Multitriangulations on ciliated surfaces

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A k-triangulation, or multitriangulation of order k, on the convex n-gon is a maximal set of edges (boundary edges and internal diagonals) such that no $k+1$ of them mutually intersect inside the polygon.

Setting $k = 1$, we recover the usual notion of a triangulation.

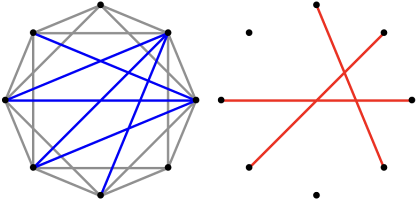

Figure 1: (left) A 2-triangulation of the octagon; (right) a forbidden 3-crossing

The simplicial complex of k-triangulations $\Delta_{n,k}$ has

- vertices corresponding to edges of the n-gon of length $> k$
- faces corresponding to $(k+1)$-crossing-free subsets of edges
- facets corresponding to k-triangulations

Theorem (Capoyleas–Pach '92; Nakamigawa '00; Dress–Koolen–Moulton '02)

All k-triangulations of the n-gon have the same number of edges, equal to $k(2n-2k-1)$.

Theorem (Nakamigawa '00, Pilaud–Santos '08)

Any edge e in a k-triangulation T of length at least $k+1$ can be flipped uniquely to obtain a k-triangulation $(T \setminus \{e\}) \cup \{f\}$. The graph of flips is regular and connected.

Stump ('11) and Pilaud–Pocchiola ('12) give a bijection between k-triangulations of the n-gon and reduced pipe dreams for the permutation $\pi_{n,k} := [1, 2, \ldots, k, n-k, n-k-1, \ldots, k+1, n-k+1, n-k+2, \ldots, n-1]$. To obtain a reduced pipe dream from a k-triangulation T:

- label the columns of the pipe dream $1, \ldots, n-1$ ($\rightarrow$) and the rows $n, \ldots, 2$ ($\downarrow$)
- for any edge $(i,j) \in T$, $i > j$, place a bump tile in row i and column j
- for any edge $(i,j) \notin T$, $i > j$, place a crossing tile in row i and column j

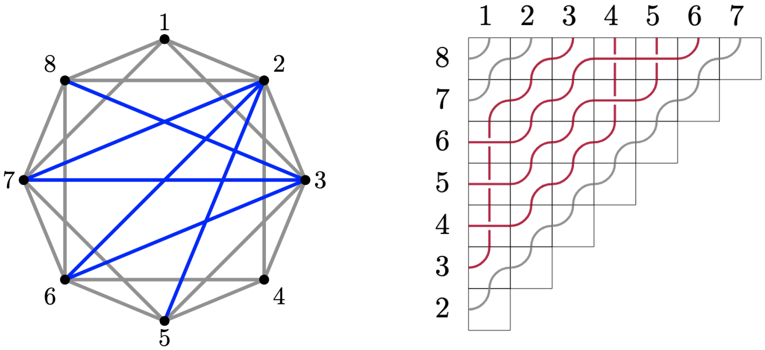

Figure 2: (left) A 2-triangulation of the octagon; (right) the corresponding reduced pipe dream for $\pi = [1, 2, 6, 5, 4, 3, 7]$

Stump ('11) and Pilaud–Pocchiola ('12) give a bijection between k-triangulations of the n-gon and reduced pipe dreams for the permutation $\pi_{n,k} := [1, 2, \ldots, k, n-k, n-k-1, \ldots, k+1, n-k+1, n-k+2, \ldots, n-1]$. To obtain a reduced pipe dream from a k-triangulation T:

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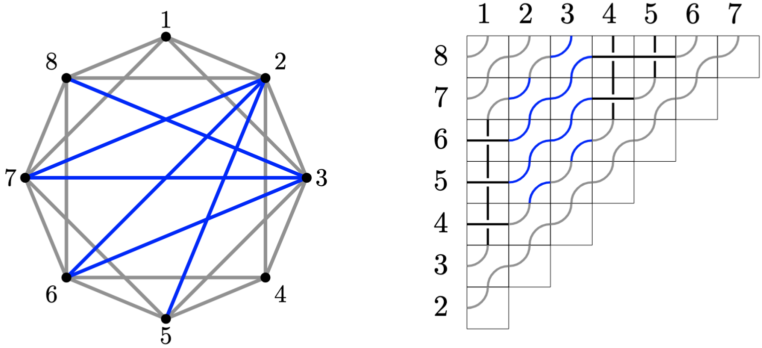


Figure 3: (left) A 2-triangulation of the octagon; (right) the corresponding reduced pipe dream for $\pi = [1, 2, 6, 5, 4, 3, 7]$

Pilaud and Santos (2008) show that k-triangulations decompose as complexes of k-star polygons.

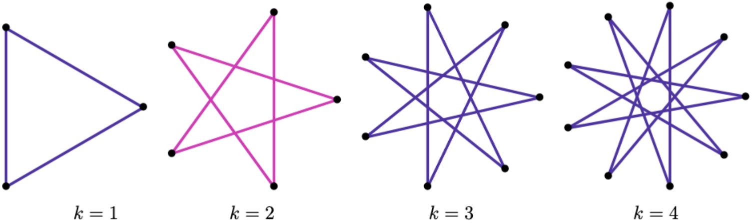


Figure 4: k-star polygons for $k = 1, 2, 3, 4$

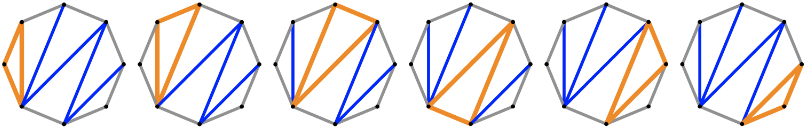


Figure 5: A triangulation on the octagon has 6 triangles. Each internal edge (length > 1) participates in exactly 2 triangles. Each boundary edge (length $= 1$) participates in exactly 1 triangle.

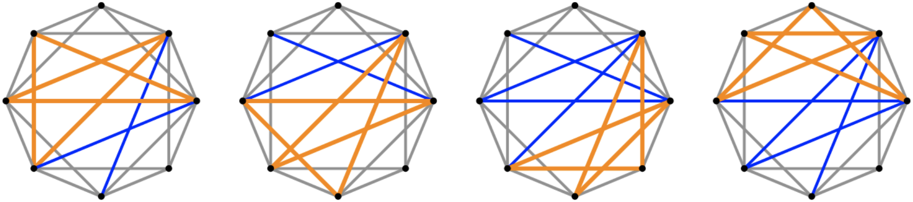


Figure 6: A 2-triangulation of the octagon has four 2-stars. Each edge of length > 2 participates in exactly 2 stars. Each edge of length 2 participates in exactly 1 star.

The edges of a star in a k-triangulation correspond directly to the bump tiles a pipe passes through.

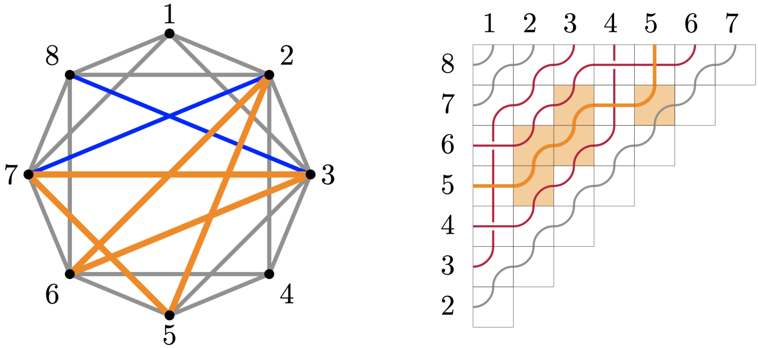


Figure 7: A 2-star in a 2-triangulation of the octagon and the corresponding pipe

Let X be a ciliated surface, let $\overline{X}$ denote its universal cover, and let $\pi : \overline{X} \to X$ be an associated covering map.

Definition

A k-triangulation on X is a maximal set E of edges such that $\pi^{-1}(E) \subseteq \overline{X}$ contains no $(k+1)$-crossing (i.e., no $k+1$ edges mutually intersect in the interior of $\overline{X}$). [a]

[a] Multitriangulations on surfaces are implicitly defined with additional conditions in Lepoutre–Pilaud ('19). S.–Tung–Yang–Zhang ('25) show that the strong and weak definitions are not equivalent; when $k = 2$ on the half-cylinder, the weak definition agrees with the definition implicitly used by Lepoutre and Pilaud.

- When X is a disc with n marked points on the boundary, we recover a k-triangulation on the convex n-gon
- Setting $k = 1$, we recover ideal triangulations on ciliated surfaces

Let $\mathcal{C}_n$ denote the half-cylinder with n marked points on a single boundary.

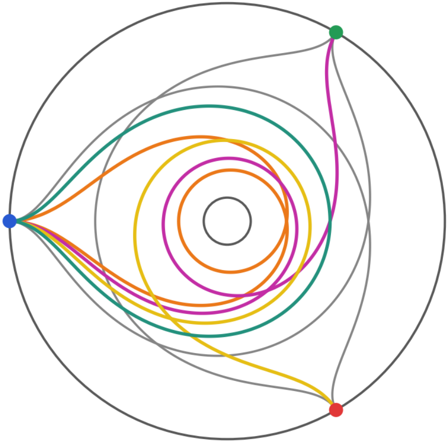
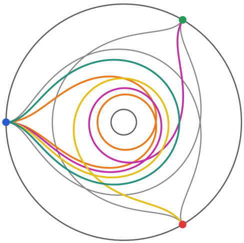


Figure 8: A 2-triangulation T on $\mathcal{C}_3$, the half-cylinder with 3 marked points on a single boundary



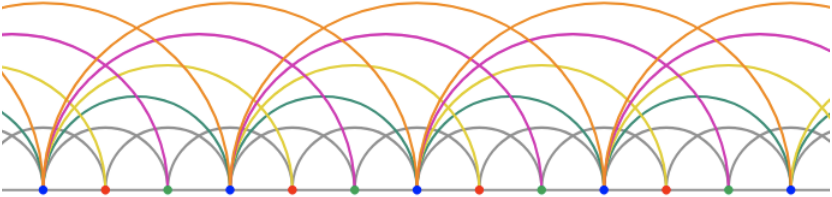


Figure 9: (top) A 2-triangulation T on $\mathcal{C}_3$, the half-cylinder with 3 marked points on a single boundary; (bottom) the lift $\pi^{-1}(T)$ to the universal cover

Theorem (S.–Tung–Yang–Zhang '25)

Every 2-triangulation on the half-cylinder $\mathcal{C}_n$ admits a 2-star decomposition: after lifting to the universal cover, each edge of length > 2 and $< 2n$ participates in exactly 2 stars. Each edge of length 2 or $2n$ participates in exactly 1 star.

Conjecture (S.–Tung–Yang–Zhang '25)

Every k-triangulation on the half-cylinder $\mathcal{C}_n$ admits a k-star decomposition: each edge of length $> k$ and $< nk$ participates in exactly 2 k-stars. Each edge of length k or nk participates in exactly 1 star.

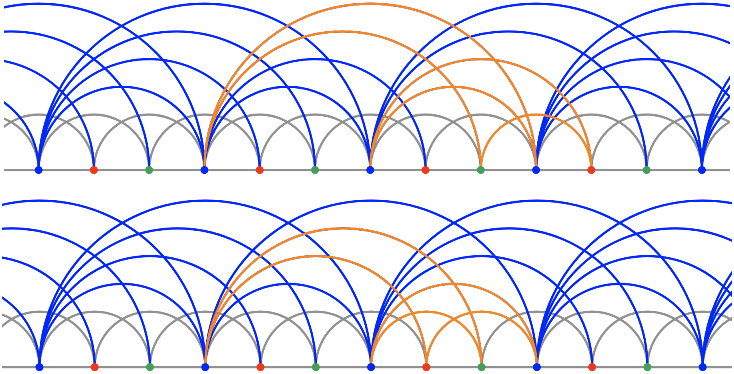


Figure 10: A 2-triangulation on $\mathcal{C}_3$ has two 2-stars. Each edge of length > 2 and < 6 participates in both stars. Each edge of length 2 or 6 participates in exactly 1 star.

The simplicial complex of k-triangulations on the half-cylinder with n marked points on a single boundary, $\Delta_{\mathcal{C}_n,k}$, has

- vertices corresponding to edges of length $> k$
- facets corresponding to k-triangulations on $\mathcal{C}_n$

The simplicial complex $\Delta_{\mathcal{C}_n,1}$ is the type B/C cluster complex, a subword complex, and a vertex-decomposable sphere. More generally, $\Delta_{X,1}$ is pure, has the flip property, and is a sphere when finite:

Theorem (Hatcher '91)

For a ciliated surface with no internal marked points, if the arc complex is finite (has finitely many vertices), then it is a sphere (the only two possibilities are n-gons and half-cylinders). Otherwise, it is contractible.

Theorem (Fomin–Shapiro–Thurston '08)

For any ciliated surface, if the tagged arc complex is finite (has finitely many vertices), then it is a sphere (the only possibilities are n-gons, half-cylinders, polygons with one internal marked point, and closed surfaces with finitely many internal marked points). Otherwise, it is contractible.

Theorem (S.–Tung–Yang–Zhang '25)

There is a canonical bijection between 2-triangulations on $\mathcal{C}_n$ and 2-triangulations of the $4n$-gon invariant under rotation by $\pi/2$.

Conjecture (S.–Tung–Yang–Zhang '25)

There is a canonical bijection between k-triangulations on $\mathcal{C}_n$ and k-triangulations on the $2nk$-gon invariant under rotation by π/k.

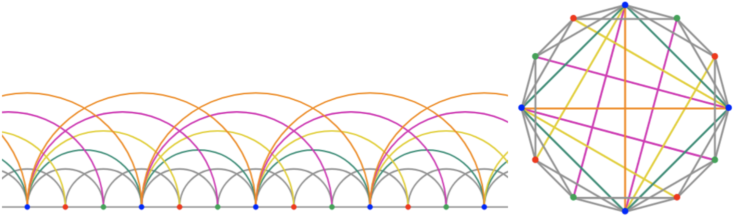


Figure 11: A 2-triangulation on $\mathcal{C}_3$ lifted to the universal cover and the corresponding 2-triangulation on the 12-gon invariant under rotation by $\pi/2$ ("3-periodic").

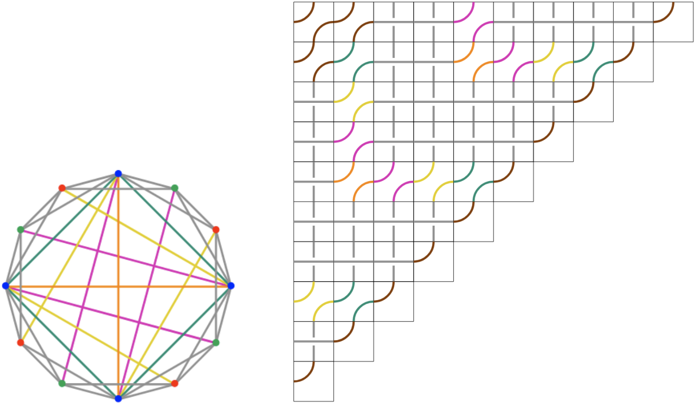


Figure 12: (left) A 3-periodic 2-triangulation of the 12-gon; (right) the corresponding reduced pipe dream for the permutation $[1, 2, 10, 9, 8, 7, 6, 5, 4, 3, 11]$

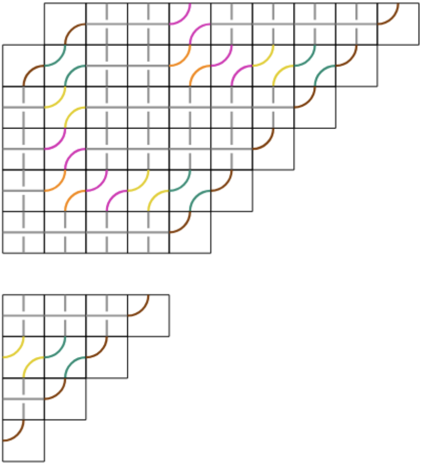


Figure 13: Step 1: The NW-most 2 pipes are pruned and the corner box is removed. The bottom 4 rows are separated from the remainder of the pipe dream.

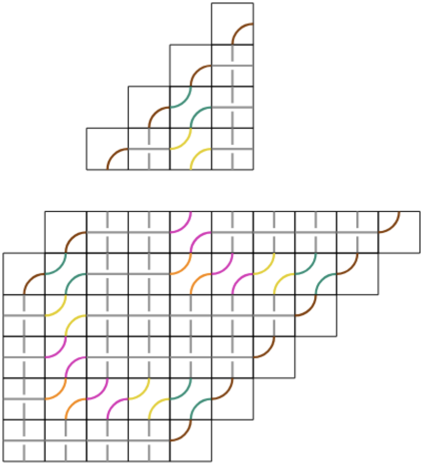


Figure 14: Step 2: The separated 4-row staircase is rotated 180°, shifted, and reflected.

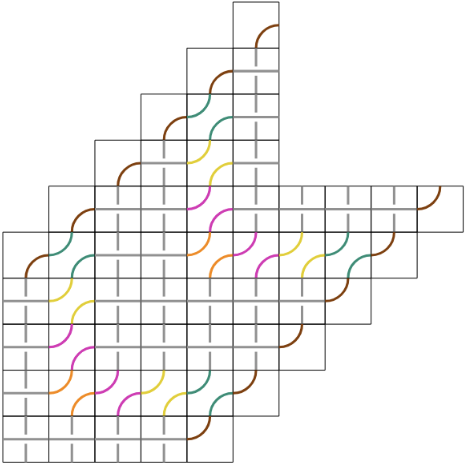
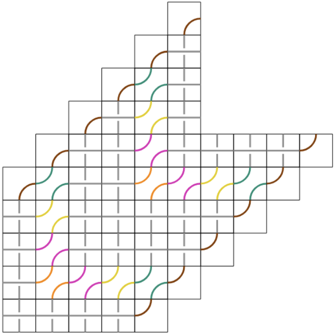


Figure 15: Step 3: The separated 4-row staircase is reattached to obtain a chevron pipe dream. [1]

[1] This construction recovers special cases of results of Ceballos, Labbe, and Stump ('11) on multi-cluster complexes

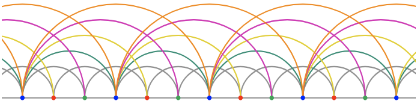


The chevron pipe dream in the above example records a subword of

$$\left(s_1 s_2 s_3 s_4\right)^6 = c^6$$

in type B_4. Deleting the two crossed-out copies of c gives a reduced expression for the longest element w_0:

$$w_0 = s_1 s_2 s_3 s_4 \cancel{s_1 s_2 s_3 s_4} s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_4 \cancel{s_1 s_2 s_3 s_4} s_1 s_2 s_3 s_4.$$



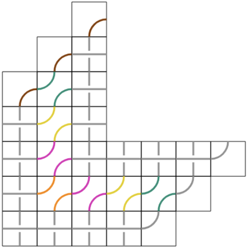


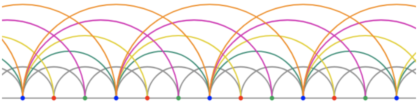
Figure 16: Left: a 2-triangulation on the half-cylinder $\mathcal{C}_3$ lifted to the universal cover; right: half of the corresponding chevron pipe dream.

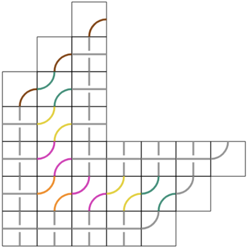
The chevron pipe dream in the above example records a subword of

$$(s_1 s_2 s_3 s_4)^3 = c^3$$

in type B_4. Deleting the crossed-out copy of c gives a reduced expression for an element w satisfying $w^2 = w_0$.

$$w = s_1 s_2 s_3 s_4 \cancel{s_1 s_2 s_3 s_4} s_1 s_2 s_3 s_4.$$





Theorem (S.–Tung–Yang–Zhang '25)

The simplicial complex $\Delta_{\mathcal{C}_n,2}$ is isomorphic to the union of subword complexes

$$\bigcup_{w^2=w_0} \mathcal{SC}(c^n, w)$$

where w_0 is the longest element of B_{2n-2} and c is the Coxeter word $s_1 s_2 \ldots s_{2n-2}$.

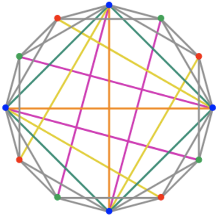
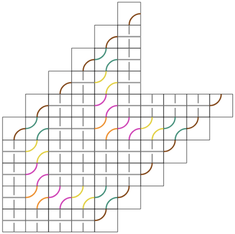
 

Figure 17: (left) A 3-periodic 2-triangulation of the 12-gon; (right) the chevron pipe dream.

Conjecture (S.–Tung–Yang–Zhang '25)

The simplicial complex $\Delta_{\mathcal{C}_{n},k}$ is isomorphic to the union of subword complexes

$$\bigcup_{w \in \mathcal{A}_k} \mathcal{SC}(c^n, w)$$

where w_0 is the longest element of B_{nk-k}, c is the Coxeter word $s_1 s_2 \dots s_{nk-k}$, and $\mathcal{A}_k = \{w \in B_{nk-k} \mid w^k = w_0, \ell(w) = \ell(w_0)/k\}$.

Theorem (S.–Tung–Yang–Zhang '25)

Every 2-triangulation on $\mathcal{C}_n$ has exactly $2(n-1)$ edges of length > 2. The simplicial complex $\Delta_{\mathcal{C}_n,2}$ is pure of dimension $2n-3$.

Theorem (S.–Tung–Yang–Zhang '25)

Every edge e of length > 2 in a 2-triangulation T on $\mathcal{C}_n$ can be flipped uniquely to obtain a 2-triangulation $(T \setminus \{e\}) \cup \{f\}$. The simplicial complex $\Delta_{\mathcal{C}_n,2}$ is a weak pseudomanifold without boundary.

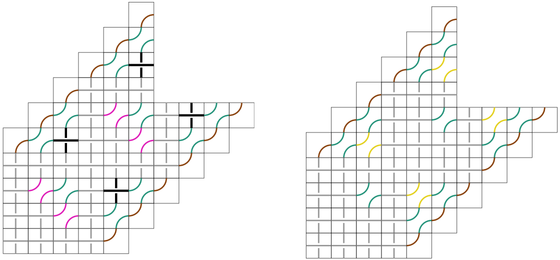


Figure 18: (left) The chevron pipe dream corresponding to a 2-triangulation T on $\mathcal{C}_3$; (right) the chevron pipe dream for $(T \setminus \{e\}) \cup \{f\}$

Conjecture (S.–Tung–Yang–Zhang '25)

The simplicial complex $\Delta_{\mathcal{C}_{n,k}}$ is a piecewise-linear sphere.

Conjecture (S.–Tung–Yang–Zhang '25)

The simplicial complex $\Delta_{\mathcal{C}_{n,k}}$ has $\binom{2(n-1)}{n-1}^k$ facets.

Thank you!