A bijection on balanced words reversing both des and maj

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What are balanced words?

Let $\mathcal{W}_{n,k}$ be the set of *balanced words*. A word $w \in \mathcal{W}_{n,k}$ is any arrangement of the letters $\{1, \ldots, k\}$ each of which must occur n times.

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Example

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- $|\mathcal{W}_{n,k}| = \binom{nk}{n,\ldots,n}$,
- $\mathcal{W}_{1,k}$ are permutations of $[k]$, and
- $\mathcal{W}_{n,2}$ are balanced binary words or lattice paths in an $n \times n$ grid.

Descents and major index

A word in $\mathcal{W}_{3,4}$ is

$$w = \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & \boxed{3} & 1 & 1 & 2 & 4 & \boxed{4} & 1 & \boxed{4} & \boxed{3} & 2 & 3 \end{array}$$

Descents and major index

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w & = & 2 & \boxed{3} & 1 & 1 & 2 & 4 & \boxed{4} & 1 & \boxed{4} & \boxed{3} & 2 & 3
\end{array}
$$

- The *descent set* of w is $\mathrm{Des}(w) = \{2, 7, 9, 10\}$,

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- The number of *descents* in w is $\mathrm{des}(w) = 4$,

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- The *descent set* of w is $\mathrm{Des}(w) = \{2, 7, 9, 10\}$,
- The number of *descents* in w is $\mathrm{des}(w) = 4$,
- The *major index* of w is the sum of the descent set $\mathrm{maj}(w) = 2 + 7 + 9 + 10 = 28$.

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Definition

The bivariate generating polynomial of des and maj on balanced words is

$$C_{n,k}(x, q) = \sum_{w \in \mathcal{W}_{n,k}} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)}.$$

For example,

$$C_{2,2}(x, q) = 1 + xq + 2xq^2 + xq^3 + x^2q^4.$$

Descents and major index

The *unique* word $w \in \mathcal{W}_{n,k}$ having the minimum value of both $\operatorname{des}(w) = 0$ and $\operatorname{maj}(w) = 0$ is

$$1 \quad 1 \quad \dots \quad 1 \quad 2 \quad 2 \quad \dots \quad 2 \quad \dots \quad k \quad k \quad \dots \quad k.$$

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The *unique* word having the maximum value of both $\mathrm{des}(w) = n(k-1)$ and $\mathrm{maj}(w) = n^2\binom{k}{2}$ is

$$k \quad k-1 \quad \dots \quad 1 \quad k \quad k-1 \quad \dots \quad 1 \quad \dots \quad k \quad k-1 \quad \dots \quad 1.$$

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For example, words in $\mathcal{W}_{2,3}$ can have a maximum des of 4 and a maximum maj of 12.

Descents and major index

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For example, words in $\mathcal{W}_{2,3}$ can have a maximum des of 4 and a maximum maj of 12.

Words in $\mathcal{W}_{2,3}$ having des $= 1$ and maj $= 3$

1	1	3	2	2	3
1	3	3	1	2	2
1	2	3	1	2	3
1	2	2	1	3	3
2	3	3	1	1	2
2	2	3	1	1	3

Descents and major index

The *unique* word $w \in \mathcal{W}_{n,k}$ having the minimum value of both $\operatorname{des}(w) = 0$ and $\operatorname{maj}(w) = 0$ is

$$1 \quad 1 \quad \dots \quad 1 \quad 2 \quad 2 \quad \dots \quad 2 \quad \dots \quad k \quad k \quad \dots \quad k.$$

The *unique* word having the maximum value of both $\operatorname{des}(w) = n(k-1)$ and $\operatorname{maj}(w) = n^2\binom{k}{2}$ is

$$k \quad k-1 \quad \dots \quad 1 \quad k \quad k-1 \quad \dots \quad 1 \quad \dots \quad k \quad k-1 \quad \dots \quad 1.$$

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Words in $\mathcal{W}_{2,3}$ having des $= 1$ and maj $= 3$

1	1	3	2	2	3
1	3	3	1	2	2
1	2	3	1	2	3
1	2	2	1	3	3
2	3	3	1	1	2
2	2	3	1	1	3

Words in $\mathcal{W}_{2,3}$ having des $= 4 - 1 = 3$ and maj $= 12 - 3 = 9$

2	1	3	2	3	1
3	2	3	1	2	1
3	2	2	1	3	1
2	1	3	1	3	2
3	1	3	2	2	1
3	1	2	1	3	2

The main theorem

Theorem (Tielker 2023)

The bivariate generating polynomial of des *and* maj *on balanced words is palindromic, i.e.*

$$C_{n,k}(x^{-1}, q^{-1}) = x^{-n(k-1)} q^{-n^2\binom{k}{2}} C_{n,k}(x, q).$$

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For example, the coefficients of $C_{2,3}(x, q)$, written as an array is the same when rotated by 180°.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 6 & 5 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 6 & 10 & 14 & 10 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 5 & 6 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

The main theorem

We prove the theorem by

- constructing a des and maj reversing bijection, f_2, on balanced binary words $\mathcal{W}_{n,2}$, and
- constructing the general bijection, f_k, for balanced words $\mathcal{W}_{n,k}$ (with $k > 2$) via recursion.

Encoding balanced binary words as pairs of boolean arrays

A *boolean array* is a word in the letters T and F. For example,

$$\mu = \mathsf{T\,F\,F\,T\,F\,T\,T}$$

is one. The *complement* of μ is written $\bar{\mu}$.

$$\bar{\mu} = \mathsf{F\,T\,T\,F\,T\,F\,F}.$$

Let $B_n = \{\mathsf{T}, \mathsf{F}\}^n$ be the set of boolean arrays of length n. Also write B_n^k to mean boolean arrays of length n having k T's.

Encoding balanced binary words as pairs of boolean arrays

Consider the following word $w \in \mathcal{W}_{5,2}$,

$$w = 1\ 2\ 2\ 2\ 1\ 1\ 2\ 1\ 1\ 2.$$

Encoding balanced binary words as pairs of boolean arrays

Consider the following word $w \in \mathcal{W}_{5,2}$,

$$w = 1\ 2\ 2\ 2\ 1\ 1\ 2\ 1\ 1\ 2.$$

Group each descent into a box like so

$$w = 1\ 2\ 2\ \boxed{2\ 1}\ 1\ \boxed{2\ 1}\ 1\ 2.$$

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$$w = 1\ 2\ 2\ 2\ 1\ 1\ 2\ 1\ 1\ 2.$$

Group each descent into a box like so

$$w = 1\ 2\ 2\ \boxed{2\ 1}\ 1\ \boxed{2\ 1}\ 1\ 2.$$

Record the 1's outside the boxes with an F and the boxes themselves with a T,

$$\mu_w = \mathrm{F}\ \boxed{\mathrm{T}}\ \mathrm{F}\ \boxed{\mathrm{T}}\ \mathrm{F}\,.$$

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Consider the following word $w \in \mathcal{W}_{5,2}$,

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Record the 1's outside the boxes with an F and the boxes themselves with a T,

$$\mu_w = \text{F}\ \boxed{\text{T}}\ \text{F}\ \boxed{\text{T}}\ \text{F}\,.$$

Record the 2's outside the boxes with an F and the boxes themselves with a T,

$$\nu_w = \text{F}\ \text{F}\ \boxed{\text{T}}\ \boxed{\text{T}}\ \text{F}\,.$$

Encoding balanced binary words as pairs of boolean arrays

We can also start with two boolean arrays of 1) the same length and 2) having the same number of T's and construct a binary word. Let,

$$\mu = \mathsf{F}\ \boxed{\mathsf{T}}\ \mathsf{F}\ \boxed{\mathsf{T}}\ \mathsf{F}. \qquad \nu = \mathsf{F}\ \mathsf{F}\ \boxed{\mathsf{T}}\ \boxed{\mathsf{T}}\ \mathsf{F}.$$

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$\mu =$ F $\boxed{\text{T}}$ F $\boxed{\text{T}}$ F. $\nu =$ F F $\boxed{\text{T}}$ $\boxed{\text{T}}$ F.

Expand each T into a 21 like so

$\boxed{2\ 1}$ $\boxed{2\ 1}$

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Expand each T into a 21 like so

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Read the number of F's before the first T in μ and write

1 $\boxed{2\ 1}$ $\boxed{2\ 1}$.

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Read the number of F's before the first T in μ and write

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Read the number of F's before the first T in ν and place it after any 1's

$$1\ 2\ 2\ \boxed{2\ 1}\ \boxed{2\ 1}.$$

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$$1\ 2\ 2\ \boxed{2\ 1}\ \boxed{2\ 1}.$$

Repeat the process for the second group of F's in array to get

$$1\ 2\ 2\ \boxed{2\ 1}\ 1\ \boxed{2\ 1}.$$

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Expand each T into a 21 like so

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Read the number of F's before the first T in μ and write

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Read the number of F's before the first T in ν and place it after any 1's

$$1\ 2\ 2 \ \boxed{2\,1} \ \boxed{2\,1}.$$

Repeat the process for the second group of F's in array to get

$$1\ 2\ 2 \ \boxed{2\,1} \ 1 \ \boxed{2\,1}.$$

Finally, repeat the process for the last group to get

$$1\ 2\ 2 \ \boxed{2\,1} \ 1 \ \boxed{2\,1} \ 1\ 2.$$

Encoding balanced binary words as pairs of boolean arrays

The process we saw just now gives us the following invertible map

$$\phi : \mathcal{W}_{n,2} \to \bigsqcup_{0 \leq k \leq n} B_n^k \times B_n^k$$

by

$$\phi(w) = (\mu_w, \nu_w).$$

The des and maj reversing bijection on $\mathcal{W}_{n,2}$

Let $f_2 : \mathcal{W}_{n,2} \to \mathcal{W}_{n,2}$ be given by

$$f_2(w) = \phi^{-1}(\bar{\mu}_w, \bar{\nu}_w).$$

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Let $f_2 : \mathcal{W}_{n,2} \to \mathcal{W}_{n,2}$ be given by

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Theorem

The map f_2 on $\mathcal{W}_{n,2}$ is an involution satisfying

$$\mathrm{des}(w) + \mathrm{des}(f_2(w)) = n,$$

and

$$\mathrm{maj}(w) + \mathrm{maj}(f_2(w)) = n^2$$

for all $w \in \mathcal{W}_{n,2}$.

The des and maj reversing bijection on $\mathcal{W}_{n,2}$

Consider the same word $w \in \mathcal{W}_{5,2}$ as earlier,

$$w = 1\ 2\ 2\ \boxed{2\ 1}\ 1\ \boxed{2\ 1}\ 1\ 2.$$

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The map ϕ sends this to

$$\mu_w = \text{F}\ \boxed{\text{T}}\ \text{F}\ \boxed{\text{T}}\ \text{F}. \qquad \nu_w = \text{F}\ \text{F}\ \boxed{\text{T}}\ \boxed{\text{T}}\ \text{F}.$$

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The complements of these are

$$\bar{\mu}_w = \boxed{\mathrm{T}}\ \mathrm{F}\ \boxed{\mathrm{T}}\ \mathrm{F}\ \boxed{\mathrm{T}}. \qquad \bar{\nu}_w = \boxed{\mathrm{T}}\ \boxed{\mathrm{T}}\ \mathrm{F}\ \mathrm{F}\ \boxed{\mathrm{T}}.$$

The des and maj reversing bijection on $\mathcal{W}_{n,2}$

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The complements of these are

$$\bar{\mu}_w = \boxed{\mathrm{T}}\ \mathrm{F}\ \boxed{\mathrm{T}}\ \mathrm{F}\ \boxed{\mathrm{T}}. \qquad \bar{\nu}_w = \boxed{\mathrm{T}}\ \boxed{\mathrm{T}}\ \mathrm{F}\ \mathrm{F}\ \boxed{\mathrm{T}}.$$

And $\phi^{-1}(\bar{\mu}_w, \bar{\nu}_w)$ gives

$$f_2(w) = \boxed{2\ 1}\ 1\ \boxed{2\ 1}\ 1\ 2\ 2\ \boxed{2\ 1}.$$

Clearly, $\mathrm{des}(w) + \mathrm{des}(f_2(w)) = n$.

The des and maj reversing bijection on $\mathcal{W}_{n,2}$

The proof that $\mathrm{maj}(w) + \mathrm{maj}(f_2(w)) = n^2$ would be trivial if $\mathrm{Des}(w) \cap \mathrm{Des}(f_2(w)) = \emptyset$; however, in our example $\mathrm{Des}(1222112112) = \{4, 7\}$ and $\mathrm{Des}(2112112221) = \{1, 4, 9\}$.

$$
\begin{array}{ccccccccccc}
w & = & 1 & 2 & 2 & \boxed{2\,1} & 1 & \boxed{2\,1} & 1 & 2 \\
\mu_w & = & F & & & T & F & T & F & \\
\nu_w & = & & F & F & T & & T & F &
\end{array}
$$

The des and maj reversing bijection on $\mathcal{W}_{n,2}$

Count 4 with respect to μ_w and ν_w.

$$
\begin{array}{lcccccccccc}
 & & & & & 4 & & & & \\
w & = & 1 & 2 & 2 & \boxed{2\,1} & 1 & \boxed{2\,1} & 1 & 2 \\
\mu_w & = & F^1 & & & T^1 & F & T & F & \\
\nu_w & = & & F^1 & F^1 & T & & T & F &
\end{array}
$$

The des and maj reversing bijection on $\mathcal{W}_{n,2}$

Count 4 with respect to μ_w and ν_w.

$$
\begin{array}{llcccccccc}
 & & & & & 4 & & & & \\
w & = & 1 & 2 & 2 & 2\,1 & 1 & 2\,1 & 1 & 2 \\
\mu_w & = & \mathrm{F}^1 & & & \mathrm{T}^1 & \mathrm{F} & \mathrm{T} & \mathrm{F} & \\
\nu_w & = & & \mathrm{F}^1 & \mathrm{F}^1 & \mathrm{T} & & \mathrm{T} & \mathrm{F} &
\end{array}
$$

Next, count 7 with respect to μ_w and ν_w.

$$
\begin{array}{llcccccccc}
 & & & & & 4 & & 7 & & \\
w & = & 1 & 2 & 2 & 2\,1 & 1 & 2\,1 & 1 & 2 \\
\mu_w & = & \mathrm{F}^2 & & & \mathrm{T}^2 & \mathrm{F}^1 & \mathrm{T}^1 & \mathrm{F} & \\
\nu_w & = & & \mathrm{F}^2 & \mathrm{F}^2 & \mathrm{T}^1 & & \mathrm{T} & \mathrm{F} &
\end{array}
$$

The des and maj reversing bijection on $\mathcal{W}_{n,2}$

$$
\begin{array}{ccccccccccc}
 & & & & & 4 & & 7 & & \\
w & = & 1 & 2 & 2 & \boxed{2\ 1} & 1 & \boxed{2\ 1} & 1 & 2 \\
\mu_w & = & \mathsf{F}^2 & & & \mathsf{T}^2 & \mathsf{F}^1 & \mathsf{T}^1 & \mathsf{F} & \\
\nu_w & = & & \mathsf{F}^2 & \mathsf{F}^2 & \mathsf{T}^1 & & \mathsf{T} & \mathsf{F} &
\end{array}
$$

The des and maj reversing bijection on $\mathcal{W}_{n,2}$

$$
\begin{array}{ccccccccccc}
 & & & & & 4 & & 7 & & \\
w & = & 1 & 2 & 2 & \boxed{2\ 1} & 1 & \boxed{2\ 1} & 1 & 2 \\
\mu_w & = & F^2 & & & T^2 & F^1 & T^1 & F & \\
\nu_w & = & & F^2 & F^2 & T^1 & & T & F &
\end{array}
$$

Similarly, compute the major index of $f_2(w)$

$$
\begin{array}{ccccccccc}
 & & 1 & & 4 & & & & 9 \\
w & = & \boxed{2\ 1} & 1 & \boxed{2\ 1} & 1 & 2 & 2 & \boxed{2\ 1} \\
\mu^c_w & = & T^3 & F^2 & T^2 & F^1 & & & T^1 \\
\nu^c_w & = & T^2 & & T^1 & & F^1 & F^1 & T
\end{array}
$$

The des and maj reversing bijection on $\mathcal{W}_{n,2}$

$$
\begin{array}{ccccccccccc}
 & & & & & 4 & & 7 & & \\
w & = & 1 & 2 & 2 & 2\,1 & 1 & 2\,1 & 1 & 2 \\
\mu_w & = & F^2 & & & T^2 & F^1 & T^1 & F & \\
\nu_w & = & & F^2 & F^2 & T^1 & & T & F &
\end{array}
$$

Similarly, compute the major index of $f_2(w)$

$$
\begin{array}{cccccccccc}
 & & 1 & & 4 & & & & 9 \\
w & = & 2\,1 & 1 & 2\,1 & 1 & 2 & 2 & 2\,1 \\
\mu_w^c & = & T^3 & F^2 & T^2 & F^1 & & & T^1 \\
\nu_w^c & = & T^2 & & T^1 & & F^1 & F^1 & T
\end{array}
$$

Tally the counts for each index of μ_w and μ_w^c.

$$
\begin{array}{ccccccc}
\mu_w & = & F^2 & T^2 & F^1 & T^1 & F \\
\mu_w^c & = & T^3 & F^2 & T^2 & F^1 & T^1 \\
 & & 5 & 4 & 3 & 2 & 1
\end{array}
$$

The des and maj reversing bijection on $\mathcal{W}_{n,2}$

$$
\begin{array}{lcccccccccc}
 & & & & & 4 & & 7 & & \\
w & = & 1 & 2 & 2 & \boxed{2\,1} & 1 & \boxed{2\,1} & 1 & 2 \\
\mu_w & = & F^2 & & & T^2 & F^1 & T^1 & F & \\
\nu_w & = & & F^2 & F^2 & T^1 & & T & F &
\end{array}
$$

Similarly, compute the major index of $f_2(w)$

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\begin{array}{lccccccccc}
 & & 1 & & 4 & & & & 9 \\
w & = & \boxed{2\,1} & 1 & \boxed{2\,1} & 1 & 2 & 2 & \boxed{2\,1} \\
\mu_w^c & = & T^3 & F^2 & T^2 & F^1 & & & T^1 \\
\nu_w^c & = & T^2 & & T^1 & & F^1 & F^1 & T
\end{array}
$$

Tally the counts for each index of μ_w and μ_w^c.

$$
\begin{array}{lcccccc}
\mu_w & = & F^2 & T^2 & F^1 & T^1 & F \\
\mu_w^c & = & T^3 & F^2 & T^2 & F^1 & T^1 \\
 & & 5 & 4 & 3 & 2 & 1
\end{array}
$$

Tally the counts for each index of ν_w and ν_w^c.

$$
\begin{array}{lcccccc}
\nu_w & = & F^2 & F^2 & T^1 & T & F \\
\nu_w^c & = & T^2 & T^1 & F^1 & F^1 & T \\
 & & 4 & 3 & 2 & 1 & 0
\end{array}
$$

Hence, $\mathrm{maj}(w) + \mathrm{maj}(f_2(w)) = \sum_{i=1}^n i + \sum_{i=1}^n (i-1) = n^2$.

Ascents and comajor index

Clearly, we can modify the map ϕ to encode *ascents* and the *comajor index* instead. Consider the same word $w \in \mathcal{W}_{5,2}$ as earlier,

$$w = \boxed{1\ 2}\ 2\ 2\ 1\ \boxed{1\ 2}\ 1\ \boxed{1\ 2}\,.$$

The map $\underline{\phi}$ sends this to

$$\underline{\mu}_w = \boxed{T}\ F\ \boxed{T}\ F\ \boxed{T}\,. \qquad \underline{\nu}_w = \boxed{T}\ F\ F\ \boxed{T}\ \boxed{T}\,.$$

The complements of these are

$$\underline{\bar{\mu}}_w = F\ \boxed{T}\ F\ \boxed{T}\ F\,. \qquad \underline{\bar{\nu}}_w = F\ \boxed{T}\ \boxed{T}\ F\ F\,.$$

And $\underline{\phi}^{-1}(\underline{\bar{\mu}}_w, \underline{\bar{\nu}}_w)$ gives

$$\underline{f}_2(w) = 2\ 1\ \boxed{1\ 2}\ 1\ \boxed{1\ 2}\ 2\ 2\ 1.$$

Clearly, $\text{asc}(w) + \text{asc}(\underline{f}_2(w)) = n$ and $\text{comaj}(w) + \text{comaj}(\underline{f}_2(w)) = n^2$.

The bijection f_2 also reverses ascents and comajor index

Theorem

For $w \in \mathcal{W}_{n,2}$*,*

$$\mathrm{asc}(w) + \mathrm{asc}(f_2(w)) = n,$$

and

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Ascents and descents in binary words are closely related. For any word $w = w_1 w_2 \dots w_n$, let $L(w) = w_2 \dots w_n$ and $R(w) = w_1 \dots w_{n-1}$. In our running example, we have

$$\begin{array}{ll} \mu_w = \mathrm{F}\ \boxed{\mathrm{T}}\ \mathrm{F}\ \boxed{\mathrm{T}}\ \mathrm{F}, & \underline{\mu}_w = \boxed{\mathrm{T}}\ \mathrm{F}\ \boxed{\mathrm{T}}\ \mathrm{F}\ \boxed{\mathrm{T}} \\ \nu_w = \mathrm{F}\ \mathrm{F}\ \boxed{\mathrm{T}}\ \boxed{\mathrm{T}}\ \mathrm{F}, & \underline{\nu}_w = \boxed{\mathrm{T}}\ \mathrm{F}\ \mathrm{F}\ \boxed{\mathrm{T}}\ \boxed{\mathrm{T}} \end{array}$$

Notice that $\underline{\mu}_w = L(\mu)\,\mathrm{T}$ and $\underline{\nu}_w = \mathrm{T}\,R(\nu)$.

The bijection f_2 also reverses ascents and comajor index

In general, the boolean encoding for ascents is related to the boolean encoding for descents as follows.

	$\mu_1 = \mathsf{F}$	$\mu_1 = \mathsf{T}$
$\nu_n = \mathsf{F}$	$\underline{\mu} = L(\mu)\,\mathsf{T}$ $\underline{\nu} = \mathsf{T}\,R(\nu)$	$\underline{\mu} = L(\mu)\,\mathsf{T}$ $\underline{\nu} = \mathsf{F}\,R(\nu)$
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The proof that f_2 also reverses ascents and comajor index as well follows from the fact that complementation leaves the table as it is.

Mirror-symmetric balanced words

We now define a subset of balanced binary words called *mirror-symmetric* words. The following word is mirror-symmetric

$$w = 1\ 2\ 2\ 2\ 2\ 1\ 2\ 2\ 2\ 1\ 2\ 1\ 1\ 1\ 2\ 1\ 1\ 1\ 1\ 2.$$

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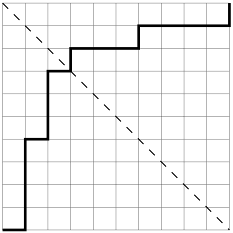
We can render it as a lattice path by taking an east step for a 1 and a north step for a 2. The path is invariant when reflected across the NW-SE diagonal.

A balanced binary word w is mirror-symmetric if

$$w_{n+1} \dots w_{2n} = \mathrm{rev}((w_1 \dots w_n)^c).$$



The bijection f_2 preserves mirror-symmetry

Theorem

Let $w \in \mathcal{W}_{n,2}$ be mirror-symmetric. Then so is $f_2(w)$.

The proof follows from the observation that

$$\mu = \text{rev}(\nu)$$

$$\nu = \text{rev}(\mu).$$

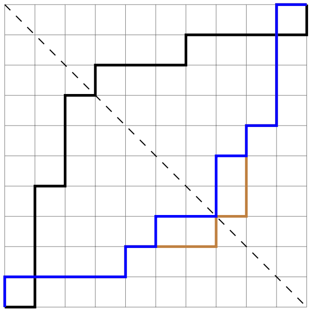


Figure: The black path is w, the blue path is $f_2(w)$, and the red path is the complement of w.

Overview of the general bijection

For $w \in \mathcal{W}_{n,k}$ and $k > 2$, we construct w in two steps.

$$w_{-k} \in \mathcal{W}_{n,k-1} \xleftrightarrow{\quad g(w_{-k}, \mu_w) \quad} \underline{w} \in \underline{\mathcal{W}}_{n,k} \xleftrightarrow{\qquad h(\underline{w}, \nu_w) \qquad} w$$

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The word w is encoded into a triple by

$$\begin{aligned} \Gamma_k(w) =& (w_{-k}, \mu_w, \nu_w) \\ \Gamma^{-1}((w_{-k}, \mu_w, \nu_w)) =& h(g(w_{-k}, \mu_w), \nu_w). \end{aligned}$$

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For example,

$$\begin{aligned} \Gamma_3(123322311321123) &= (1222112112, \mathrm{F\,F\,T\,F\,F\,T\,F\,F}, \mathrm{T\,F\,T\,F\,F\,F\,T}) \\ 123322311321123 &= h(g(1222112112, \mathrm{F\,F\,T\,F\,F\,T\,F\,F}), \mathrm{T\,F\,T\,F\,F\,F\,T}). \end{aligned}$$

Overview of the general bijection

We then define the bijection as

$$f_k(w) = \Gamma^{-1}(f_{k-1}(w_{-k}), \nu_w, \mu_k).$$

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In other words,

$$f_{k-1}(w_{-k}) \xleftrightarrow{g(f_{k-1}(w_{-k}), \nu_w)} \underline{f_k(w)} \xleftrightarrow{h(\underline{f_k(w)}, \mu_w)} f_k(w)$$

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For example,

$$\begin{aligned}
f_3(123322311321123) =& \Gamma^{-1}(1222112112, \mathrm{T\,F\,T\,F\,F\,F\,T}, \mathrm{F\,F\,T\,F\,F\,T\,F\,F}) \\
=& h(g(f_2(1222112112), \mathrm{T\,F\,T\,F\,F\,F\,T}), \mathrm{F\,F\,T\,F\,F\,T\,F\,F}) \\
=& h(g(2112112221, \mathrm{T\,F\,T\,F\,F\,F\,T}), \mathrm{F\,F\,T\,F\,F\,T\,F\,F}) \\
=& h(3211321122321, \mathrm{F\,F\,T\,F\,F\,T\,F\,F}) \\
=& 323113231122321.
\end{aligned}$$

The subword w_{-k} of w

Consider the word

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With this, we refine the descents of $w \in \mathcal{W}_{n,k}$ having $\operatorname{des}(w) = d$ as

$$\begin{gathered} \operatorname{des}(w) = d_1 + d_2, \\ d_1 = \operatorname{des}(w_{-k}) \\ d_2 = d - d_1 \end{gathered}$$

and say that $w \in W_{n,k}^{d_1,d_2}$.

The unique subword $\underline{w}$ of w

We construct a minimal subword $\underline{w}$ containing w_{-k} such that $\mathrm{des}(\underline{w}) = \mathrm{des}(w)$. Consider the word

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Finally, we can remove all but one 3 from a group of more than one 3's that is sandwiched between a non-descent.

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When all such 3's have been removed, we get

$$\underline{w} = 1\ 1\ 3\ 2\ 1\ 1\ 2\ 2\ 2.$$

Encoding $\underline{w}$ with respect to w_k with μ_w

Continuing the previous example, we annotate the non-descents in w_{-k}.

$$\begin{array}{cccccccccccccccccc} w_{-k} & = & \bullet & 1 & \bullet & 1 & \bullet & 2 & 1 & \bullet & 1 & \bullet & 2 & \bullet & 2 & \bullet & 2 \\ \underline{w} & = & \bullet & 1 & \bullet & 1 & 3 & 2 & 1 & \bullet & 1 & \bullet & 2 & \bullet & 2 & \bullet & 2. \end{array}$$

Represent the non-descents with the set $\mathrm{NDes}(w_{-k}) = \{0, 1, 2, 5, 6, 7, 8\}$.

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$$\mu_w \in B_{(k-1)n-d_1}^{d_2},$$

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and the map g has the form

$$g : \mathcal{W}_{n,k-1}^{d_1} \times B_{(k-1)n-d_1}^{d_2} \longrightarrow \underline{\mathcal{W}}_{n,k}^{d_1,d_2}.$$

Encoding w with respect to $\underline{w}$ with ν_w

Let $v \in \underline{\mathcal{W}}_{3,5}^{6,1}$ be the following word where we annotate the descents in v and the position at the end.

$$v = 3 \bullet 2\ 4\ 4 \bullet 1\ 5 \bullet 3 \bullet 1\ 2 \bullet 1\ 4 \bullet 3 \bullet 2 \bullet .$$

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We can construct a balanced word $w \in \mathcal{W}_{3,5}^{6,1}$ by inserting two 5's into any of the markers.

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As a boolean array, we can represent these arrangements as

$$\nu = \mathrm{F\,T\,T\,F\,F\,F\,F\,F\,F} \text{ and } \nu = \mathrm{F\,F\,F\,F\,F\,F\,T\,F\,T}.$$

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In general, given $w \in \mathcal{W}_{n,k}^{d_1,d_2}$,

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The map Γ_k and f_k

Lemma

The function Γ_k given by $\Gamma_k(w) = (w_{-k}, \mu_w, \nu_w)$ is a bijection.

Formally, define the map $f_k : \mathcal{W}_{n,k} \to \mathcal{W}_{n,k}$ for $k \geq 3$ by

$$f_k(w) = \Gamma_k^{-1}((f_{k-1}(w_{-k}), \nu_w, \mu_w)).$$

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$$f_k(w) = \Gamma_k^{-1}((f_{k-1}(w_{-k}), \nu_w, \mu_w)).$$

Theorem

For $k \geq 3$, f_k satisfies

$$\mathrm{des}(w) + \mathrm{des}(f_k(w)) = (k-1)n$$

and

$$\mathrm{maj}(w) + \mathrm{maj}(f_k(w)) = n^2 \binom{k}{2} \text{ for all } w \in \mathcal{W}_{n,k}.$$

The map f_k

To prove that the major index is reversed by f_k, we make use of the following lemma.

Lemma

Let $w \in \mathcal{W}_{n,k}$ for $k \geq 3$. Then,

$$\mathrm{maj}(w) - \mathrm{maj}(w_{-k}) + \mathrm{maj}(f_k(w)) - \mathrm{maj}(f_k(w)_{-k}) = (k-1)n^2.$$

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Corollary

The bijections satisfy the property $f_{k-1}(w_{-k}) = f_k(w)_{-k}$ for all $w \in \mathcal{W}_{n,k}$ for all $n \geq 1, k \geq 3$.

Applying f_k on permutations

Unlike the cases where $n > 1$, the descent sets of permutations behave in the following way under f_k.

Theorem

Let $w \in S_k$. *Then* $\mathrm{Des}(f_k(w)) = [k-1] \setminus \mathrm{Des}(w)$.

Example

$$\begin{array}{rcccccccccc} w & = & 4 & 6 & \color{red}{8} & \color{red}{7} & 2 & \color{red}{9} & \color{red}{5} & 1 & 3 \\ f_8(w) & = & \color{red}{7} & \color{red}{6} & 1 & 5 & \color{red}{8} & 3 & 4 & \color{red}{9} & 2 \end{array}$$

An example of f_k on $\mathcal{W}_{3,5}$

$w = 325544135121432$

$\underline{w} = 3254413121432$

$\mu_w = \mathrm{F\,T\,F\,F\,F\,F}$

$\nu_w = \mathrm{F\,T\,F\,F\,T\,F\,F\,F\,F}$

$f_5(w) = 154524225331134$

$w = 321312132$

$\underline{w} = 321312132$

$\mu_w = \mathrm{T\,T\,F\,T}$

$\nu_w = \mathrm{F\,F\,F\,F\,F}$

$f_3(w) = 122233113$

$w = 324413121432$

$\underline{w} = 3213121432$

$\mu_w = \mathrm{F\,F\,F\,T}$

$\nu_w = \mathrm{F\,T\,T\,F\,F\,F\,F\,F}$

$f_4(w) = 142422331134$

$w = 211212$

$\mu_w = \mathrm{T\,F\,T}$

$\nu_w = \mathrm{T\,T\,F}$

$f_2(w) = 122211$

Thank you!