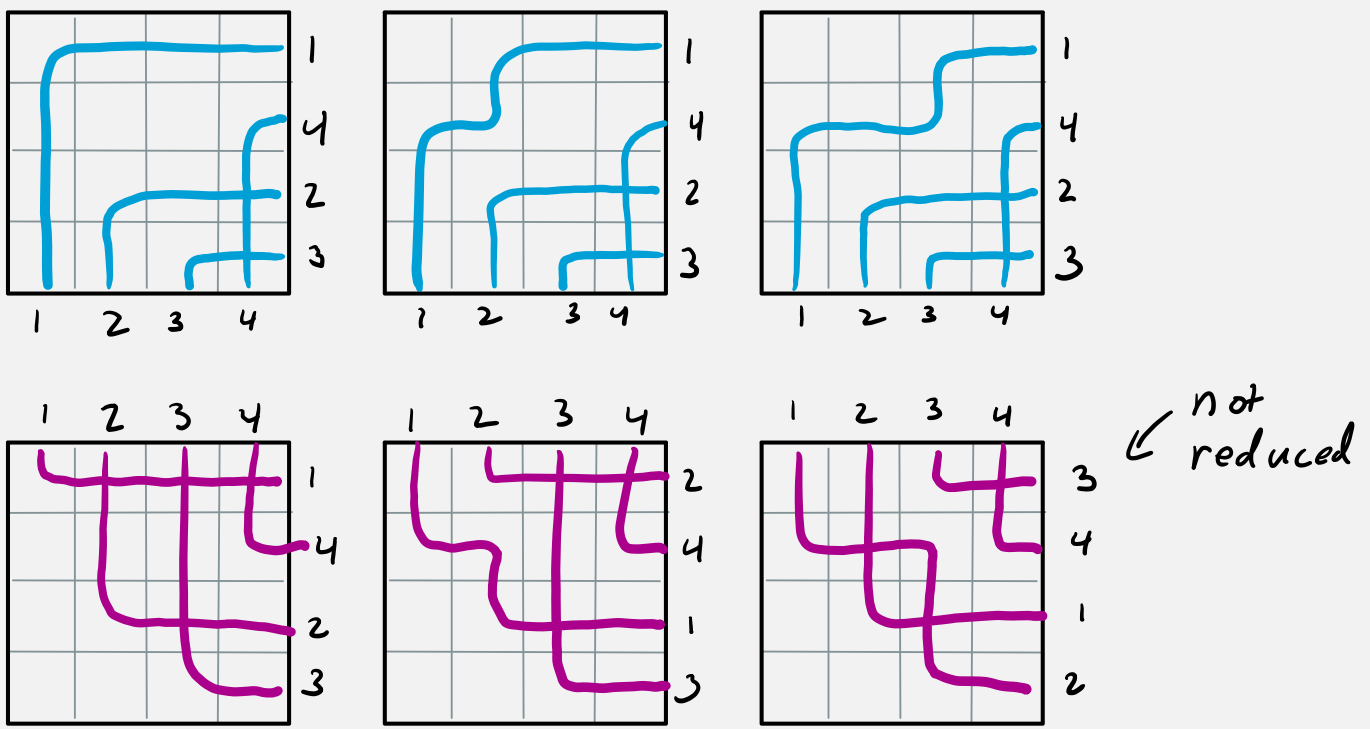
THEOREM (WEIGANDT 2025)

The Grothendieck polynomial is $\mathfrak{G}_w = \sum_v (-1)^{\ell(v)-\ell(w)} a_{w,v}\, \mathfrak{S}_v$, where $a_{w,v}$counts BPDs for w with co-BPDs that are reduced and trace out v.



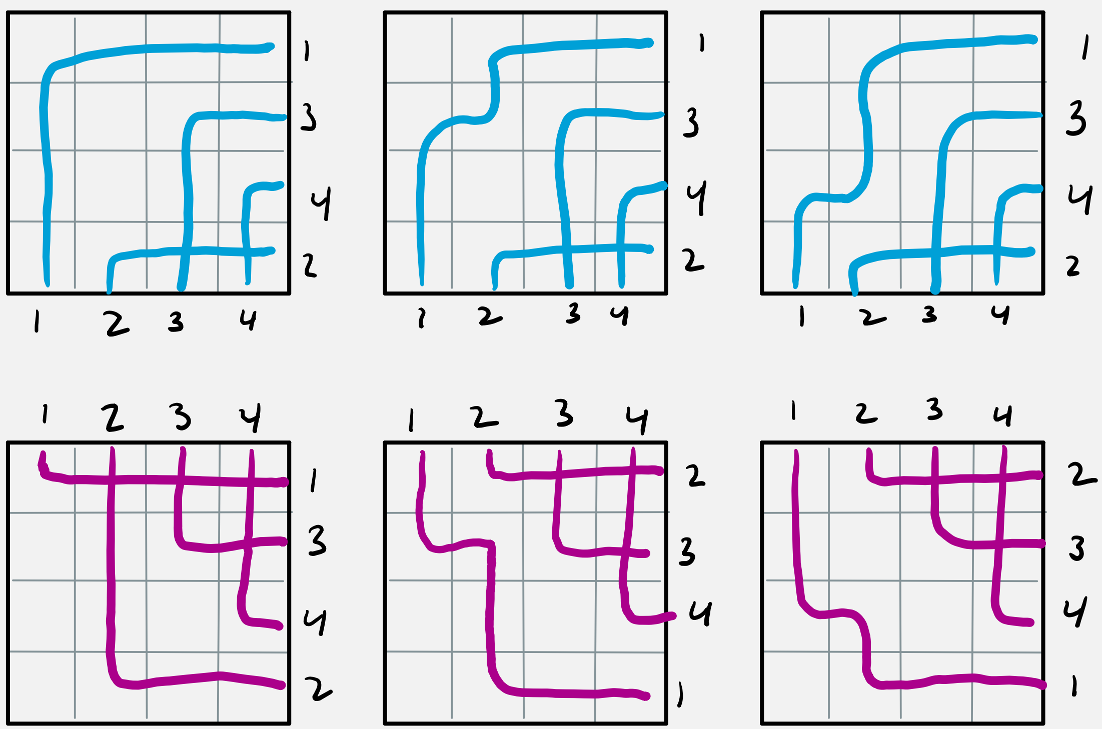
$\mathfrak{G}_{2143} = \mathfrak{S}_{2143} - \mathfrak{S}_{2341} - \mathfrak{S}_{3142} + \mathfrak{S}_{3241}$

EXAMPLE



$\mathfrak{G}_{1423} = \mathfrak{S}_{1423} - \mathfrak{S}_{2413}$

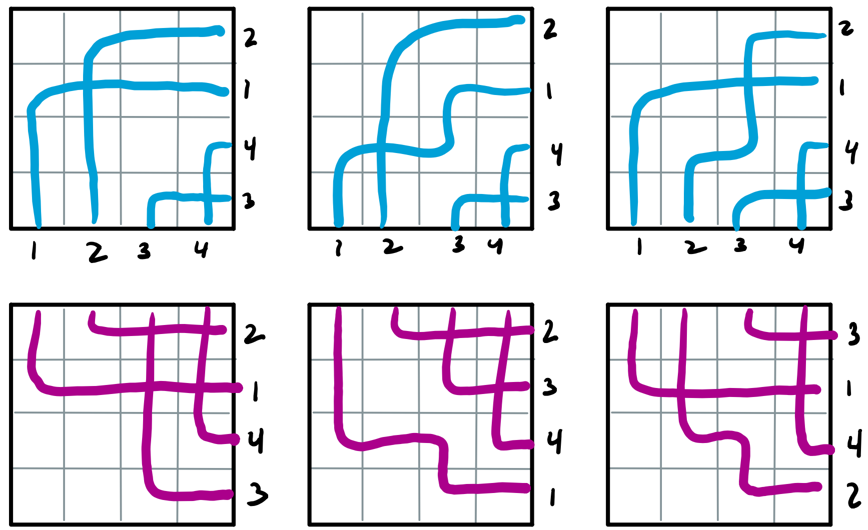
EXAMPLE



$\mathfrak{G}_{1342} = \mathfrak{S}_{1342} - 2\,\mathfrak{S}_{2341}$

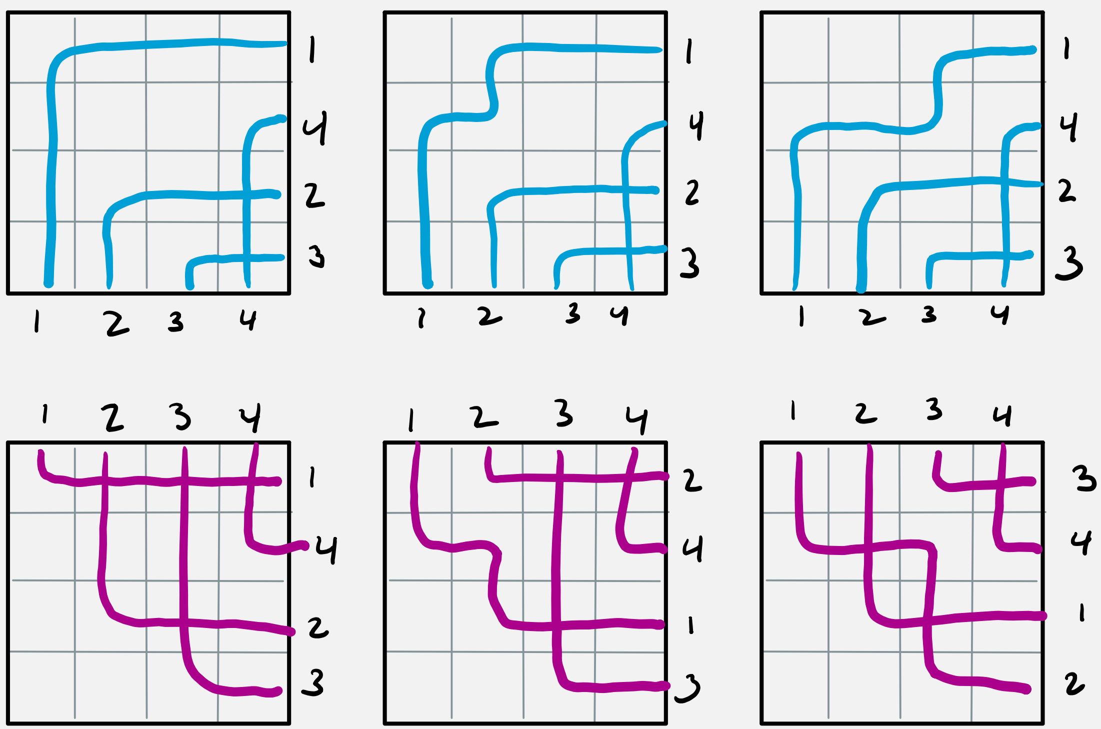
THEOREM (WEIGANDT 2025)

The Schubert polynomial is $\mathfrak{S}_w = \sum_v b_{w,v} \mathfrak{G}_v$, where $b_{w,v}$ counts reduced BPDs for w with co-BPDs that trace out v.



$\mathfrak{S}_{2143} = \mathfrak{G}_{2143} + \mathfrak{G}_{2341} + \mathfrak{G}_{3142}$

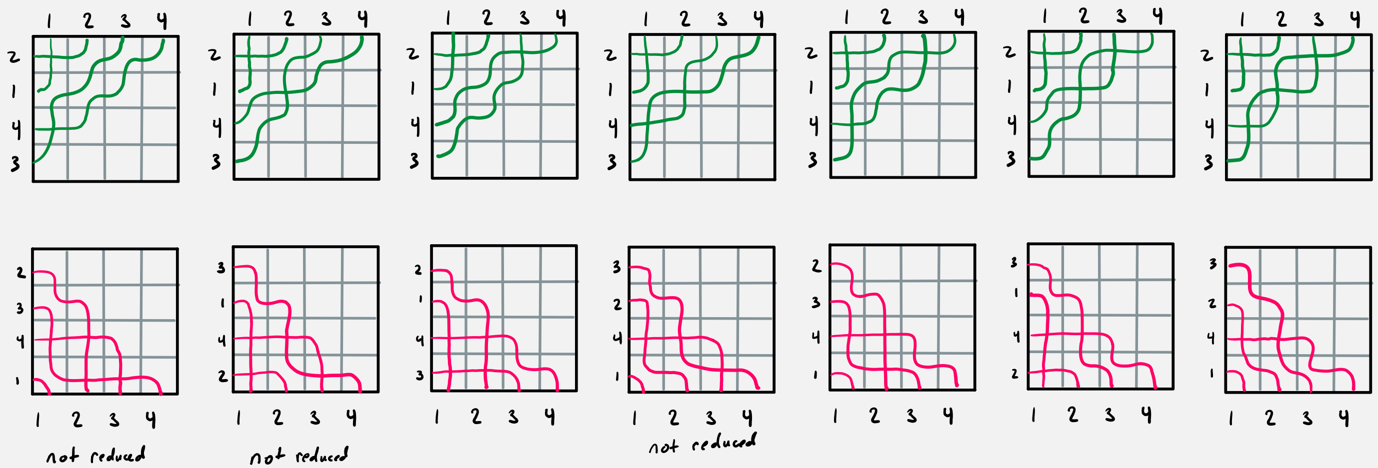
EXAMPLE



$\mathfrak{S}_{1423} = \mathfrak{G}_{1423} + \mathfrak{G}_{2413} + \mathfrak{G}_{3412}$

COMPARING FORMULAS

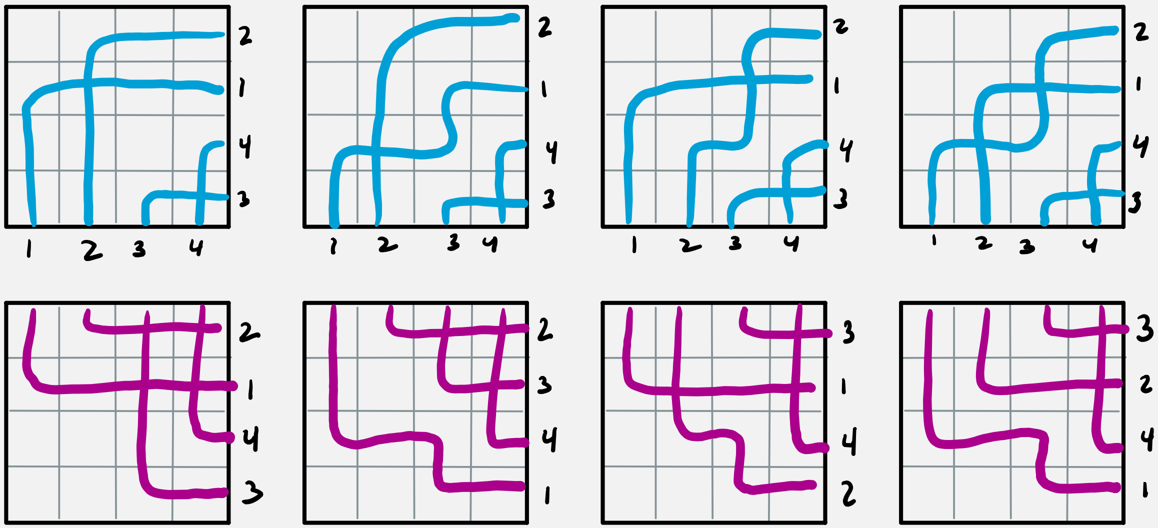
EXAMPLE: 2143



$$\underbrace{\mathfrak{G}_{2143} = \mathfrak{S}_{2143} - \mathfrak{S}_{2341} - \mathfrak{S}_{3142} + \mathfrak{S}_{3241}}_{\text{Grothendieck to Schubert}}$$

$$\underbrace{\mathfrak{S}_{2143} = \mathfrak{G}_{2143} + \mathfrak{G}_{2341} + \mathfrak{G}_{3142}}_{\text{Schubert to Grothendieck}}$$

EXAMPLE: 2143



$$\mathfrak{G}_{2143} = \mathfrak{S}_{2143} - \mathfrak{S}_{2341} - \mathfrak{S}_{3142} + \mathfrak{S}_{3241}$$

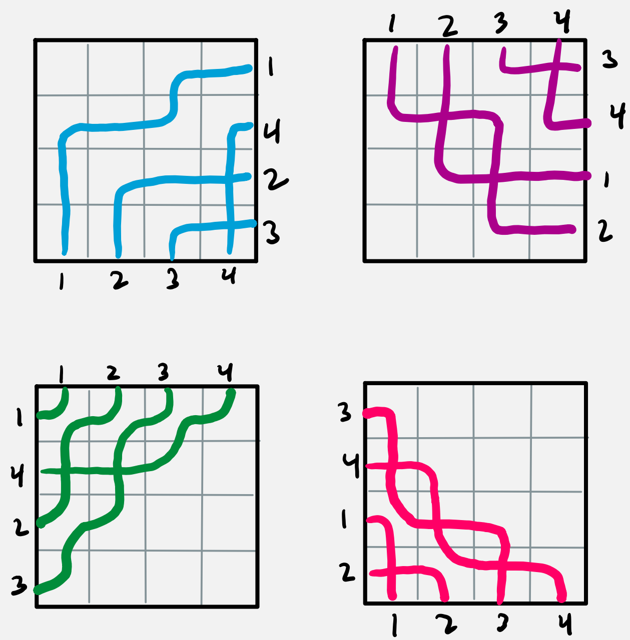
Grothendieck to Schubert

$$\mathfrak{S}_{2143} = \mathfrak{G}_{2143} + \mathfrak{G}_{2341} + \mathfrak{G}_{3142}$$

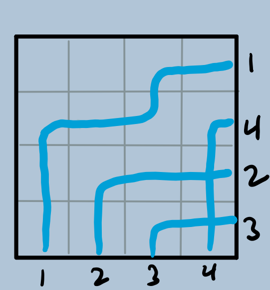
Schubert to Grothendieck

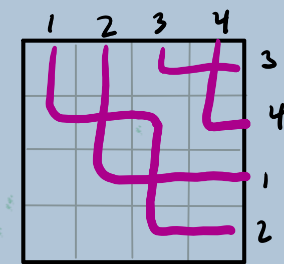
IDEA OF THE PROOF

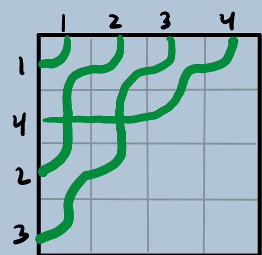
- Gao and Huang (2023) gave a (weight preserving) bijection between reduced BPDs and reduced pipe dreams
- Here we need "column weight" version
- Proposition: Permutations of the co-diagrams are preserved

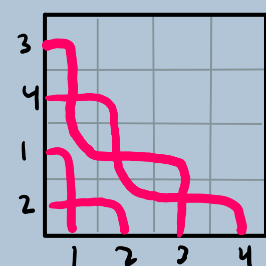


THANKS!



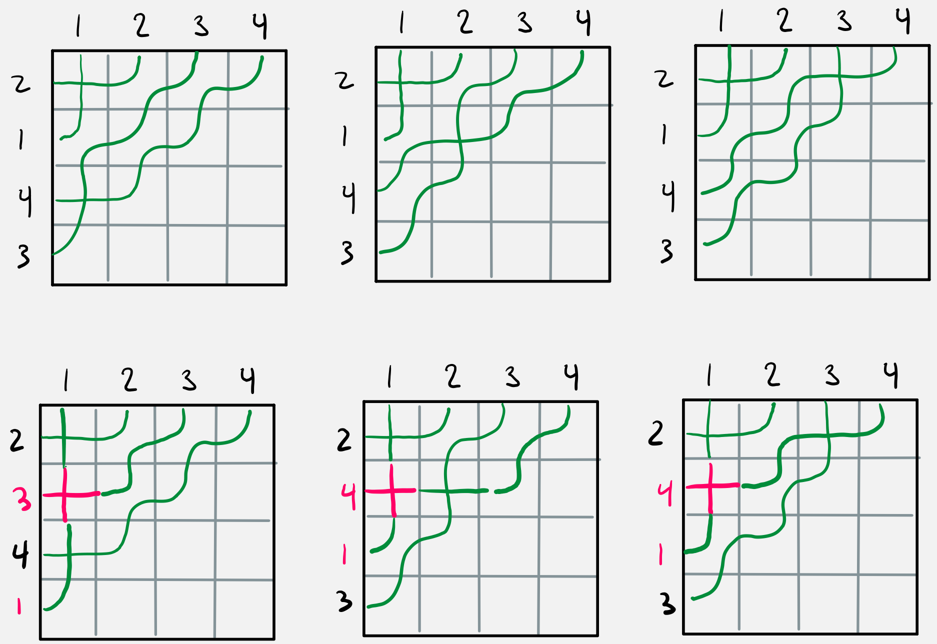






IDEA OF PROOF

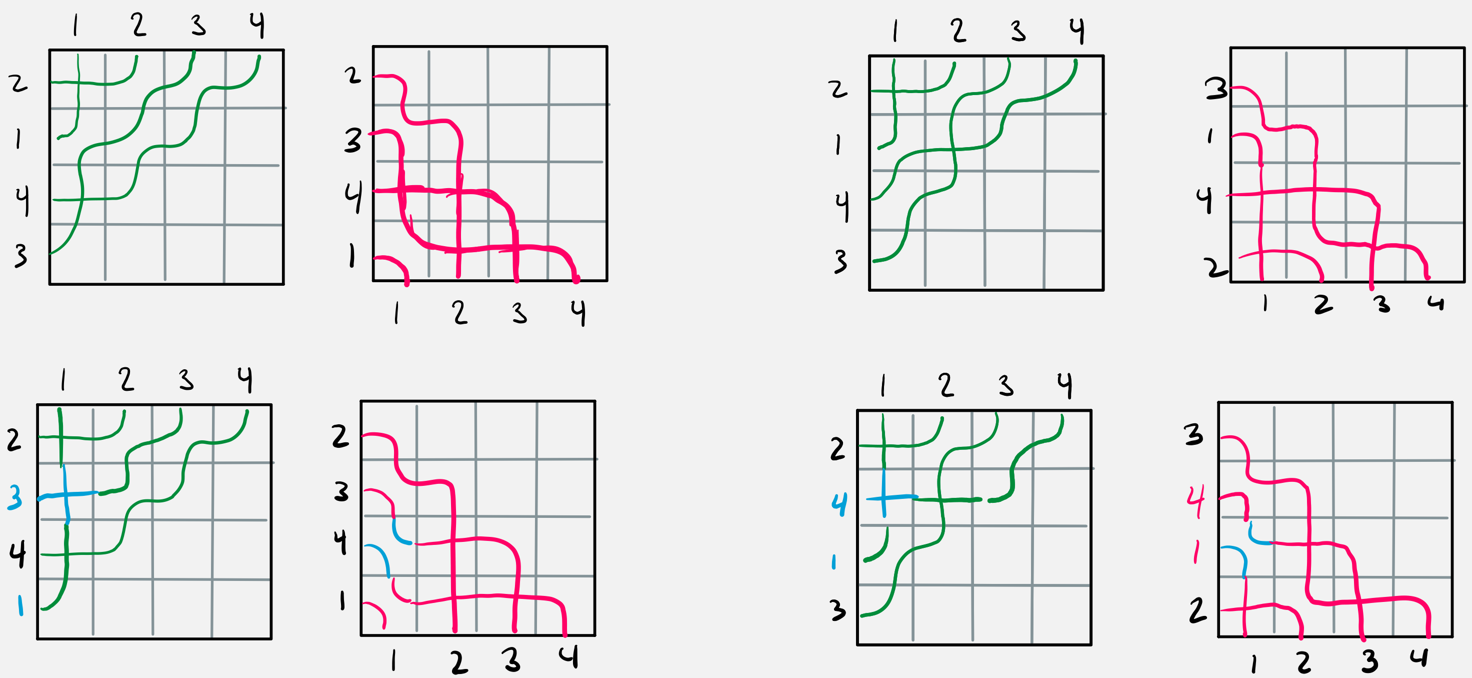
COTRANSITION ON PIPE DREAMS



$$x_2 \mathfrak{S}_{2143} = \mathfrak{S}_{2341} + \mathfrak{S}_{2413}$$

- Knutson (2022) gave very simple combinatorial interpretation!

CO-PERMUTATION CHANGES IN PREDICTABLE WAY

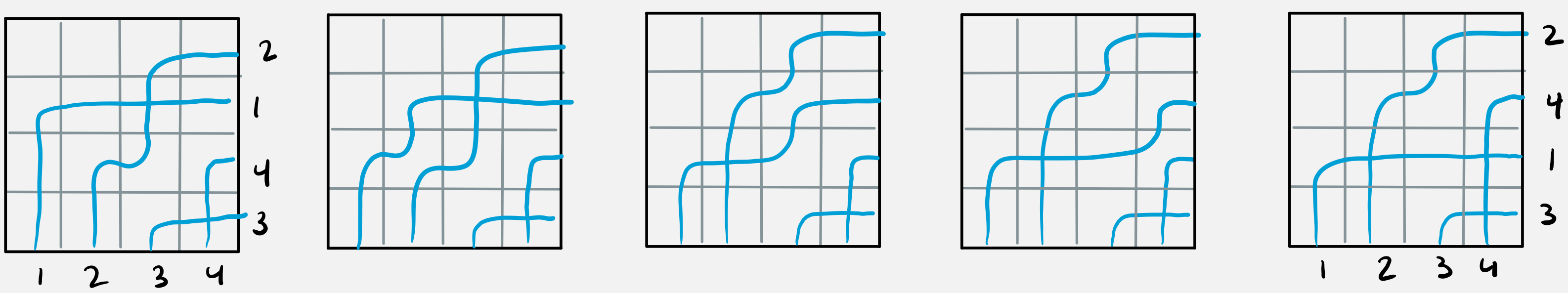


no change

increase in weak Bruhat order

COTRANSITION ON BPDS

- Work of Daoji Huang (2020) tells us how to do this!



COTRANSITION ON BPDS

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