

Descent sets of permutations with only even or only odd cycle lengths

Ron Adin, Sergi Elizalde, Pál Hegedűs and Yuval Roichman

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$$(a, b)(a, b)(a, a, b, c) \longleftrightarrow (b)(a)(a, b, a)(a, c, b)$$

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$$|O_4| = |C_{(1,1,1,1)}| + |C_{(3,1)}| = 1 + 8 = 9.$$

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Theorem (Adin-Hegedűs-R 2025)

For every even n and $J \subseteq [n-1]$,

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Representation theoretic proof

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it follows that our main result is equivalent to the claim

$$\sum_{\pi \in \mathcal{O}_n} \mathcal{F}_{n, \text{Des}(\pi)} = \omega \left(\sum_{\pi \in E_n} \mathcal{F}_{n, \text{Des}(\pi)} \right).$$

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For $\lambda \vdash n$, let $C_\lambda \subset \mathfrak{S}_n$ be the conjugacy class of permutations of cycle type λ . The **Lyndon symmetric function** indexed by λ is

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Theorem [Gessel-Reutenauer '93] For every $\lambda \vdash n$, $L_\lambda = \mathcal{Q}_{C_\lambda}$ is symmetric and Schur-positive.

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Examples

$$L_{(4)} = L_{(2,1,1)}$$

$$L_{(4,1)} = L_{(3,2)} + L_{(2,1,1,1)}$$

$$L_{(6)} + L_{(3,3)} = L_{(3,2,1)} + L_{(3,1,1,1)}$$

Higher Lie characters

Let Z_λ be the centralizer in $\mathfrak{S}_n$ of a permutation of cycle type λ .

Observation If k_i denotes the number of parts of λ equal to i , then

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For each i , let ω_i be the linear character on $\mathbb{Z}_i \wr \mathfrak{S}_{k_i}$ which is trivial on $\mathfrak{S}_{k_i}$ and primitive on $\mathbb{Z}_i$. Let $\omega^\lambda := \bigotimes_i \omega_i$, a linear character on Z_λ . The **higher Lie character** indexed by λ is

$$\psi^\lambda := \omega^\lambda \uparrow_{Z_\lambda}^{\mathfrak{S}_n}.$$

Theorem [Garsia, Gessel, Reutenauer]

$$\text{ch}(\psi^\lambda) = L_\lambda \quad (\forall \lambda \vdash n).$$

Main theorem restated (plus)

Theorem (Adin-Hegedűs-R '25)

For every $n = 2m > 0$,

$$\sum_{\substack{\lambda \vdash 2m \\ \text{all parts odd}}} L_{\lambda} = \omega \left(\sum_{\substack{\lambda \vdash 2m \\ \text{all parts even}}} L_{\lambda} \right)$$

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where $\psi^{(\mu^+, \mu^-)}$ is the type B higher Lie character corr. to (μ^+, μ^-) .

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Note that $|\{\pi \in B_m : \text{all cycles of } \pi \text{ are positive}\}| = (2m-1)!!$,
and also the index of B_m in $\mathfrak{S}_{2m}$ is $(2m-1)!!$. Thus the theorem
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Combining the theorem with the Gessel-Reutenauer theorem implies

$$|\{\pi \in O_n : \text{Des}(\pi) = J\}| = |\{\pi \in E_n : \text{Des}(\pi) = [n-1] \setminus J\}|.$$

Explicit generating functions

Let $\underline{x} = (x_1, x_2, \dots)$ be a countable set of indeterminates. For a permutation $\pi \in \mathfrak{S}_n$, with n_i cycles of size i for each $i \geq 1$, denote

$$\underline{x}^{c(\pi)} := \prod_i x_i^{n_i}.$$

Let OP_n (EP_n) be the set of all **partitions** of n with only odd (respectively, even) parts. Then

$$\begin{aligned} & \sum_{n \geq 0} \sum_{\lambda \in OP_n} \frac{1}{|\mathfrak{S}_n|} \sum_{\pi \in \mathfrak{S}_n} \psi_{\mathfrak{S}_n}^\lambda(\pi) \underline{x}^{c(\pi)} \\ &= \sum_{n \geq 0} \sum_{\lambda \in EP_n} \frac{1}{|\mathfrak{S}_n|} \sum_{\pi \in \mathfrak{S}_n} \text{sign}(\pi) \psi_{\mathfrak{S}_n}^\lambda(\pi) \underline{x}^{c(\pi)} \\ &= \prod_{p \geq 0} \left(\frac{1 + x_{2^p}}{1 - x_{2^p}} \right)^{1/2^{p+1}}. \end{aligned}$$

Note: The equality of the first and last expressions was proved already in [Calderbank-Hanlon-Sundaram '94].

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Corollary For any positive integer n ,

$$\text{ch}(\rho_{\text{odd}}^{\mathfrak{S}_n}) = \sum_{\substack{\lambda \vdash n \\ \text{all parts odd}}} L_\lambda = \mathcal{Q}_{O_n}.$$

Theorem [Adin-Hegedűs-R '25] For any integers $k \geq 0$ and $n > 0$

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Corollary For every classical Weyl group W and any $k \geq 0$, the k -th root enumerator ρ_k^W is a proper character (i.e., positively spanned by the irreducible characters).

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We will construct an explicit bijection

$$\{\pi \in O_n : \text{Asc}(\pi) \subseteq S\} \longleftrightarrow \{\pi \in E_n : \text{Des}(\pi) \subseteq S\}.$$

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Multisets of
odd, distinct necklaces

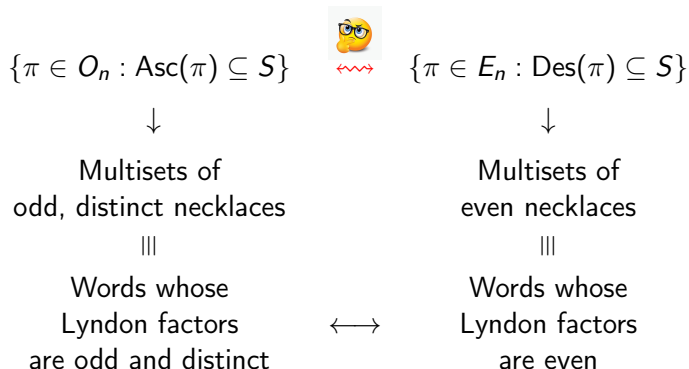


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$\mathcal{M}_n$ = multisets of necklaces of total length n .

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$\mathcal{M}_n$ = multisets of necklaces of total length n .

The **weight** of $M \in \mathcal{M}_n$ is $\text{wt}(M) = a^{\#a's} b^{\#b's} c^{\#c's} \dots$

Multisets of necklaces

Let $A = \{a, b, c, \dots\}$, $\mathcal{W}$ = finite words over A .

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Define the **weight** of a set $S = \{s_1, s_2, \dots, s_k\} \subseteq [n-1]$ to be the monomial with exponents $s_1, s_2 - s_1, \dots, n - s_k$.

Ex: If $n = 6$ and $S = \{2, 3\} \subseteq [5]$, then $\text{wt}(S) = a^2 b c^3$.

From permutations to multisets of necklaces: $\text{Des} \subseteq S$

In 1993, [Gessel](#) and [Reutenauer](#) described a bijection

$$\{\pi \in \mathfrak{S}_n : \text{Des}(\pi) \subseteq S\} \rightarrow \{M \in \mathcal{M}_n : \text{wt}(M) = \text{wt}(S)\}$$

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Let $n = 8$ and $S = \{4, 7\}$, so $\text{wt}(S) = a^4b^3c$.

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From permutations to multisets of necklaces: $\text{Des} \subseteq S$

$\mathcal{M}_n^e$ = multisets of necklaces of even length.

$$\{\pi \in O_n : \text{Asc}(\pi) \subseteq S\}$$



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However, those of *odd length* are primitive and **distinct**.

$\mathcal{M}_n^o$ = multisets of distinct necklaces of odd length.

We get a bijection

$$\{\pi \in O_n : \text{Asc}(\pi) \subseteq S\} \rightarrow \{M \in \mathcal{M}_n^o : \text{wt}(M) = \text{wt}(S)\}.$$

The bijections so far

$\mathcal{M}_n^e$ = multisets of necklaces of even length.

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$$8632 \cdot 541 \cdot 7 = (5)(3)(2,6,4)(1,8,7)$$

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We can interpret multisets of necklaces as words.

Lyndon words

A **Lyndon word** is a primitive word that is lexicographically smaller than all of its cyclic rotations. Let $\mathcal{L}$ be the set of Lyndon words.

Example: $aabab \in \mathcal{L}$, but $ababa \notin \mathcal{L}$.

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Theorem (Lyndon '58)

Every $w \in \mathcal{W}$ has a unique **Lyndon factorization** $w = \ell_1 | \ell_2 | \dots | \ell_m$ where $\ell_i \in \mathcal{L}$ for all i , and $\ell_1 \geq \ell_2 \geq \dots \geq \ell_m$ lexicographically.

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$w = \text{dedccedcdbdbdaabd}$

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$$\begin{aligned} w = dedccedcddbdbdaabd &= de | d | ccedcd | bd | bd | aabd \\ &\leftrightarrow (d, e)(d)(c, c, e, d, c, d)(b, d)(b, d)(a, a, b, d). \end{aligned}$$

We identify multisets of necklaces with words.

The bijections so far

Define the following sets of length- n words:

$\mathcal{W}_n^e$ = words all of whose Lyndon factors have even length.

$\mathcal{W}_n^o$ = words all of whose Lyndon factors have odd length
and are distinct.

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$$\{M \in \mathcal{M}_n^o : \text{wt}(M) = \text{wt}(S)\}$$



$$\{w \in \mathcal{W}_n^o : \text{wt}(w) = \text{wt}(S)\}$$

$$(b)(a)(a,b,a)(a,c,b) \equiv b|acb|aab|a$$



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We want a weight-preserving bijection between $\mathcal{W}_n^o$ and $\mathcal{W}_n^e$.

A weight preserving bijection $\Psi : \mathcal{W}_n^o \rightarrow \mathcal{W}_n^e$

Given $w \in \mathcal{W}_n^o$,

Example

$w =$ dadcdebccc

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Given $w \in \mathcal{W}_n^o$, initially set $\alpha = w$ and $\beta = -$.

Example

	α		β
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Given $w \in \mathcal{W}_n^o$, initially set $\alpha = w$ and $\beta = -$.

Repeat until α is empty:

- Let $\alpha = o_1 | o_2 | \dots | o_m$ be the Lyndon factorization of α .

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	d adcde	bccc
	d c _i de	adbccc

A weight preserving bijection $\Psi : \mathcal{W}_n^o \rightarrow \mathcal{W}_n^e$

Given $w \in \mathcal{W}_n^o$, initially set $\alpha = w$ and $\beta = -$.

Repeat until α is empty:

- Let $\alpha = o_1|o_2|\dots|o_m$ be the Lyndon factorization of α .
- If $|o_m| > 1$, write $o_m = r's$ where $s = \text{lex. smallest proper suffix}$.
- Say that o_m is *splittable* if $s < o_{m-1}$.
- Update $(\alpha, \beta) \mapsto \begin{cases} (o_1 o_2 \dots o_{m-1} r, s\beta) & \text{if } o_m \text{ splittable and } |s| \text{ even,} \\ (o_1 o_2 \dots o_{m-1} s, r\beta) & \text{if } o_m \text{ splittable and } |r| \text{ even,} \\ (o_1 o_2 \dots o_{m-2}, o_m o_{m-1} \beta) & \text{if } o_m \text{ not splittable.} \end{cases}$

Example

	α	β
$w =$	d adcdebccc	—
	d adcde	bccc
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Example

	α	β
$w =$	d adcdebccc	—
	d adcde	bccc
	d c de	adbccc
	—	cdedadbccc

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Set $\Psi(w) = \beta$.

Example

	α	β
$w =$	d adcdebccc	—
	d adcde	bccc
	d cde	adbccc
	—	cdedadbccc = $\Psi(w)$

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A weight preserving bijection $\Psi : \mathcal{W}_n^o \rightarrow \mathcal{W}_n^e$

Theorem

The map $\Psi : \mathcal{W}_n^o \rightarrow \mathcal{W}_n^e$ is a weight-preserving bijection.

A weight preserving bijection $\Psi : \mathcal{W}_n^o \rightarrow \mathcal{W}_n^e$

Theorem

The map $\Psi : \mathcal{W}_n^o \rightarrow \mathcal{W}_n^e$ is a weight-preserving bijection.

Composing the maps, we obtain the desired bijection:

$\{\pi \in O_n : \text{Asc}(\pi) \subseteq S\}$		$\{\pi \in E_n : \text{Des}(\pi) \subseteq S\}$
$\updownarrow$		$\updownarrow$
$\{M \in \mathcal{M}_n^o : \text{wt}(M) = \text{wt}(S)\}$		$\{M \in \mathcal{M}_n^e : \text{wt}(M) = \text{wt}(S)\}$
$\{w \in \mathcal{W}_n^o : \text{wt}(w) = \text{wt}(S)\}$	$\xrightarrow{\Psi}$	$\{w \in \mathcal{W}_n^e : \text{wt}(w) = \text{wt}(S)\}$

Thank you