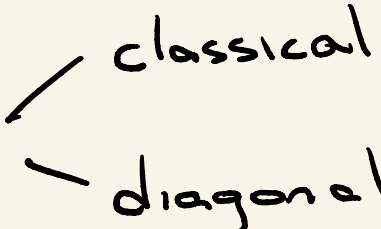


The Superspace Coinvariant Ring

Brendon Rhoades (UCSD)

┌ joint w/ Robert Angarone (WMN)
Sutanay Bhattacharya (UCSD)
Patricia Commins (UQAM)
Trevor Karn (TAMU)
Satoshi Murai (Waseda)
Andy Wilson (Kennesaw State) └

Outline

① Coinvariant history 
classical
diagonal

② Superspace coinvariants

③ Future directions

SYMMETRIC POLYNOMIALS

$$S := \mathbb{C}[x_1, \dots, x_n]$$

$$S_n \curvearrowright S$$

$$w \cdot x_i = x_{w(i)}$$

SYMMETRIC POLYNOMIALS

$$S := \mathbb{C}[x_1, \dots, x_n]$$

$$\mathfrak{S}_n \curvearrowright S \quad w \cdot x_i = x_{w(i)}$$

$$S^{\mathfrak{S}_n} = \text{symmetric polynomials}$$

$$= \{f \in S : w \cdot f = f \text{ for all } w \in \mathfrak{S}_n\}$$

$$e_d = \sum_{i_1 < \dots < i_d} x_{i_1} \dots x_{i_d}$$

NEWTON $\{e_1, \dots, e_n\}$ is an algebraically independent
generating set of $S^{\mathfrak{S}_n}$.

CLASSICAL COINVARIANT RING

Let $I_n = (S_+^{\mathfrak{S}_n}) = (e_1, \dots, e_n) \subset S$.

DEF $R_n = \text{COINVARIANT RING} = S/I_n$.

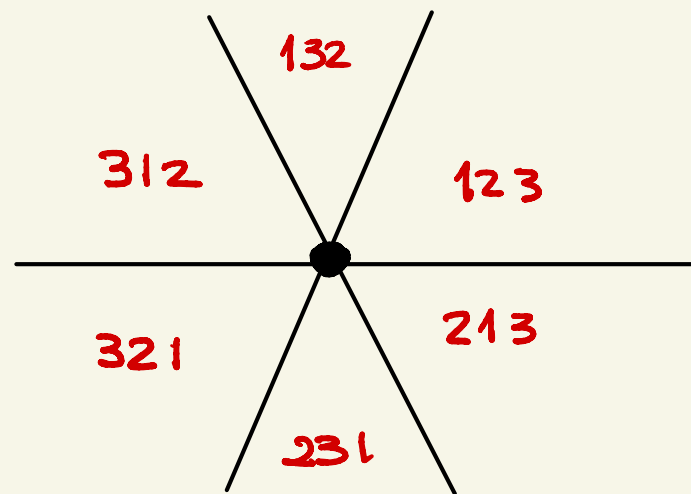
* graded $\mathfrak{S}_n$ -module

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ALGEBRA OF $R_n \longleftrightarrow$ COMBINATORICS OF $\mathfrak{S}_n$



CLASSICAL COINVARIANT RING

Let $I_n = (S_+^{G_n}) = (e_1, \dots, e_n) \subset S$.

DEF $R_n = \text{COINVARIANT RING} = S/I_n$.

ALGEBRA OF $R_n \longleftrightarrow$ COMBINATORICS OF G_n

CHEVALLEY: $R_n \cong_{G_n} \mathbb{C}[G_n]$

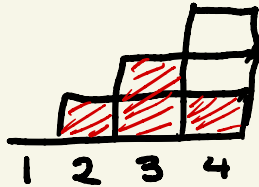
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ALGEBRA OF $R_n \longleftrightarrow$ COMBINATORICS OF G_n

E. ARTIN $\{x_1^{a_1} \dots x_n^{a_n} : a_i < i\}$ is a basis of R_n .



CLASSICAL COINVARIANT RING

$$\text{Let } I_n = (S_+^{\mathfrak{S}_n}) = (e_1, \dots, e_n) \subset S.$$

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ALGEBRA OF $R_n \longleftrightarrow$ COMBINATORICS OF $\mathfrak{S}_n$

LUSZTIG - STANLEY

$$\text{gr Frob}(R_n; q) = \sum_{T \in \text{SYT}(n)} q^{\text{maj}(T)} s_{\text{shape}(T)}$$

DIAGONAL COINVARIANT RING

$$\mathbb{G}_n^c \mathbb{C}[x_n, y_n] = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$$

$$w \cdot x_i = x_{w(i)} \quad w \cdot y_i = y_{w(i)}$$

DIAGONAL COINVARIANT RING

$$\mathfrak{S}_n \curvearrowright \mathbb{C}[x_n, y_n] = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$$

$$w \cdot x_i = x_{w(i)} \quad w \cdot y_i = y_{w(i)}$$

$$\text{GARSIA-HAIMAN} \quad DI_n = (\mathbb{C}[x_n, y_n]_{+}^{\mathfrak{S}_n}) \subset \mathbb{C}[x_n, y_n].$$

$$DR_n = \text{DIAGONAL COINVARIANT RING} = \mathbb{C}[x_n, y_n] / DI_n.$$

* bigraded $\mathfrak{S}_n$ -module

DIAGONAL COINVARIANT RING

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ALGEBRA OF $DR_n \longleftrightarrow$ COMBINATORICS OF $Perk_n$

$Perk_3$

1 2 3	2 1 3	1 3 2	2 3 1	3 1 2	3 2 1
1 2 2	2 1 2	2 2 1			
1 1 3	1 3 1	3 1 1			
1 1 2	1 2 1	2 1 1			
1 1 1					

DIAGONAL COINVARIANT RING

GARSIA-HAIMAN $DI_n = (\mathbb{C}[x_n, y_n]_+^{S_n}) \subset \mathbb{C}[x_n, y_n]$.

$DR_n = \text{DIAGONAL COINVARIANT RING} = \mathbb{C}[x_n, y_n] / DI_n$.

ALGEBRA OF $DR_n \longleftrightarrow$ COMBINATORICS OF Park_n

HAIMAN $DR_n \cong_{S_n} \mathbb{C}[\text{Park}_n] \otimes \text{sign}$

CARLSSON-OBLOMKOV Monomial basis of DR_n

$\text{grFrob}(DR_n; q, t) \stackrel{\text{HAIMAN}}{=} \nabla e_n \stackrel{\text{CARLSSON-MELLIT}}{=} \sum_{P \in \text{Park}_n} q^{\text{area}(P)} t^{\text{dinv}(P)} F_{\text{idcs}(P)}$

SUPERSPACE

$$\Omega = \mathbb{C}[x_1, \dots, x_n] \otimes \wedge \{\theta_1, \dots, \theta_n\}$$

PHYSICS $x_1, \dots, x_n$ "BOSONS" $\theta_1, \dots, \theta_n$ "FERMIONS"

Ω = regular differential forms on $\mathbb{C}^n$

$$d: \Omega \longrightarrow \Omega \quad df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \theta_i \quad \text{"EULER OPERATOR"}$$

$$G_n \curvearrowright \Omega$$

$$w \cdot x_i = x_{w(i)} \quad w \cdot \theta_i = \theta_{w(i)}$$

SUPERSPACE COINVARIANTS

$$* SI_n = (\Omega_+^{\mathfrak{G}_n}) \underset{\text{SOLOMON}}{=} (e_1, \dots, e_n, de_1, \dots, de_n) \subset \Omega.$$

$$* SR_n = \underset{\text{RING}}{\text{SUPERSPACE COINVARIANT}} = \Omega / SI_n.$$

* bigraded $\mathfrak{G}_n$ -module

SUPERSPACE COINVARIANTS

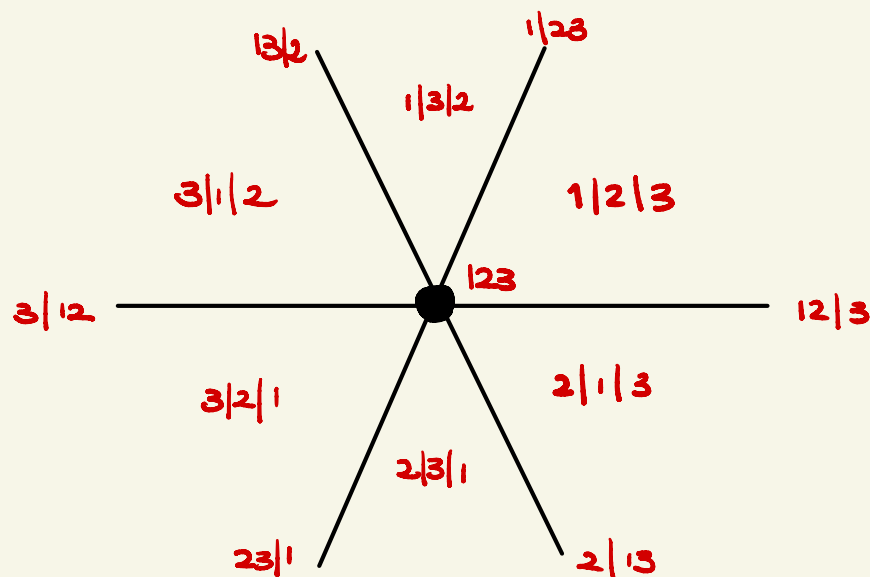
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* bigraded $\mathfrak{G}_n$ -module

$$* \mathcal{OP}_n = \text{ORDERED SET PARTITIONS OF } [n]$$

$\mathcal{OP}_3$



FIELDS CONJECTURES

CONT [N. BERGERON, COLMENAREJO, LI, MACHACEK, SULZGRUBER, ZABROCKI]

ALGEBRA OF $SR_n \longleftrightarrow$ COMBINATORICS OF $\mathcal{OP}_n$

$$\textcircled{1} \dim_{\mathbb{C}} SR_n = \# \mathcal{OP}_n$$

$$\textcircled{2} \text{Hilb}(SR_n; q, z) = \sum_{k=1}^n z^{n-k} [k]!_q \text{Stir}_q(n, k)$$

$$\textcircled{3} SR_n \cong_{G_n} \mathbb{C}[\mathcal{OP}_n] \otimes \text{sign}$$

$$\begin{aligned} \textcircled{4} \text{grFrob}(SR_n; q, z) &= \sum_{k=1}^n z^{n-k} \Delta'_{e_{k-1} e_n} \Big|_{t \rightarrow 0} \\ &= \sum_{k=1}^n z^{n-k} \sum_{T \in \text{SYT}(n)} q^{maj T + \binom{n-k}{2} - (n-k) \text{des} T} \begin{bmatrix} \text{des} T \\ n-k \end{bmatrix}_q S_{\text{shape}(T)} \end{aligned}$$

FIELDS ~~CONJECTURES~~ THEOREMS

CONT [N. BERGERON, COLMENAREJO, LI, MACHACEK, SULZGRUBER, ZABROCKI]

ALGEBRA OF $SR_n \longleftrightarrow$ COMBINATORICS OF $\mathcal{OP}_n$

$$\textcircled{1} \dim_{\mathbb{C}} SR_n = \# \mathcal{OP}_n \quad [\text{R-WILSON}]$$

$$\textcircled{2} \text{Hilb}(SR_n; q, z) = \sum_{k=1}^n z^{n-k} [k]!_q \text{Stir}_q(n, k) \quad [\text{R-WILSON}]$$

$$\textcircled{3} SR_n \cong_{G_n} \mathbb{C}[\mathcal{OP}_n] \otimes \text{sign} \quad [\text{MURAI-R-WILSON}]$$

$$\textcircled{4} \text{grFrob}(SR_n; q, z) = \sum_{k=1}^n z^{n-k} \Delta'_{e_{k-1} e_n} \Big|_{t \rightarrow 0}$$

$$[\text{MURAI-R-WILSON}] \quad = \sum_{k=1}^n z^{n-k} \sum_{T \in \text{SYT}(n)} q^{maj T + \binom{n-k}{2} - (n-k) \text{des} T} \begin{bmatrix} \text{des} T \\ n-k \end{bmatrix}_q S_{\text{shape}(T)}$$

Two Main Tools

① R.-WILSON "OPERATOR THM" [CONJ. BY SWANSON-WALLACH]

$SI_n^\perp$ = smallest subspace of $\mathcal{Q}$ containing $\prod_{1 \leq i < j \leq n} (x_i - x_j)$,

closed under $\partial/\partial x_1, \dots, \partial/\partial x_n$

+ "HIGHER EULER OPERATORS" $d_j = \sum_{i=1}^n \frac{\partial^j}{\partial x_i^j} \theta_i$

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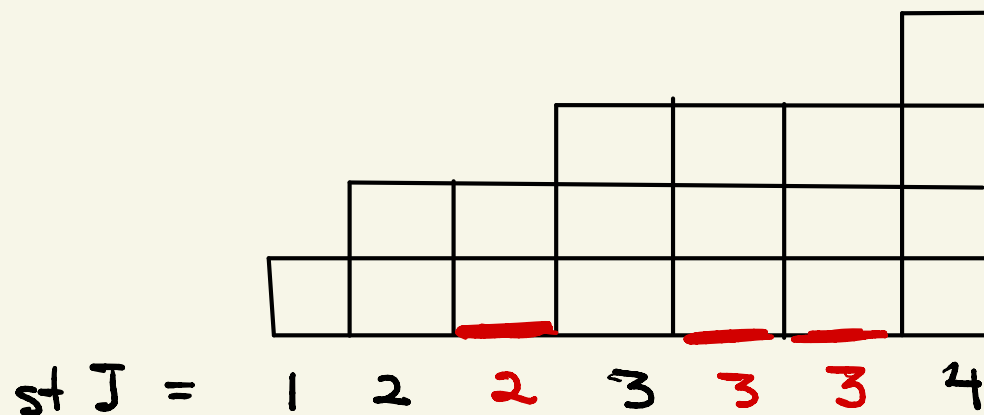
+ "HIGHER EULER OPERATORS" $d_j = \sum_{i=1}^n \frac{\partial^j}{\partial x_i^j} \theta_i$

② FINDING A GOOD BASIS of SR_n

SAGAN - SWANSON MONOMIALS

$J \subseteq [n] \rightsquigarrow$ **J-STAIRCASE** $(st_1 J, st_2 J, \dots, st_n J)$

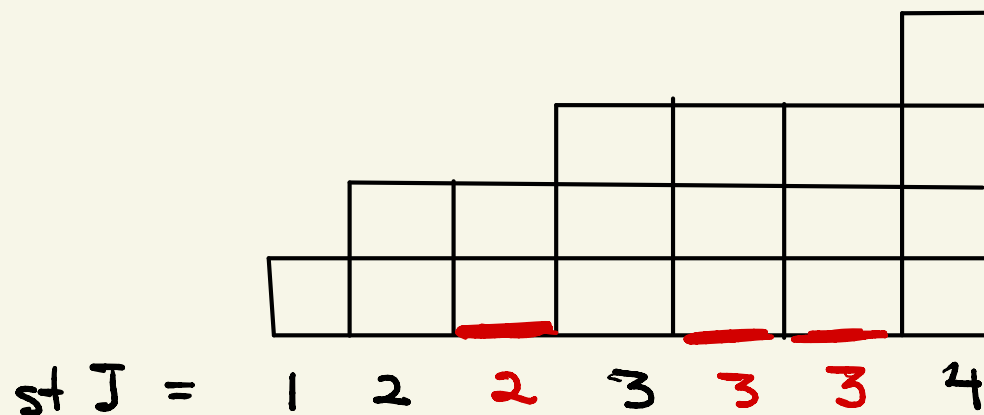
$n = 7$ $J = \{3, 5, 6\}$



SAGAN-SWANSON MONOMIALS

$$J \in [n] \rightsquigarrow \text{J-STAIRCASE } (st_1 J, st_2 J, \dots, st_n J)$$

$$n=7 \quad J = \{3, 5, 6\}$$



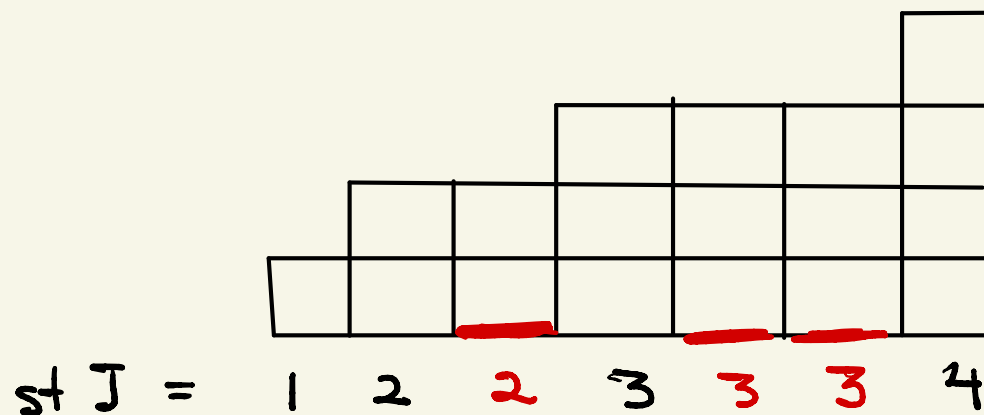
$$\mathcal{A}_n(J) = \{x_1^{a_1} \dots x_n^{a_n} : a_i < st_i J\}$$

CONJ [SAGAN-SWANSON] $\bigsqcup_J \mathcal{A}_n(J) \cdot \Theta_J$ is a basis of SR_n .

SAGAN-SWANSON MONOMIALS

$$J \in [n] \rightsquigarrow \text{J-STAIRCASE } (st_1 J, st_2 J, \dots, st_n J)$$

$$n=7 \quad J = \{3, 5, 6\}$$



ANGARONE-
COMMINS-KARN-
MURAI-TR:

THEY WERE
RIGHT!

$$\mathcal{A}_n(J) = \{x_1^{a_1} \dots x_n^{a_n} : a_i < st_i J\}$$



CONJ [SAGAN-SWANSON] $\bigsqcup_J \mathcal{A}_n(J) \cdot \Theta_J$ is a basis of SR_n .

SAGAN - SWANSON BASIS

THM [ANGARONE - GUMINS - KARN - MURAI - R]

$$\bigsqcup_J \Lambda_n(J) \cdot \Theta_J \text{ is a basis of } SR_n.$$

RMK • This basis is **USEFUL** in studying $\mathfrak{S}_n$ -structure of SR_n .

• Facilitates calculation of $\text{Hilb}(\varepsilon_\mu SR_n; q, z)$ ($\mu \vdash n$)

$$\varepsilon_\mu = \sum_{w \in \mathfrak{S}_\mu} (\text{sign } w) \cdot w$$

SAGAN - SWANSON BASIS

THM [ANGARONE - GUMINS - KARN - MURAI - R]

$\bigsqcup_J \Lambda_n(J) \cdot \Theta_J$ is a basis of SR_n .

* No elementary proof known.

SAGAN - SWANSON BASIS

THM [ANGARONE - GUMINS - KARN - MURAI - R]

$$\bigsqcup_J \Lambda_n(J) \cdot \theta_J \text{ is a basis of } SR_n.$$

Pf uses... ① Supercommutative $\leadsto$ Commutative [R-WILSON]
TRANSFER PRINCIPLE

$$\Lambda_n(J) \text{ basis of } S / (I_n; f_J) \text{ for all } J \subseteq [n] \Rightarrow \bigsqcup_J \Lambda_n(J) \cdot \theta_J \text{ basis of } SR_n$$

$f_J = \prod_{j \in J} x_j \left(\prod_{j < k} (x_j - x_k) \right)$

SAGAN - SWANSON BASIS

THM [ANGARONE - GUMINS - KARN - MURAI - R]

$$\bigsqcup_J \Lambda_n(J) \cdot \Theta_J \text{ is a basis of } SR_n.$$

Pf uses... ② **SALMON-TERAO ALGEBRAS** of hyperplane arrangements [ABE-MAENO-MURAI-NUMATA]

[Embeds the $S/(I_n: f_J)$ into **SHORT EXACT**

SEQUENCES from **FREE TRIPLS** $(\Lambda, \Lambda \setminus H, \Lambda \setminus H)$]

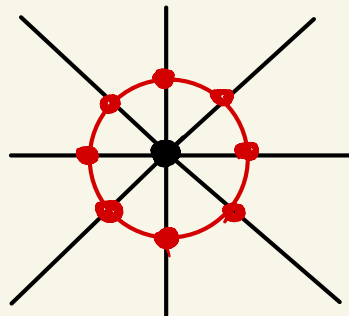
OTHER GROUPS

W : WEYL GROUP $\hookrightarrow V$ refln rep.

$W \hookrightarrow \Omega$ (regular differential forms on V)

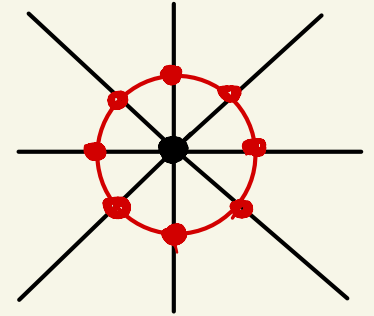
$SR_W = \Omega / (\Omega_+^W)$ (bigraded W -module)

$\mathcal{CP}_W =$ Coxeter complex



OTHER GROUPS

W : WEYL GROUP $\mathcal{CP}_W = \text{Coxeter complex}$



CONJ. [MURAI-R-WILSON]

For any $d \geq 0$, there is a W -equivariant surjection

$$\left(\begin{array}{l} \text{exterior degree } d \\ \text{part of } SR_W \end{array} \right) \longrightarrow \mathbb{C} \left[\begin{array}{l} \text{codim. } d \text{ faces} \\ \text{of } \mathcal{CP}_W \end{array} \right] \otimes \det$$

[MURAI-R-WILSON] Isomorphism in type A.

[BHATTACHARYA-R] Isomorphism in type BC.

TYPE F_4

$$\text{Hilb}(\text{SR}_{F_4}; 1, z) = 1152 + 2304z + 1396z^2 + 244z^3 + z^4$$

reversed f-vector
of $\mathcal{OP}_{F_4}$ $(1152, 2304, 1392, 240, 1)$

F_4 has 25 irreducible characters $\chi_1, \dots, \chi_{25}$

2304 0 4 2 2 6 2 6 2 4 8 8 12 8 12 12 12 20 12 20 24 18 18 12 24 32

2304 0 4 2 2 6 2 6 2 4 8 8 12 8 12 12 12 20 12 20 24 18 18 12 24 32

1396 0 6 1 1 6 0 6 0 0 5 5 12 5 6 8 4 16 4 16 21 10 10 3 14 18

1392 0 6 1 1 6 0 6 0 0 5 5 12 4 6 8 4 16 4 16 21 10 10 3 14 18

244 0 4 0 0 2 0 2 0 0 1 1 4 1 0 2 0 4 0 4 6 1 1 0 2 2

240 0 4 0 0 2 0 2 0 0 1 1 4 0 0 2 0 4 0 4 6 1 1 0 2 2

↑
 χ_{13} , degree 4

FINITE FIELDS

$$G = GL_n(\mathbb{F}_q) \curvearrowright \Omega = \mathbb{F}_q[x_1 \dots x_n] \otimes \wedge \{\theta_1 \dots \theta_n\}$$

$$SR_G = \Omega / (\Omega_+^G) \quad (\text{bigraded } G\text{-module})$$

FINITE FIELDS

$$G = GL_n(\mathbb{F}_q) \curvearrowright \Omega = \mathbb{F}_q[x_1 \dots x_n] \otimes \wedge \{\theta_1 \dots \theta_n\}$$

$$SR_G = \Omega / (\Omega_+^G) \quad (\text{bigraded } G\text{-module})$$

R-WILSON
$$\text{Hilb}(SR_G; t, z) = \prod_{i=1}^{n-1} [q^n - q^i]_t \cdot \sum_{J \subseteq [n]} z^{|J|} [q^n - 1 - \sum_{j \in J} q^{j-1}]_t.$$

「uses work of HARTMANN-SHEPHERD on Ω^G 」

Q MODULE STRUCTURE?

MORE SETS OF VARIABLES

		# OF BOSONS $x_n, y_n, \dots$		
		0	1	2
# OF FERMIONS $\theta_n, \xi_n, \dots$	0	$\mathbb{C}$ (EXERCISE)		
	1			
	2			

MORE SETS OF VARIABLES

		# OF BOSONS $x_n, y_n, \dots$		
		0	1	2
# OF FERMIONS $\theta_n, \xi_n, \dots$	0	$\mathbb{C}$ (EXERCISE)	$\mathbb{R}_n$ (ARTIN, CHEVALER, BOREL...)	
	1			
	2			

MORE SETS OF VARIABLES

OF BOSONS $x_n, y_n, \dots$

		0	1	2
# OF FERMIONS $\theta_n, \xi_n, \dots$	0	$\mathbb{C}$ (EXERCISE)	$\mathbb{R}_n$ (ARTIN, CHEVALLEY, BOREL...)	DR_n (HAIMAN)
	1			
	2			

MORE SETS OF VARIABLES

OF BOSONS $x_n, y_n, \dots$

		0	1	2
# OF FERMIONS $\theta_n, \xi_n, \dots$	0	$\mathbb{C}$ (EXERCISE)	R_n (ARTIN, CHEVALLEY, BOREL...)	DR_n (HAIMAN)
	1	$R^{(0 1)}$ (EXERCISE)		
	2			

MORE SETS OF VARIABLES

OF BOSONS $x_n, y_n, \dots$

		0	1	2
# OF FERMIONS $\theta_n, \xi_n, \dots$	0	$\mathbb{C}$ (EXERCISE)	R_n (ARTIN, CHEVALLEY, BOREL...)	DR_n (HAIMAN)
	1	$R^{(0 1)}$ (EXERCISE)	SR_n (THIS TALK)	
	2			

MORE SETS OF VARIABLES

		# OF BOSONS $x_n, y_n, \dots$		
		0	1	2
# OF FERMIONS $\theta_n, \xi_n, \dots$	0	$\mathbb{C}$ (EXERCISE)	R_n (ARTIN, CHEVALLEY, BOREL...)	DR_n (HAIMAN)
	1	$R^{(0 1)}$ (EXERCISE)	SR_n (THIS TALK)	
	2	FDR_n [JESSE KIM, JONGWON KIM, R.]		

MORE SETS OF VARIABLES

OF BOSONS $x_n, y_n, \dots$

		0	1	2
# OF FERMIONS $\theta_n, \xi_n, \dots$	0	$\mathbb{C}$ (EXERCISE)	R_n (ARTIN, CHEVALLEY, BOREL...)	DR_n (HAIMAN)
	1	$R^{(0 1)}$ (EXERCISE)	SR_n (THIS TALK)	$\sum_k z^{n-k} \Delta_{e_{k-1}} e_n$ (ZABROTKI CONT.)
	2	$FD R_n$ [JESSE KIM, JONGWON KIM, R.]		

MORE SETS OF VARIABLES

OF BOSONS $x_n, y_n, \dots$

OF
FERMIONS

$\theta_n, \xi_n, \dots$

	0	1	2
0	$\mathbb{C}$ (EXERCISE)	$\mathbb{R}_n$ (ARTIN, CHEVALLEY, BOREL...)	$\mathbb{DR}_n$ (HAIMAN)
1	$\mathbb{R}^{(011)}$ (EXERCISE)	$\mathbb{SR}_n$ (THIS TALK)	$\sum_k z^{n-k} \Delta'_{e_{k-1}} e_n$ (ZABROCKI CONJ.)
2	$\mathbb{FDR}_n$ [JESSE KIM, JONGWON KIM, R.]	$\mathbb{R}^{(112)}$ [LENTFER CONJ.]	

MORE SETS OF VARIABLES

OF BOSONS $x_n, y_n, \dots$

OF
FERMIONS

$\theta_n, \xi_n, \dots$

	0	1	2
0	$\mathbb{C}$ (EXERCISE)	$\mathbb{R}_n$ (ARTIN, CHEVALLEY, BOREL...)	$\mathbb{D}\mathbb{R}_n$ (HAIMAN)
1	$\mathbb{R}^{(0 1)}$ (EXERCISE)	$S\mathbb{R}_n$ (THIS TALK)	$\sum_k z^{n-k} \Delta'_{e_{k-1}} e_n$ (ZABROCKI CONT.)
2	$F\mathbb{D}\mathbb{R}_n$ [JESSE KIM, JONGWON KIM, R.]	$\mathbb{R}^{(1 2)}$ [LENTFER CONT.]	$\oplus$ - ops. [D'ADDERIO - IRACI - VANDEN WYNGAERT CONT.]

MORE SETS OF VARIABLES

		# OF BOSONS $x_n, y_n, \dots$			
		0	1	2	∞ →
# OF FERMIONS $\theta_n, \xi_n, \dots$	0	$\mathbb{C}$ (EXERCISE)	$\mathbb{R}_n$ (ARTIN, CHEVALLEY, BOREL...)	$\mathbb{D}\mathbb{R}_n$ (HAIMAN)	COMBIN. SUPERSTYMETRY
	1	$\mathbb{R}^{(0 1)}$ (EXERCISE)	$S\mathbb{R}_n$ (THIS TALK)	$\sum_k z^{n-k} \Delta'_{e_{k-1}} e_n$ (ZABROSKI CONT.)	CONJ: F. BERGERON THM: LENTFER ↓
	2	$F\mathbb{D}\mathbb{R}_n$ JESSE KIM, JONGWON KIM, R.	$\mathbb{R}^{(1 2)}$ LENTFER CONT. ↓	$\oplus$ - ops. D'ADDERIO - IRACI - VANDEN WYNGAERT CONT. ↓	
		∞ ↓			

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YOURS TO DISCOVER

COINVARIANT THEORY



THANKS FOR LISTENING!

B. Rhoades and A. Wilson. The Hilbert series of the superspace coinvariant ring. Forum Math. Pi. (2024) Vol. 12: e16, 1–35.

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S. Bhattacharya and B. Rhoades. Superspace coinvariants and wreath products. arXiv: 2606.30977.