Vanishing of Schubert coefficients in probabilistic polynomial time

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joint work with Igor Pak

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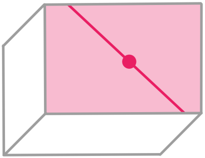
Schubert varieties

Let $\mathcal{F}l_n(\mathbb{C})$, the **complete flag variety**, be the set of complete flags $V_\bullet$:

$$0 = V_0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n = \mathbb{C}^n, \quad \text{where } \dim V_i = i.$$



The opposite Borel subgroup B_- acts on $\mathcal{F}l_n(\mathbb{C})$ with finitely many orbits X_w° called **Schubert cells**, where $w \in S_n$. The **Schubert varieties** X_w are the closures $\overline{X_w^\circ}$.

Schubert Intersections

Suppose $\ell(u) + \ell(v) + \ell(w) = \binom{n}{2}$. For generic $g, g' \in \mathrm{GL}_n(\mathbb{C})$, **Schubert structure constants** are

$$c_{u,v,w} := \#(X_u \cap gX_v \cap g'X_w) \in \mathbb{Z}_{\geq 0}.$$

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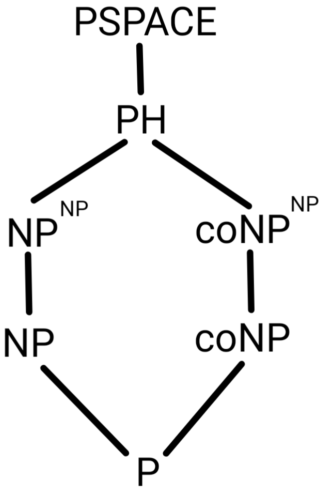
Problem

1. Determine a manifestly positive combinatorial rule for $c_{u,v,w}$.
2. Determine a combinatorial rule to decide when $c_{u,v,w} >^{?} 0$.

Partial results for (2): Knutson '01, Purbhoo '06, Billey–Vakil '08, St. Dizier–Yong '22, + many others!

Complexity classes



Initial Results

We use Hein–Sottile '17 and Purbhoo '06 to obtain systems of equations for both $[c_{u,v,w} >^{?} 0]$ and $[c_{u,v,w} =^{?} 0]$.

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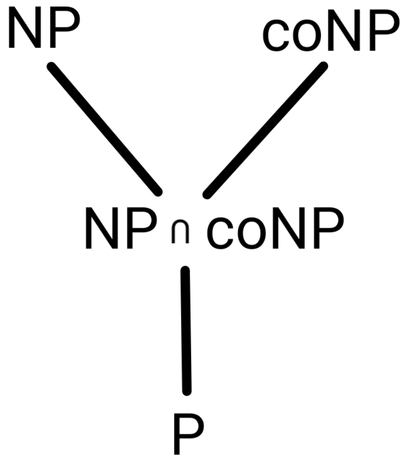
Theorem (Pak–R. '25)

$[c_{u,v,w} >^? 0] \in \mathsf{NP} \cap \mathsf{coNP}$, *assuming* GRH *and strong derandomization.*

Zooming in: Complexity Intuition



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This gave us hope that we can do better...

we just need to be OK with being wrong sometimes!

Example: Bipartite Perfect Matchings

Problem (BPM)

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Take $E(G) = (E_{ij})_{i,j \in [n]}$, where

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Proposition

G has a perfect matching if and only if $\det E(G) \neq 0$.

Main Ingredients

The proposition says to solve BPM, we only need to check if $\det E(G) \overset{?}{\equiv} 0$.

Lemma (Schwarz–Zippel)

Let $Q \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ be non-zero with degree $d \geq 0$ over $\mathbb{C}$. Take $S \subset \mathbb{C}$ be a finite set. Then:

$$\mathbf{P}\left[Q(c_1, c_2, \ldots, c_n) = 0\right] \leq \frac{d}{|S|},$$

over random, independent and uniform choices $c_1, c_2, \ldots, c_n \in S$.

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Can we apply a similar technique to $[c_{u,v,w} \overset{?}{>} 0]$?

Main Ingredients

Lemma (Purbhoo '06)

Let $u, v, w \in S_n$. For $A, B, C \in \mathrm{N} \subset \mathrm{GL}_n(\mathbb{C})$ generic unipotent:

$$c_{u,v,w} > 0 \iff A \cdot R_u + B \cdot R_v + C \cdot R_w = A \cdot R_u \oplus B \cdot R_v \oplus C \cdot R_w.$$

Here $R_\rho := \mathfrak{n} \cap (\rho \cdot \mathrm{B}_{-}\rho^{-1})$, for $\mathfrak{n}$ the Lie algebra of N and $\rho \in S_n$.

This lemma arises by translating conditions for transverse intersections into Lie algebraic statements via the Killing form.

Overview

1. Restrict to 0-dimensional case (*i.e.* $\ell(u)+\ell(v)+\ell(w)=\binom{n}{2}$).
2. Use Purbhoo's Lemma to construct a matrix $M_{u,v,w}$ that is non-singular precisely when $c_{u,v,w}>0$.
3. Determine when $\det(M_{u,v,w})\equiv 0$.
4. Test a random evaluation $\overrightarrow{\det(M_{u,v,w})} \stackrel{?}{=} 0$.
5. Use Schwartz–Zippel to determine the likelihood of a faulty test.

Main Results

Theorem (Pak–R. '25)

Schubert positivity can be decided in probabilistic poly-time for all classical G and arbitrary k-fold intersections, with one-sided error.

Corollary (Pak–R. '25)

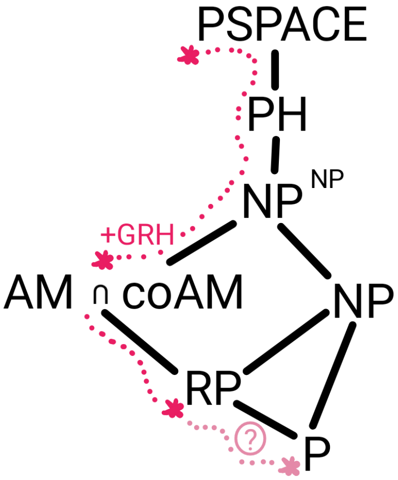
$[c_{u,v,w} >^{?} 0] \in \mathrm{NP}$

Our NP certificates are random bits where $\overrightarrow{\det(M_{u,v,w})} \neq 0$.

Corollary (Pak–R. '25)

$[c_{u,v,w} >^{?} 0] \in \mathrm{P}$, *assuming derandomization (i.e.,* $\mathrm{BPP} = \mathrm{P}$*).*

Conclusion



Thank you!