Chow polynomials of totally nonnegative matrices and posets

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Joint work with Petter Brändén

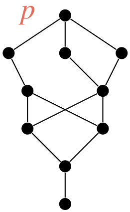
FPSAC 2026 – July 17 2026

Chow polynomials of posets

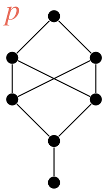
Definition

For every finite bounded graded poset P, define $\mathrm{H}_{\text{point}}(t) = 1$ and

$$\mathrm{H}_P(t) = 1 + t \sum_{\mathrm{rk}(p)>1} \mathrm{H}_{\tau([\hat{0},p])}(t), \qquad \mathrm{rk}(P) > 0.$$



P



$\tau([\hat{0}, p])$

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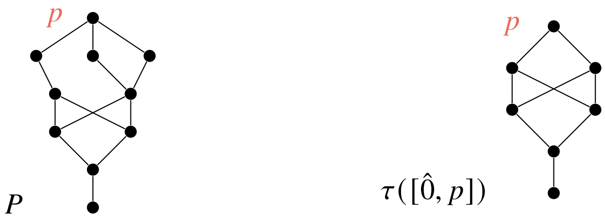

What properties should $\mathrm{H}_P(t)$ satisfy?

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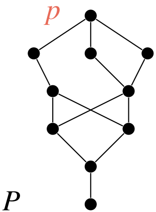

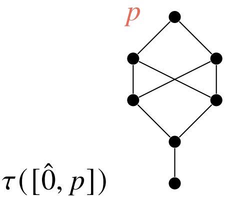

$$\mathrm{H}_P(t) = 1 + 8t + 16t^2 + 8t^3 + t^4.$$

Palindromicity, unimodality, and real-rootedness

Theorem (Larson)

If $\mathcal{L}$ is a geometric lattice, $\mathrm{H}_{\mathcal{L}}(t)$ is the Hilbert series of the Chow ring of the underlying matroid.

Theorem (Adiprasito–Huh–Katz)

For every geometric lattice, $\mathrm{H}_{\mathcal{L}}(t)$ is palindromic and unimodal.

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Theorem (Ferroni–Matherne–V.)

For every finite bounded graded poset P, $\mathrm{H}_P(t)$ is palindromic and unimodal. For every Cohen–Macaulay poset, it is γ-positive.

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For every finite bounded graded poset P, $\mathrm{H}_P(t)$ is palindromic and unimodal. For every Cohen–Macaulay poset, it is γ-positive.

A previous conjecture, now disproven, proposed real-rootedness for every Cohen–Macaulay poset.

Why extend to arbitrary posets?

- For geometric lattices, palindromicity and unimodality follow from deep theorems (Poincaré duality and Hard Lefschetz)
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Why extend to arbitrary posets?

- For geometric lattices, palindromicity and unimodality follow from deep theorems (Poincaré duality and Hard Lefschetz)
- For arbitrary posets, the proof is fully combinatorial.
- Separate the real-rootedness question from the geometry of matroids.
- Several classes are known to have real-rooted Chow polynomials:
  - uniform geometric lattices (Brändén–V.);
  - simplicial posets with positive h-vector (Hoster–Stump) **this FPSAC**;
  - totally nonnegative posets **this talk**;
  - UMEL-shellable posets (Coron–Ferroni–Li).

Goal for today

Prove real-rootedness using totally nonnegative matrices.

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Prove real-rootedness using totally nonnegative matrices.

Theorem

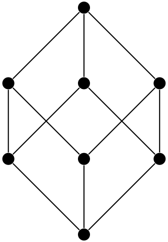
Let P be a TN rank-uniform poset. Then $\mathrm{H}_P(t)$ has only real zeros.

Joint work with Petter Brändén (KTH).

Boolean lattices and Eulerian polynomials

Theorem (Well-known fact)

$$P = \mathrm{B}_n \qquad \mathrm{H}_{\mathrm{B}_n}(t) = A_n(t).$$



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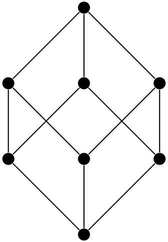

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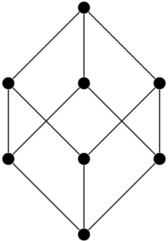

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In $\mathfrak{S}_3$: 123 132 213

231 312 321.

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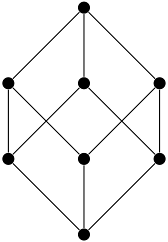

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In $\mathfrak{S}_3$: $123 \quad 132 \quad 213$

$231 \quad 312 \quad 321.$

$$\mathrm{H}_{\mathrm{B}_3}(t) = 1 + 4t + t^2.$$

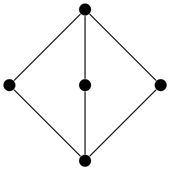
Theorem (Frobenius)

For every n, $A_n(t)$ has only real zeros.

Derangement polynomials

Theorem (Hameister–Rao–Simpson)

$$P = \tau(\mathrm{B}_n) = \mathrm{U}_{n-1,n} \qquad \mathrm{H}_{\mathrm{U}_{n-1,n}}(t) = \frac{1}{t} d_n(t).$$



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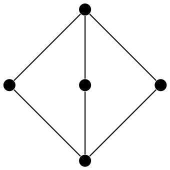

Let $\mathcal{D}_n$ be the set of **derangements** in $\mathfrak{S}_n$.

$$d_n(t) = \sum_{\omega \in \mathcal{D}_n} t^{\mathrm{exc}(\omega)}.$$

In $\mathfrak{S}_3$ $\qquad 312 \quad 231.$

$$\mathrm{H}_{\mathrm{U}_{2,3}}(t) = \tfrac{1}{t}(t + t^2) = 1 + t.$$

$$A_n(t) = \sum_{k=0}^{n} \binom{n}{k} d_k(t).$$

How does one prove real-rootedness?

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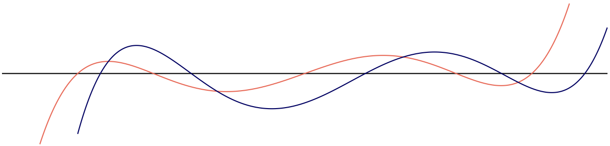
Let $p, q \in \mathbb{R}[t]$ have positive leading coefficient and only real zeros (β_i) and (α_i).
We say that p **interlaces** q if

$$p \prec q \iff \cdots \leqslant \beta_2 \leqslant \alpha_2 \leqslant \beta_1 \leqslant \alpha_1.$$



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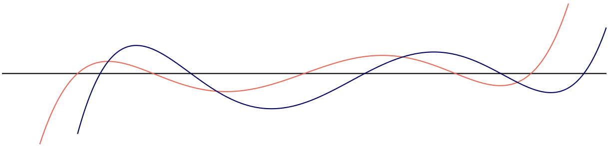

Theorem (Brändén–Solus)

$$A_n(t) \prec d_n(t).$$

Main results

A new recursion

Theorem (Brändén–V.)

Let P be a finite bounded graded poset. There exist unique polynomials $d_{[\hat{0},x]}(t)$, palindromic with center $\mathrm{rk}(x)/2$, such that

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The theorem comes with a recursive way of computing d. We call d_P the *Chow-derangement polynomial* of P.

$$\mathrm{H}_P(t) = \sum_{\hat{0} \leqslant x \leqslant \hat{1}} d_{[\hat{0},x]}(t). \qquad A_n(t) = \sum_{k=0}^{n} \binom{n}{k} d_k(t).$$

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Goal: Prove interlacing between H_P and d_P.

What makes the Boolean case “easy”?

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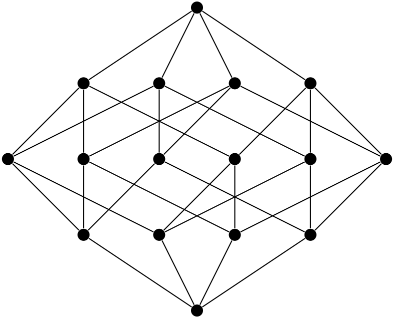
Symmetry!



Rank-uniform posets: Boolean case

A bounded graded poset P is *rank uniform* if

$$|\{z \le x : \mathrm{rk}(z) = k\}| = r_{n,k} \qquad \text{for all } x, \text{ where } n = \mathrm{rk}(x).$$

$$R = (r_{n,k})_{n,k}.$$

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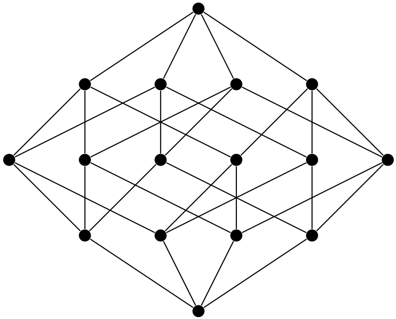
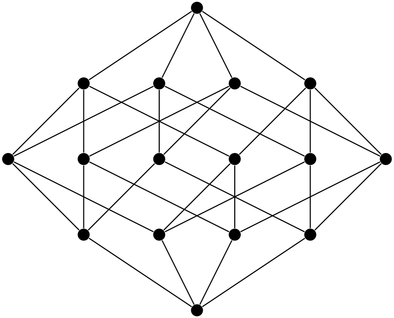


$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}.$$

Rank-uniform posets



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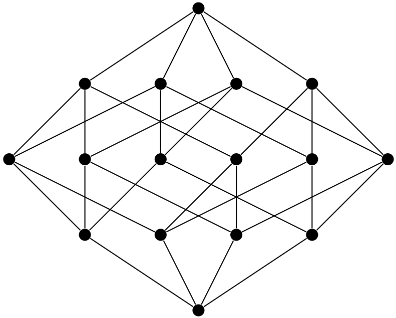

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If P is rank uniform, H and d only depend on the rank:

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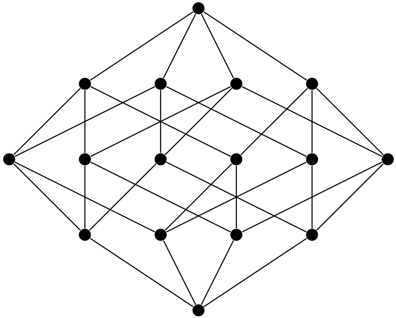

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The recursions become one-dimensional:

$$\mathrm{H}_n(t) = \sum_{\hat{0} \le z \le x} d_{[\hat{0},z]}(t) = \sum_{k=0}^{n} r_{n,k} d_k(t).$$

New question

Which matrices R force real-rootedness?

Totally nonnegative matrices

Definition

A matrix is *totally nonnegative* (TN) if all minors are nonnegative.
A rank-uniform poset is TN if its matrix R is TN.

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Examples:

- Pascal's triangle and Boolean lattices;
- face lattices of simplicial and cubical polytopes (with positive h-vector);
- perfect matroid designs;
- projective and affine geometries;
- dual partition lattices;
- their truncations;
- and their rank selections.

Why total nonnegativity?

Theorem (Brändén–Saud)

$$R \text{ is } \mathrm{TN} \quad \Longleftrightarrow \quad R \text{ is resolvable}.$$

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Let $R_n(t) = \sum_{k=0}^{n} r_{n,k} t^k$.
Row polynomials $R_n(t)$ can be refined into polynomials $R_{n,k}(t)$ satisfying

$$R_{n,0} = R_n, \qquad R_{n,n} = t^n,$$

$$R_{n+1,k} = R_{n+1,k+1} + \lambda_{n,k} R_{n,k}, \qquad \lambda_{n,k} \geq 0.$$

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This positive recurrence is what preserves interlacing.
For example, this led [Brändén–Saud] to prove real-rootedness for chain polynomials of TN posets.

The Chow-deranged map

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\end{array}$$

From the positive recursion for $R_{n,k}$ we can prove

Theorem (Brändén–V.)

For every $f(t) = \sum_{k=0}^{n} h_k R_{n,k}(t)$ with $h_k \geqslant 0$,

$$\mathrm{H}_n \prec \mathcal{D}(f) \prec d_n.$$

Real-rooted TN posets

Theorem (Brändén–V.)

If P or its dual is TN, then

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In particular, $\mathrm{H}_P(t)$ and $d_P(t)$ have only real zeros.

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Moreover,

- $\mathrm{H}_{\tau(P)}(t) \prec \mathrm{H}_P(t)$,
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- $\mathrm{H}_{\tau(P)}(t) \prec \mathrm{H}_P(t)$,
- for every corank-one element $x \in P$, $\mathrm{H}_{[\hat{0},x]}(t) \prec \mathrm{H}_P(t)$;
- the construction extends to every TN lower-triangular matrix with diagonal entries equal to one.

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- for projective geometries, recovery of the real-rootedness of q-Eulerian polynomials (Shareshian–Wachs)

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

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  and all their truncations;
- real-rootedness for paving geometric lattices (almost all geometric lattices?);
- a Pólya frequency is a sequence $(a_n)_{n\geqslant 0}$ such that the corresponding Toeplitz matrix $R = (a_{n-k})_{n,k}$ is TN. We now have new families of real-rooted polynomials and study their generating functions.

Recap

- $\mathrm{H}_P(t)$ arises from combinatorial algebraic geometry, but is conjectured to have strong combinatorial properties.
- Rank-uniformity makes the recursions one-dimensional.
- Total nonnegativity gives a refined positive recurrence.
- This recurrence gives interlacing, hence real-rootedness.

Thank you! ☺

Toeplitz matrices and symmetric functions

$$E(\mathbf{x}) = \begin{pmatrix} 1 & & & \\ e_1(\mathbf{x}) & 1 & & \\ e_2(\mathbf{x}) & e_1(\mathbf{x}) & 1 & \\ e_3(\mathbf{x}) & e_2(\mathbf{x}) & e_1(\mathbf{x}) & 1 \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\sum_{n \geqslant 0} \mathrm{H}_n(t; \mathbf{x}) z^n = \frac{(1-t)E(\mathbf{x}; z)}{E(\mathbf{x}; tz) - tE(\mathbf{x}; z)}, \qquad E(\mathbf{x}; z) = \sum_{n \geqslant 0} e_n(\mathbf{x}) z^n.$$

Theorem (Stanley)

$$\mathrm{H}_n(t; \boldsymbol{x}) = \sum_{w \in W_n} t^{\mathrm{des}(w)} \prod_{i=1}^{n} x_{w(i)}.$$

We study also the case $e(\mathbf{x} \setminus \mathbf{y})$.