Tropical Ideals

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Matroids

Fix K an infinite field.

The support of a linear form $\ell = \sum_{i=1}^{n} a_i \, x_i \in (K^n)^*$ is

$$\text{supp}(\ell) := \{x_i : a_i \neq 0\}$$

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A collection M of subsets of $\{x_1, \ldots, x_n\}$ satisfying (M0) - (M2) is called a matroid.
If $M = M(L)$ then M is realizable over K.

Tropical ideals

A polynomial $f = \sum_{\mathbf{u} \in \mathbb{N}^n} c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in K[x_1, \ldots, x_n]$ has support

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Definition (Maclagan-R '16)

A collection I of finite subsets of $\mathbb{N}^n$ satisfying (TI0) - (TI3) is called a tropical ideal.
If $I = \mathrm{supp}(J)$ then I is realizable over K.

Tropical ideals

If Δ_d = monomials of deg $\leq d$, then $I_{\leq d} := \{S \in I : S \subset \Delta_d\}$ is a matroid $M_d(I)$ on Δ_d.

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A tropical ideal is an "infinite tower of compatible matroids" $(M_d)_{d \geq 0}$:

- M_d is a matroid on the ground set Δ_d
- $M_d = M_{d+1}|_{\Delta_d}$
- For any variable x_i we have a matroid quotient $x_i \cdot M_d \to M_{d+1}|_{x_i \cdot \Delta_d}$

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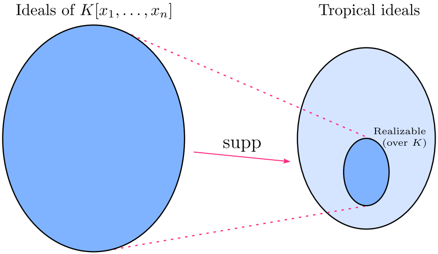

Example

Take $n = 1$ and I the collection of finite subsets of at least 2 monomials (plus $\{\emptyset\}$).

Examples of tropical ideals

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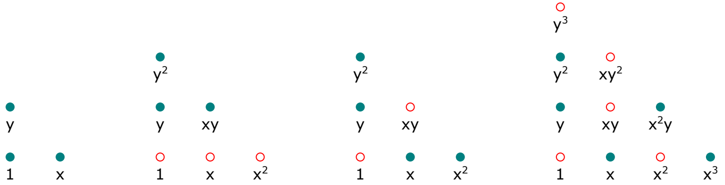
Example

Take $n = 2$, and I the collection of finite unions of subsets C consisting of $k + 2$ monomials inside a standard triangle of size k (and C is minimal with this property).



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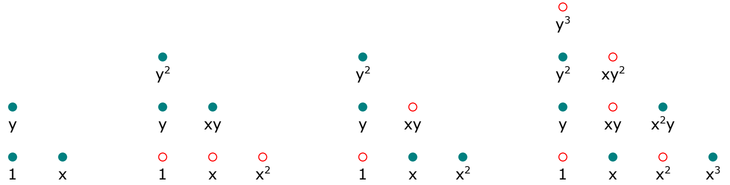

It is non-realizable: If $I = \text{supp}(J)$ for $J \subset K[x, y]$ then

$$x + y + 1 \in J \quad \text{and} \quad (x + y + 1)(x^2 + y^2 - xy - x - y + 1) = (x^3 + y^3 - 3xy + 1) \in J$$

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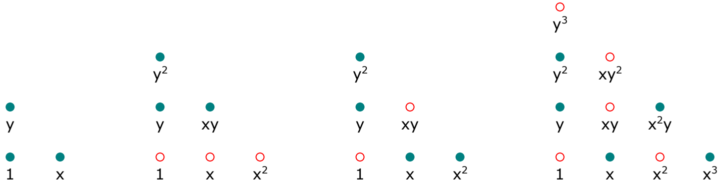

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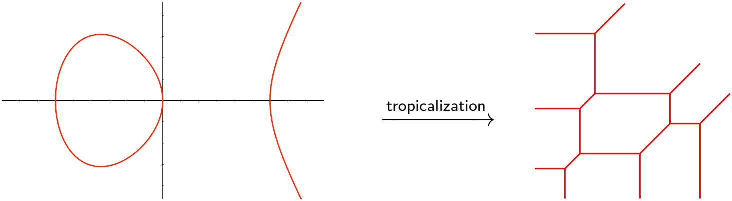
Theorem (Anderson-R '21)

There are uncountably many tropical ideals in one variable (while only countably many are realizable).

Why tropical ideals?

Tropical geometry is a piecewise-linear shadow of algebraic geometry.



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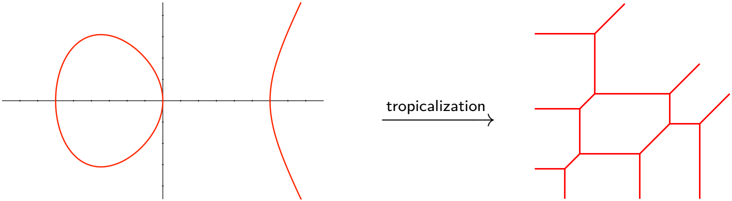
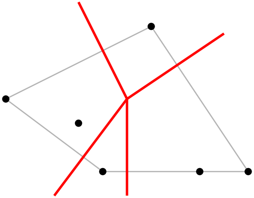


If $V \subset \mathbb{A}^n$ is the zero locus of an ideal $J \subset K[x_1, \ldots, x_n]$,

$$\textcolor{red}{\mathrm{trop}(V)} := \{\mathbf{a} \in \mathbb{R}^n : \forall\, S \in \mathrm{supp}(J),\ \text{the} \max_{\mathbf{u} \in S}(\mathbf{a} \cdot \mathbf{u}) \text{ is attained at least twice}\}$$

$$= \bigcap_{S \in \mathrm{supp}(J)} \text{codimension-1 skeleton of the normal fan } \mathcal{N}(\mathrm{convex}(S))$$

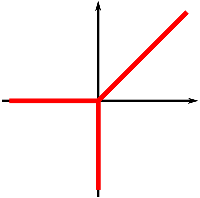


Example

Take $J = \langle x + y + 1 \rangle \subset K[x, y]$, V is a line in $\mathbb{A}^2$

$$J = \{\, x + y + 1,\; x^2 + xy + x,\; xy + y^2 + y,\; x^2 - y^2 + x - y,\; \dots \}$$

$$\mathrm{supp}(J) = \{\, \{x, y, 1\},\; \{x^2, xy, x\},\; \{xy, y^2, y\},\; \{x^2, y^2, x, y\},\; \dots \}$$



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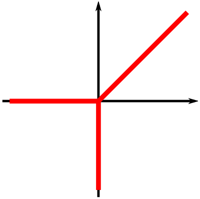
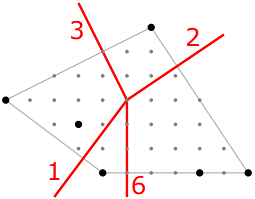


trop(V)

Structure Theorem in Tropical Geometry

The set trop(V) is a finite polyhedral fan, of the same dimension and degree as V.
Maximal cones of trop(V) come with multiplicities that make it balanced.



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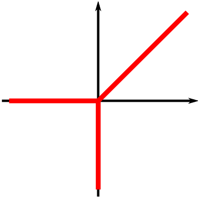

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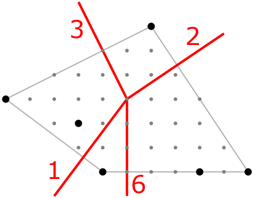

Giansiracusa-Giansiracusa '13: We should look at algebraic information as well.

Maclagan-R '14: This information is equivalent to the collection supp(J).

Hilbert functions

The (affine) Hilbert function of an ideal $J \subset K[x_1, \ldots, x_n]$ is the function $H_J : \mathbb{N} \to \mathbb{N}$

$$H_J(d) = \dim_K \left(K[x_1, \ldots, x_n]_{\leq d} \, / \, J_{\leq d} \right)$$

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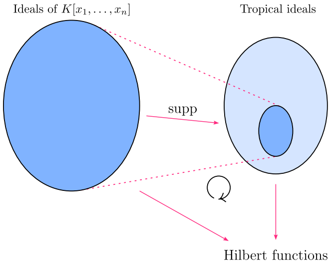
Definition

The Hilbert function of a tropical ideal I is the function $H_I : \mathbb{N} \to \mathbb{N}$

$$\begin{aligned} H_I(d) &:= \operatorname{rank} M_d(I) \\ &= \text{size of } \textbf{any} \text{ maximal } B \subset \Delta_d \text{ with } B \not\supset S \text{ for any } S \in I \end{aligned}$$

If $I = \operatorname{supp}(J)$ then $H_I = H_J$.



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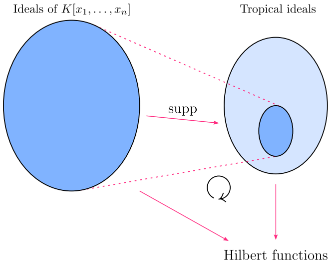
Theorem (Maclagan-R '16)

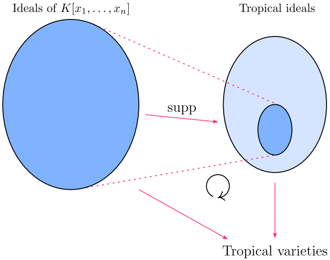
The Hilbert function of any tropical ideal is eventually polynomial.



Varieties of tropical ideals

The variety of a tropical ideal I is

$$V(I) := \{\mathbf{a} \in \mathbb{R}^n : \forall\, S \in I, \text{ the} \max_{\mathbf{u}\in S}(\mathbf{a}\cdot\mathbf{u}) \text{ is attained at least twice}\}$$



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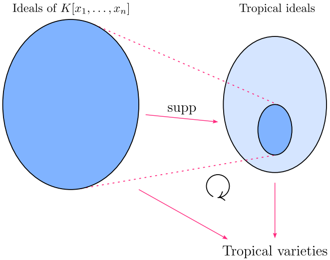
The set $V(I)$ is a finite polyhedral fan of dimension

$\dim(V(I))$ = *degree of the Hilbert polynomial of I*



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Theorem (Maclagan-R '16, '20)

The set $V(I)$ is a finite balanced polyhedral fan of dimension

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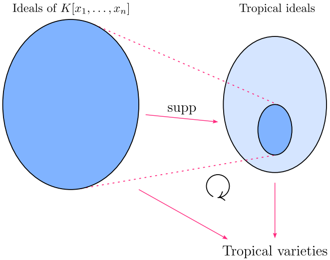

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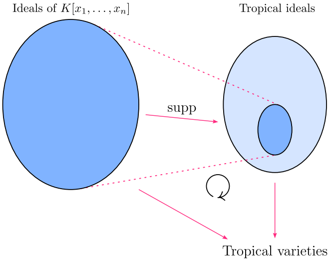


Problem: Find a useful algebraic/combinatorial property that ensures $V(I)$ is pure.

Tropically realizable matroids

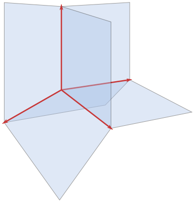
Question: How general can the polyhedral fans $V(I)$ be?

Definition

A matroid M on the set $\{1, \ldots, n\}$ is tropically realizable if there is a tropical ideal I on the variables $\{x_1, \ldots, x_n\}$ s.t.

$$V(I) = \text{Bergman fan of } M.$$

All (linearly) realizable matroids are tropically realizable.



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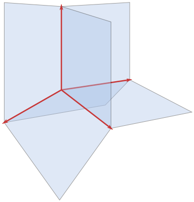

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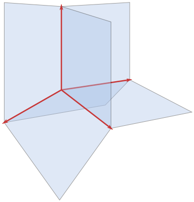

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Theorem (Draisma-R '19)

The matroid $V_8 \oplus U_{2,3}$ is not tropically realizable.

If M is a matroid on $[m]$ and N is a matroid on $[n]$, a tensor product of M and N is a matroid P on $[m] \times [n]$ such that

- For any i, the map $(i,j) \mapsto j$ is an isomorphism of matroids $P|_{\{i\}\times[n]} \cong N$
- For any j, the map $(i,j) \mapsto i$ is an isomorphism of matroids $P|_{[m]\times\{j\}} \cong M$
- $\operatorname{rank}(P) = \operatorname{rank}(M) \cdot \operatorname{rank}(N)$

Theorem

The matroid $V_8 \oplus U_{2,3}$ is not tropically realizable.

Proof idea

- If $V(I) = \mathrm{Bergman}(V_8 \oplus U_{2,3})$ then $M_1(I)$ must be $V_8 \oplus U_{2,3}$
- $M_2(I)$ restricted to the monomials $x_i\, y_j$ must be a (quasi)product of V_8 and $U_{2,3}$
- For the variety to be right, this restriction must have rank 8, but [Las Vergnas '81] proved there is no tensor product of V_8 and $U_{2,3}$

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Problem: Is the Vámos matroid V_8 tropically realizable?

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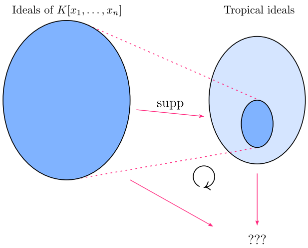

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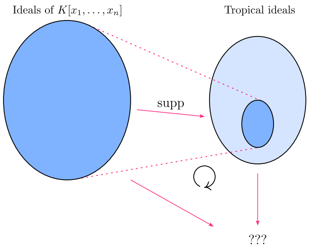

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