Electroid Varieties and the Lagrangian Grassmannian

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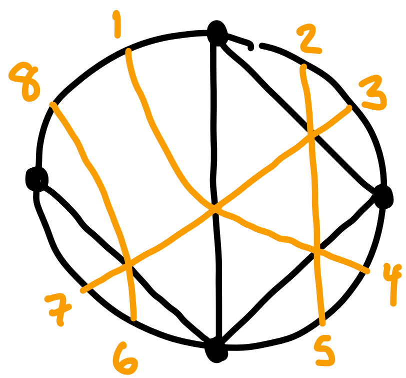
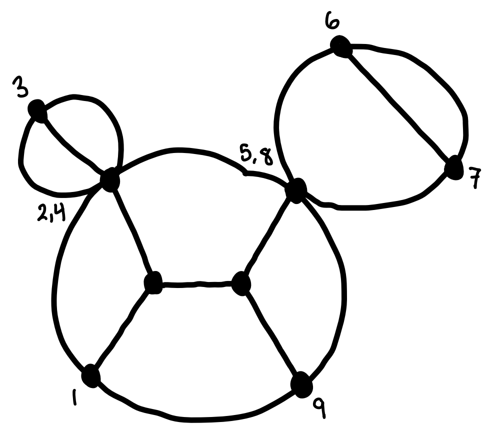
Joint work w/ Dawei Shen + David Speyer

FPSAC 2026





[ArXiV 2607.05576]

	Grassmannian	Lagrangian Grassmannian
$\mathbb{R}_{\geq 0}$ (TNN)	Positroid cells stratify $Gr(k,n)_{\geq 0}$ [Postnikov '06]	Electroid cells stratify $LG(n,2n)_{\geq 0}$ [Lam '14] [Chepuri-George-Speyer '21] [Bychkov-Gorbounov-Kazakov-Talalaev '21]
$\mathbb{C}$	Positroid varieties stratify $Gr(k,n)$ [Knutson-Lam-Speyer '11]	?

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The Grassmannian

$Gr(k,n) = \{k\text{-dim subspaces of an } n\text{-dim vspace}\}$

Plücker embedding

$$Gr(k,n) \hookrightarrow \mathbb{P}^{\binom{[n]}{k}-1}$$

$$V \longmapsto \big(\underbrace{\Delta_I(V)}_{\substack{\text{minor} \\ \text{indexed by cols } I}}\big)_{I \in \binom{[n]}{k}}$$

Ex $\begin{bmatrix} a & 0 & 0 & c \\ 0 & b & 0 & 0 \end{bmatrix} \longmapsto [\underset{\Delta_{12}}{ab} : \underset{\Delta_{13}}{0} : \underset{\Delta_{14}}{0} : \underset{\Delta_{23}}{0} : \underset{\Delta_{24}}{-bc} : \underset{\Delta_{34}}{0}]$

Positroid varieties [Knutson-Lam-Speyer '11]

$\Pi_f = \{ \Delta_I = 0 \text{ for specific } I \}$ closed positroid variety

$\uparrow$

bounded affine perm (or Grassmann necklace, Le diagram, etc)

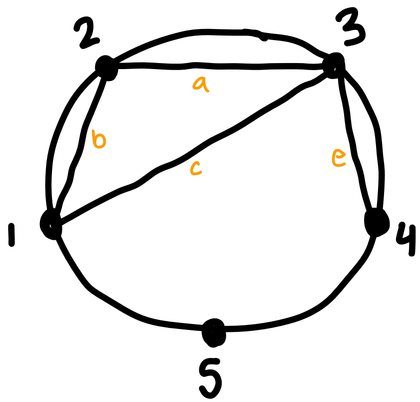
$\mathring{\Pi}_f = \{ \Delta_I = 0 \text{ for specific } I,\ \Delta_J \neq 0 \text{ for specific } J \}$ open positroid variety

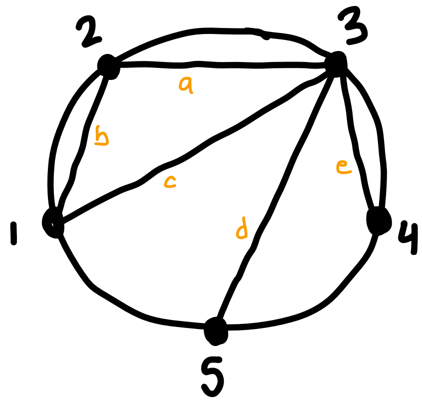
$\mathring{\Pi}_{f,\geq 0} = \mathring{\Pi}_f \cap \underbrace{Gr(k,n)_{\geq 0}}_{\{V \mid \Delta_I(V) \geq 0 \ \forall I\}}$ positroid cell

<u>Thm</u> The Grassmannian admits a stratification by positroid varieties

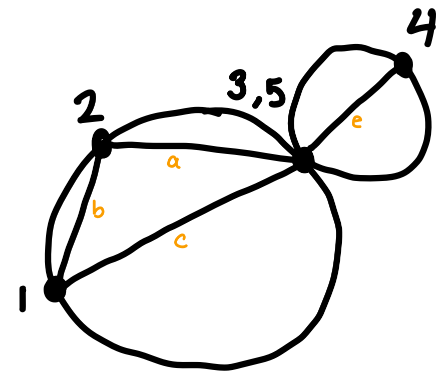
$$Gr(k,n) = \bigsqcup_f \mathring{\Pi}_f \qquad \Pi_f = \overline{\mathring{\Pi}_f} = \bigsqcup_{g \leq f} \mathring{\Pi}_g$$

Electrical networks





electrical network



cactus network

$$0 \longleftarrow d \longrightarrow \infty$$

Def/Thm [Lam '14]

E_n = (compactified) space of electrical networks w/ n boundary vts $/\sim_{elec}$

= space of cactus networks " ———————— "

Connecting electrical networks + the Grassmannian

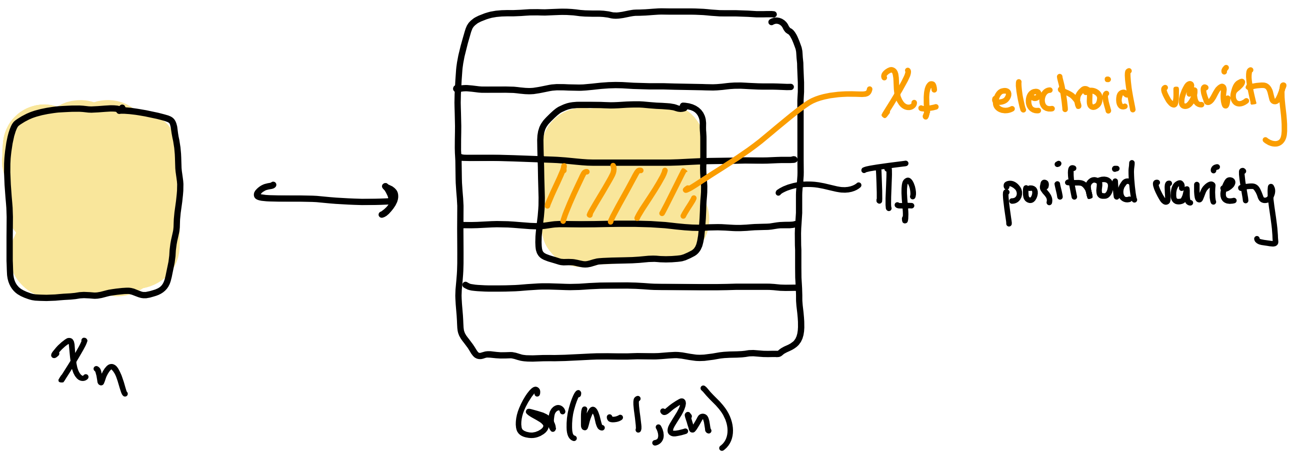
[Lam '14] constructed map

$$i : E_n \xhookrightarrow{\text{linear} \,\circ\, \text{grove map}} \mathbb{P}^{\binom{[2n]}{n-1}-1}$$

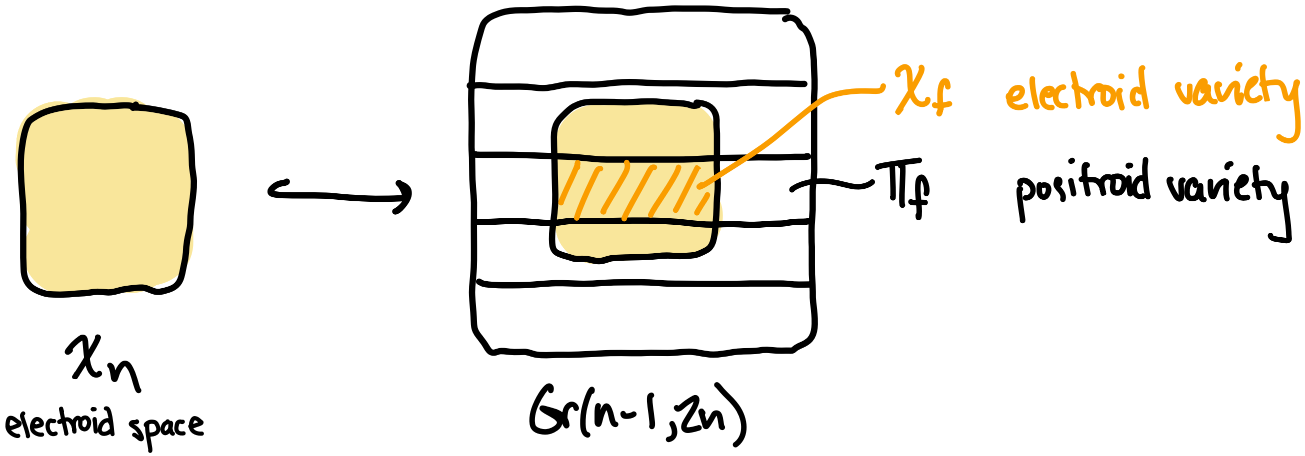
<u>Thm</u> [Lam '14] There is an embedding $i : E_n \hookrightarrow Gr(n-1, 2n)_{\geq 0}$.

<u>Def</u> Electroid space is $\mathcal{X}_n := \overline{i(E_n)}$ (= linear slice of $Gr(n-1,2n)$)

[Gao-Lam-Xu '24]



Electroid Varieties



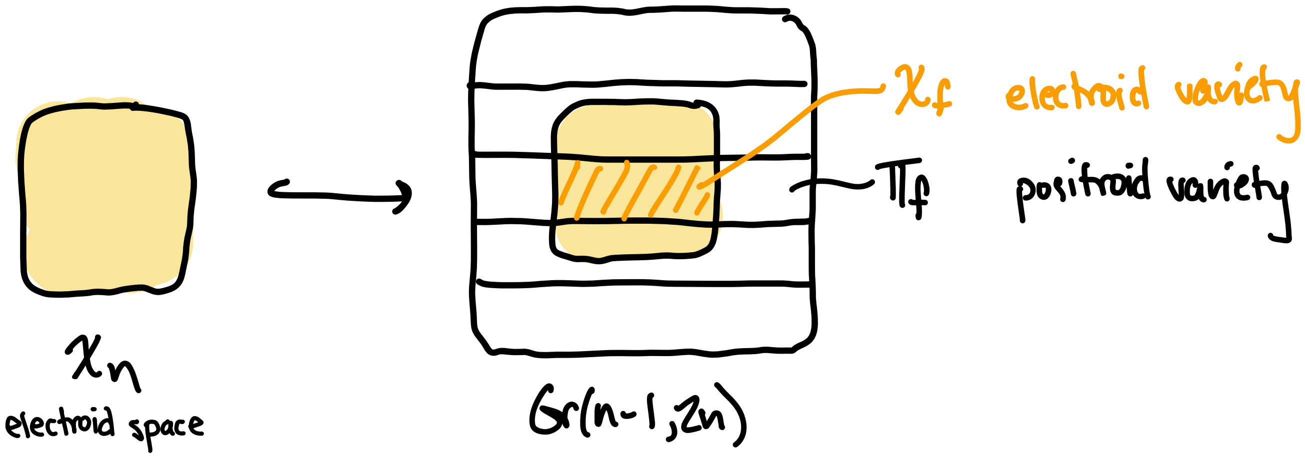
Closed electroid variety: $\mathcal{X}_f := \mathcal{X}_n \cap \Pi_f$

Open electroid variety: $\mathring{\mathcal{X}}_f := \mathcal{X}_n \cap \mathring{\Pi}_f$

Electroid cell: $\mathring{\mathcal{X}}_{f,\geq 0} := \mathring{\mathcal{X}}_f \cap Gr(n-1,2n)_{\geq 0}$

Prop [Shen-S.-Speyer, '26] $\mathcal{X}_f = \overline{\mathring{\mathcal{X}}_f} = \overline{\mathring{\mathcal{X}}_{f,\geq 0}}$

Electroid Varieties



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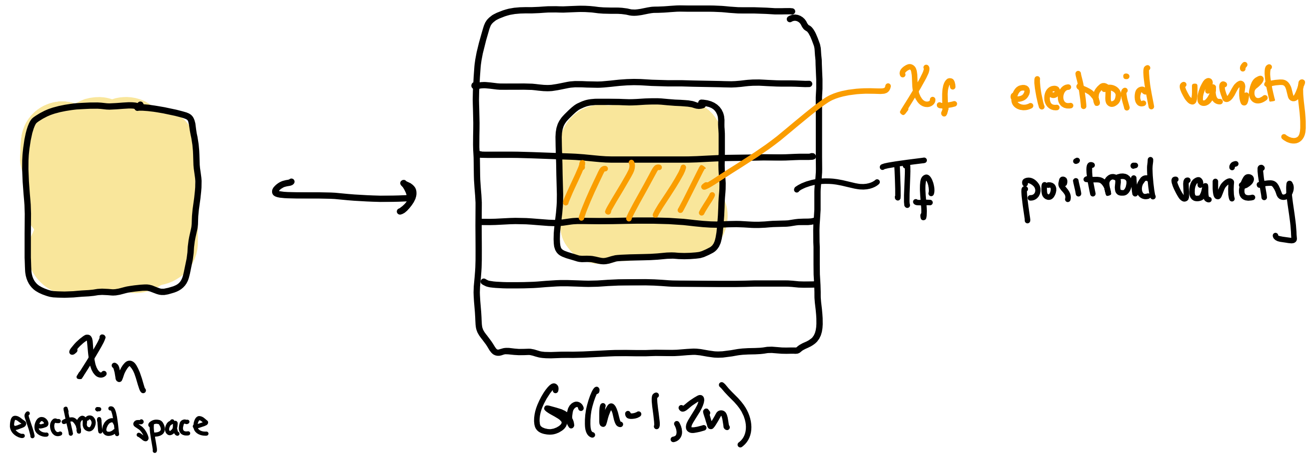
Q: Correct indexing set for electroid varieties of $\mathcal{X}_n$?

Electroid Varieties



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Prop [Shen-S.-Speyer, '26] $\mathcal{X}_f = \overline{\mathring{\mathcal{X}}_f} = \overline{\mathring{\mathcal{X}}_{f,\geq 0}}$

Q: Correct indexing set for electroid varieties of $\mathcal{X}_n$? Matchings on $[2n]$

Connecting electrical networks + the Lagrangian Grassmannian

<u>Thm</u> [Chepuri-George-Speyer '21, Bychkov-Gorbounov-Kazakov-Talalaev '21]

Electroid space $\mathcal{X}_n$ is a Lagrangian Grassmannian.

$$\underbrace{LG(n-1,2n-2)}_{\text{type C}} \cong \mathcal{X}_n \subset \underbrace{Gr(n-1,2n)}_{\text{type A}}$$

<u>Upshot</u> Electroid varieties are the type-C analog of positroid varieties

⚠ Warning! Although $\mathcal{X}_n$ is isomorphic to Karpman's Lagrangian Grassmannian, it's embedded differently in the Grassmannian and so inherits a different decomposition.

	Grassmannian	Lagrangian Grassmannian
$\mathbb{R}_{\geq 0}$ (TNN)	Positroid cells stratify $Gr(k,n)_{\geq 0}$ [Postnikov '06]	Electroid cells stratify $LG(n,2n)_{\geq 0}$ [Lam '14] [Chepuri-George-Speyer '21] [Bychkov-Gorbounov-Kazakov-Talalaev '21]
$\mathbb{C}$	Positroid varieties stratify $Gr(k,n)$ [Knutson-Lam-Speyer '11]	Electroid varieties stratify $LG(n,2n)$ [Shen-S.-Speyer '26]

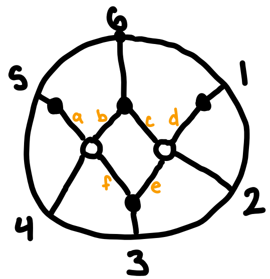
Stratification of the TNN Grassmannian: Type A

Thm [Postnikov'06] $Gr(k,n)_{\geq 0}$ has a stratification into positroid cells parameterized by plabic graphs.

$\mathring{\Pi}_{f,\geq 0}$

positroid cell

cell



plabic graph

parameterizing graph

$f = [2\,4\,6\,5\,7\,9]$

bounded affine perm

indexing object

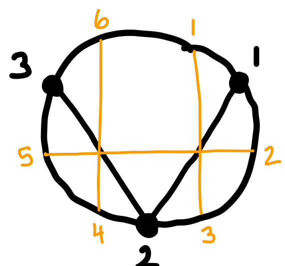
Stratification of the TNN Lagrangian Grassmannian: Type C

Thm [Lam'14] $(\mathcal{X}_n)_{\geq 0} \cong LG(n-1, 2n-2)_{\geq 0}$ has a stratification into electroid cells parameterized by cactus networks.

$\mathring{\mathcal{X}}_{\tau,\geq 0}$

electroid cell

cell



cactus network

parameterizing graph

$\tau = 13, 25, 46$

matching on $[2n]$

indexing object

Positroid Varieties

Thm [Knutson-Lam-Speyer '11] The Grassmannian admits a stratification by open positroid vars,

$$\mathrm{Gr}(k,n) = \bigsqcup \mathring{\Pi}_f$$

(i) $(\mathring{\Pi}_f)_{\geq 0}$ is Zariski dense in $\mathring{\Pi}_f$

(ii) $\mathring{\Pi}_f$ is irred, smooth, has expected dim

(iii) Π_f is reduced, irred, normal

(iv) There's a Frobenius splitting on $\mathrm{Gr}(k,n)$ under which all Π_f are compatibly split

Thm [Muller-Speyer '16]

The boundary measurement map can be extended algebraically to embed a torus $(\mathbb{C}^*)^\ell$ as a dense open subscheme of $\mathring{\Pi}_f$

Thm [Gorsky-Scroggin '24, Gorsky-Kim-Scroggin-Simental '25]

There exists a splicing open embedding, $\mathring{\Pi}_{\lambda_1/\mu_1} \times \mathring{\Pi}_{\lambda_2/\mu_2} \hookrightarrow \mathring{\Pi}_{\lambda/\mu}$.

Electroid Varieties

Thm [Shen-S.-Speyer '26] The electroid space $\mathcal{X}_n$ admits a stratification by open electroid vars,

$$\mathcal{X}_n = \bigsqcup \mathring{\mathcal{X}}_\tau$$

(i) $(\mathring{\mathcal{X}}_\tau)_{\geq 0}$ is Zariski dense in $\mathring{\mathcal{X}}_\tau$

(ii) $\mathring{\mathcal{X}}_\tau$ is irred, smooth, has expected dim

(iii) $\mathcal{X}_\tau$ is reduced, irred, reg in codim 1

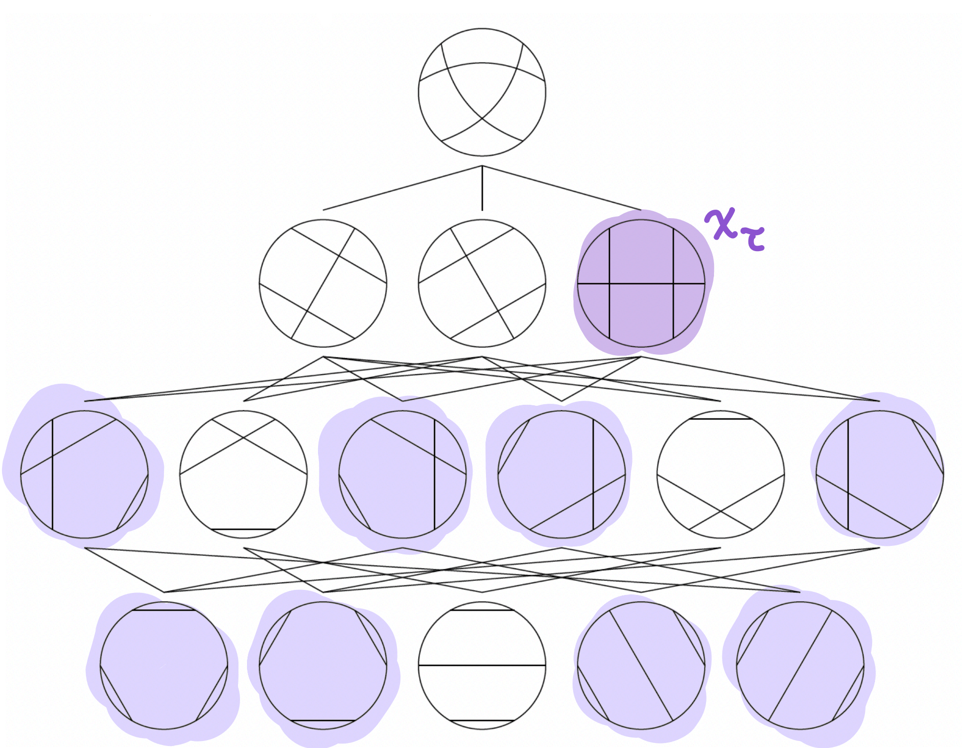
(iv) There is a Frobenius splitting on $\mathcal{X}_n$ under which all $\mathcal{X}_\tau$ are compatibly split.

(v) The grove measurement map can be extended to embed a torus $(\mathbb{C}^*)^\ell$ as a dense open subscheme of $\mathring{\mathcal{X}}_\tau$.

(vi) There's a splicing isomorphism $\mathcal{X}_{\tau_1} \times \mathcal{X}_{\tau_2} \cong \mathcal{X}_{\tau_1 \sqcup \tau_2}$ respecting stratifications on both sides.

Combinatorial Tool

~ Matchings underlie everything ~



Poset of matchings on [6]

dictates combinatorics of $\mathcal{X}_3$

Thm [Shen-S.-Speyer '26]

The dimension of $\mathring{\mathcal{X}}_\tau$ and $\mathcal{X}_\tau$ is $\underbrace{c(\tau)}_{\text{min \# crossings in } \tau}$

Thm [KLS '11]

$$\Pi_f = \bigsqcup_{f' \leq f} \mathring{\Pi}_{f'}$$

Bruhat order

Thm [Lam '14, Shen-S.-Speyer '26]

$$\mathcal{X}_\tau = \bigsqcup_{\tau' \leq \tau} \mathring{\mathcal{X}}_{\tau'}$$

uncrossing

Combinatorial Tool

~ Matchings underlie everything ~

Def The pair $\tau \lessdot \tau'$ is an *i-crossing pair* if τ' is formed from τ by crossing strands $i, i+1$ s.t. $\tau'(i) = \tau(i+1)$ and $\tau'(i+1) = \tau(i)$.

Ex

$\tau \lessdot \tau'$

Prop [Shen-S.-Speyer '26] Let $\tau \lessdot \tau'$ be an i-crossing pair. Then, we have an open embedding,

$$\psi : \mathring{X}_\tau \times \mathbb{C}^* \longrightarrow \mathring{X}_{\tau'}$$

onto the dense open subscheme of $\mathring{X}_{\tau'}$ given by the vanishing of a Plücker $\Delta_{I'}$.

Thm [Shen-S.-Speyer '26]
$\mathring{X}_\tau$ is irred, smooth, and has dimension $c(\tau)$.

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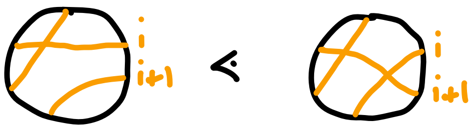

$\tau \quad \lessdot \quad \tau'$

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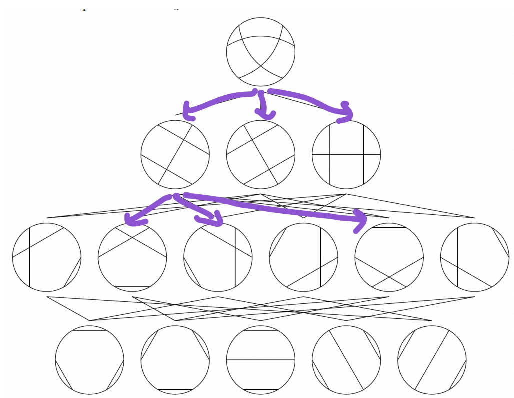

Pf/ Induction + proposition.

Summary

1. $\left\{\begin{matrix}\text{Positroid}\\ \text{Electroid}\end{matrix}\right\}$ varieties stratify $\left\{\begin{matrix}Gr(k,n)\\ LG(n-1,2n)\end{matrix}\right\}$

2. $\left\{\begin{matrix}\text{Positroid}\\ \text{Electroid}\end{matrix}\right\}$ varieties have "nice" algebro-geometric properties.

   $\hookrightarrow$ $\left\{\begin{matrix}\Pi_f\\ \mathcal{X}_g\end{matrix}\right\}$ reduced, irred, compatibly F-split, $\left\{\begin{matrix}\text{normal}\\ \text{regular in codim } 1\end{matrix}\right\}$

3. Thus far, the story of electroid varieties has paralleled that of positroid varieties. We hope to continue developing these parallels in the future!

References

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3. Yibo Gao, Thomas Lam, and Zixuan Xu, *Electrical networks and the grove algebra*, Canad. J. Math. **77** (2025), no. 2, 631–664.
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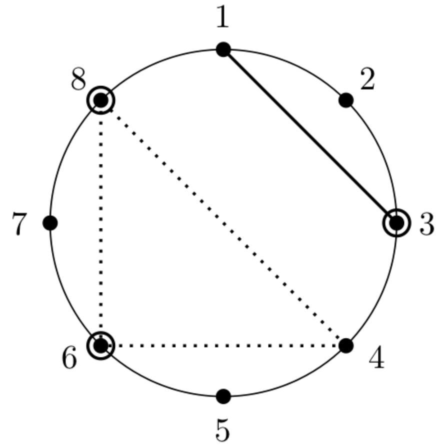

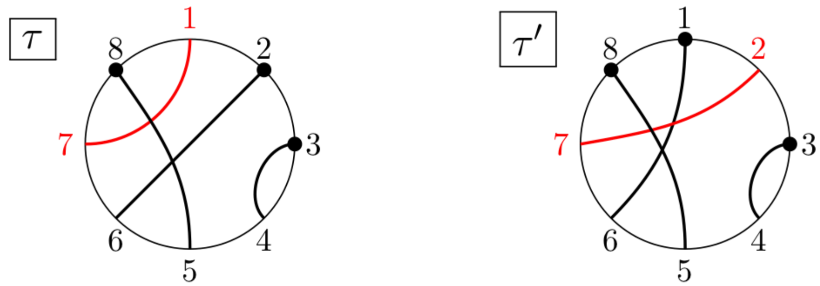

THANKS!

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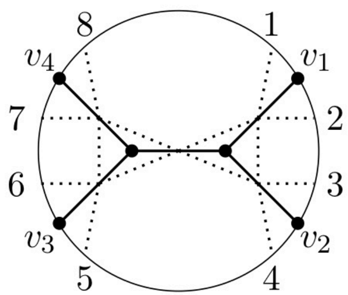

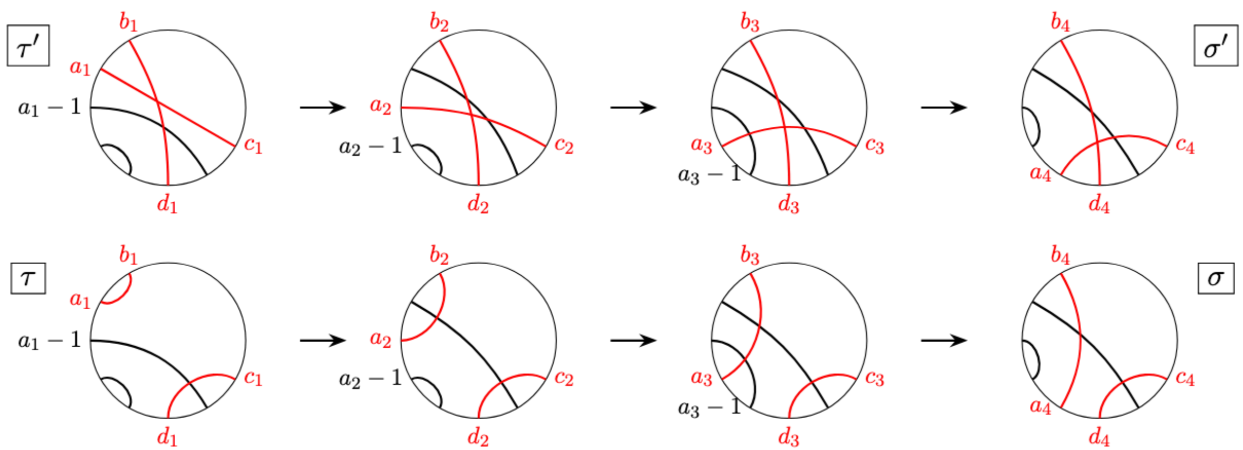

	Grassmannian	Lagrangian Grassmannian	Orthogonal Grassmannian
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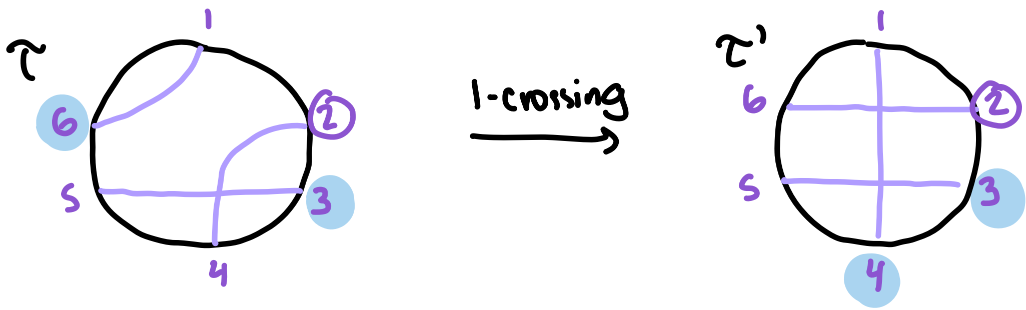
$\mathbb{C}$	Positroid varieties stratify $Gr(k,n)$ [Knutson-Lam-Speyer '11]	Electroid varieties stratify $LG(n,2n)$ [Shen-S.-Speyer '26]	?

Combinatorial Tool

~ Matchings underlie everything ~

Key ingredient in embedding proposition

★ From τ, we can read off which Plückers **must** be zero | nonzero.



For each strand which isn't $(i, \tau(i))$, add endpt which is reached first when going c-wise from i to $I_i + 1$.

$I + 1 = (23, 36, 46, 56, 62, 23)$

$I = (12, 25, 35, 45, 51, 12)$

$I + 1 = (23, 34, 46, 56, 61, 13)$

$\tilde{I} = (12, 23, 35, 45, 56, 62)$ || These Plückers nonzero

|| Subsets "less" have $\Delta_J = 0$

$$I_{i+1}(\tau') + 1 = I_{i+1}(\tau) + 1 - \tau(i) \cup \tau(i+1)$$

$$I_{i+1}(\tau') = I_{i+1}(\tau) - f(i) \cup f(i+1)$$

Combinatorial Tool

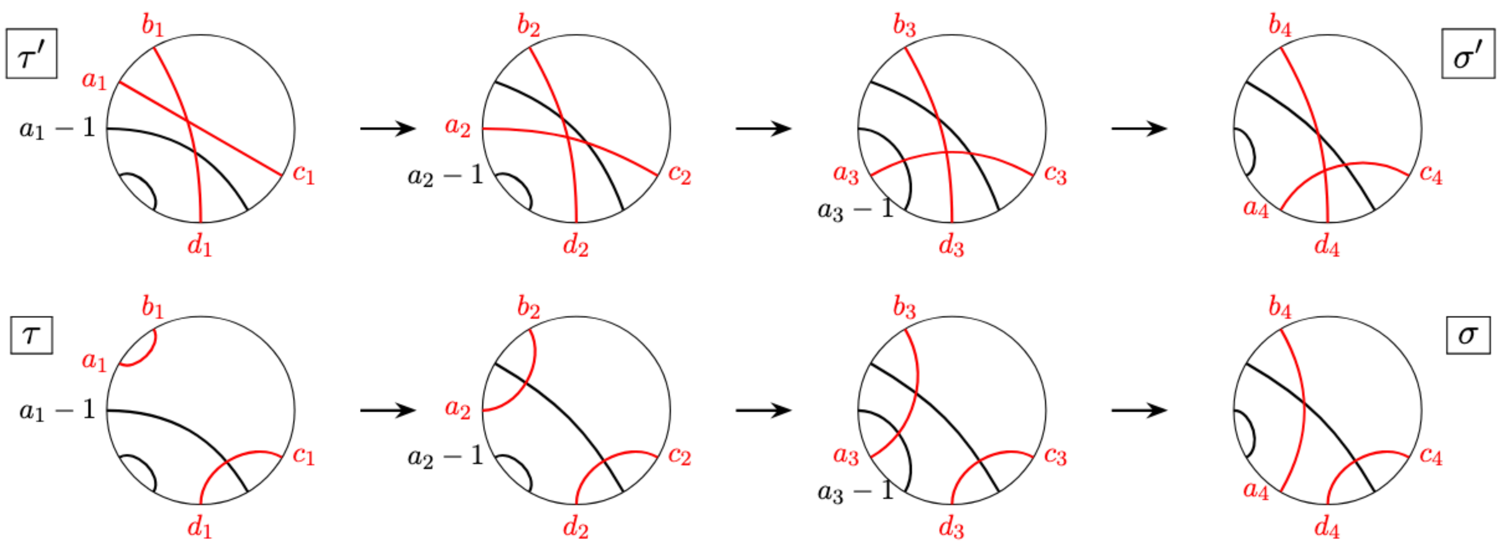
~ Matchings underlie everything ~

Key ingred in pf of RI

Def A cover rel $\tau \lessdot \tau'$ is regular pair if $X_{\tau'}$ is regular along the generic point of $\mathring{X}_{\tau}$.

↳ If $\tau \lessdot \tau'$ is i-crossing, it's regular.

Idea Show you can get from and $\tau \lessdot \tau'$ to some $\sigma \lessdot \sigma'$ i-crossing via "regular pair-preserving moves"



Electroid varieties are compatibly split

<u>Thm</u> [Lam'15] The poset of pairings on $[2n]$ is Eulerian.

<u>Cor</u> In the poset of pairings on $[2n]$, every interval of length 2 is a diamond.

$$\begin{tikzcd}[column sep=small] & g \arrow[dl, dash] \arrow[dr, dash] & \\ h_1 \arrow[dr, dash] & & h_2 \arrow[dl, dash] \\ & f & \end{tikzcd}$$

<u>Thm</u> [Shen-S.-Speyer '26] Electroid varieties are compatibly split.

Pf/ ① Induct on codim c.

② Let X_τ have codim >1. Since poset of matchings Eulerian + has unique max'l elem, get

$$\begin{tikzcd}[column sep=small] & \tau_2 \arrow[dl, dash] \arrow[dr, dash] & \\ \tau_1' \arrow[dr, dash] & & \tau_1 \arrow[dl, dash] \\ & \tau & \end{tikzcd}$$

③ By IH, $X_{\tau_1'}$ and X_{τ_1} comp. split $\Rightarrow X_{\tau_1'} \cap X_{\tau_1}$ comp split

$\Rightarrow X_\tau$ comp split.

∎