Plabic Tangles and Cluster Promotion Maps

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Based on joint work with C. Even-Zohar, M. Sherman-Bennett, R. Tessler and L. Williams.

Outline

Grassmannian and Plabic Graphs

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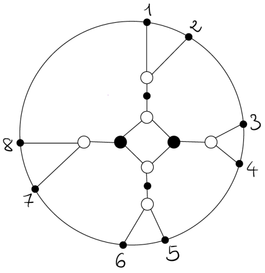

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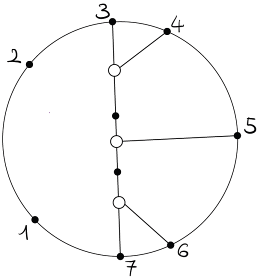

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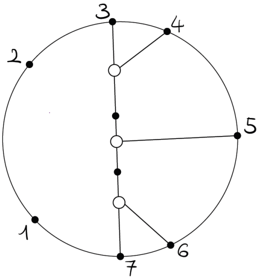
Later

A plabic graph G gives a rank-k matroid $\mathcal{M}(G)$. $\mathcal{M}(G)$ will label a *positroid variety* Π_G and a *positroid cell* S_G inside $\mathrm{Gr}_{k,n}$.



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For the white vertex w in the picture,

$r_1 z_3 + r_2 z_4 + r_3 v_1 = 0$, hence $v_1 \in \langle z_3, z_4 \rangle$.

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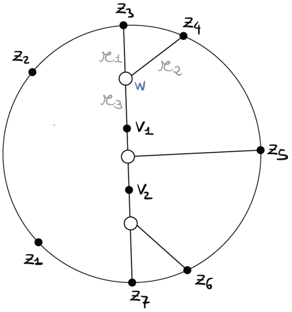

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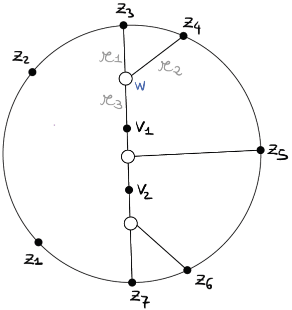

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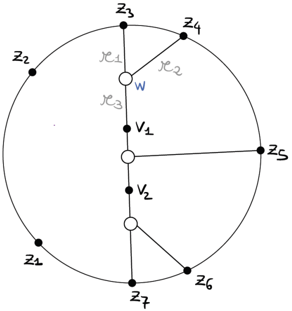

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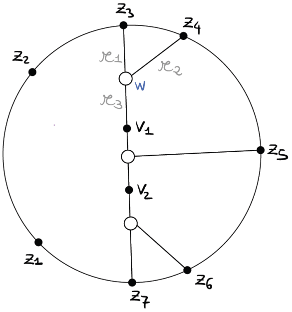
The *boundary* is the point $\mathbf{z} = [z_1 \cdots z_n] \in Gr_{m,n}$.



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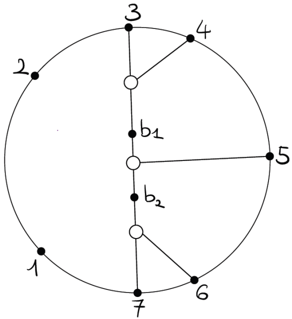

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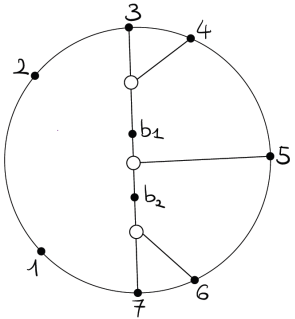

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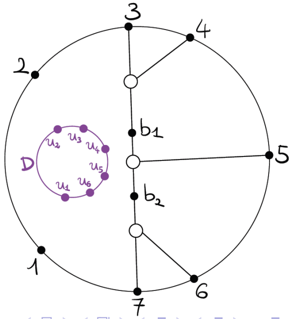

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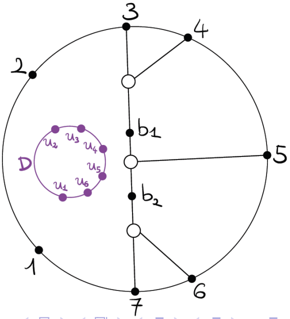

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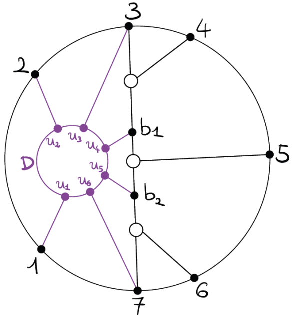

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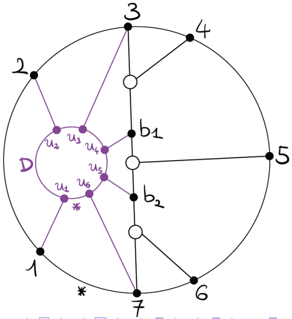

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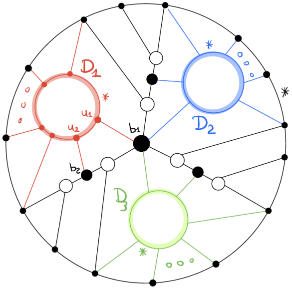

Solvable tangles

Solvable graphs and solvable tangles

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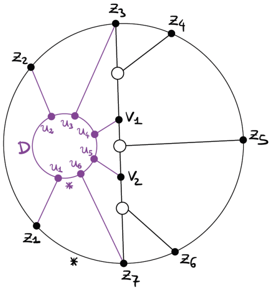

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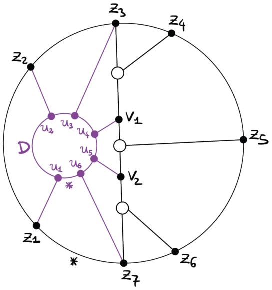

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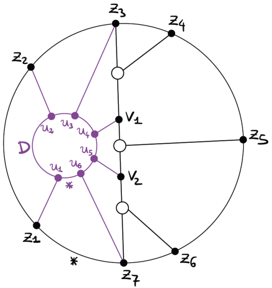

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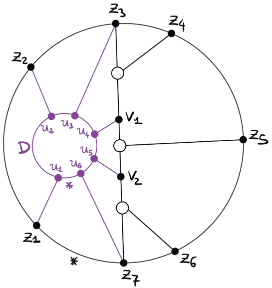

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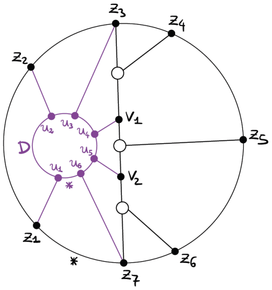

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Solvable tangles

Solvable graphs and solvable tangles

A plabic graph G with n boundary vertices is *m-solvable* if, for generic $\mathbf{z} = [z_1 \cdots z_n] \in \mathrm{Gr}_{m,n}$, there is a unique m-VRC on G with boundary $\mathbf{z}$, modulo gauge.

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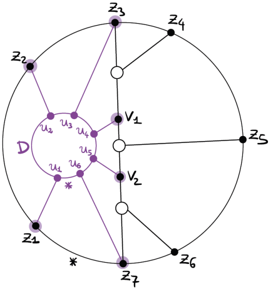

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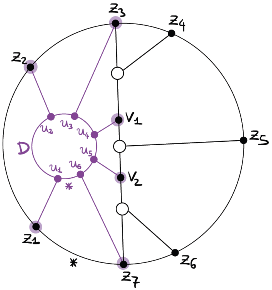

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Promotion Maps

Let $(G, \mathbf{D})$ be an m-solvable plabic tangle with $\mathbf{D} = \{D_1, \ldots, D_\ell\}$ with a pinning.
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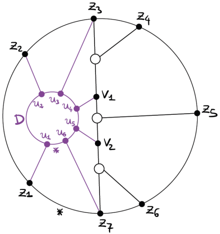

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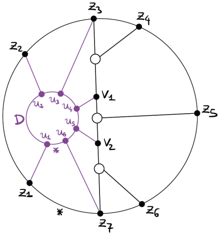

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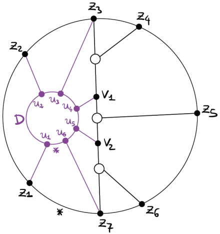

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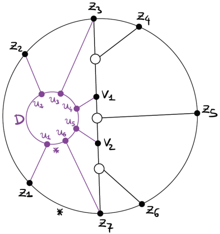

Promotion maps interact nicely with cluster structure (and positivity) on the Grassmannian!

Cluster algebras and quasi-homomorphisms

Cluster algebras [Fomin, Zelevinsky '02]

Cluster algebras are a class of commutative rings with remarkable combinatorial structure. They have distinguished generators called *cluster variables*, grouped in (overlapping) sets of the same size called *clusters*, and relations among them are encoded by *mutations* and *quivers*.

Seeds and mutations

Seed Σ: quiver with cluster var x_i at each node.
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$$
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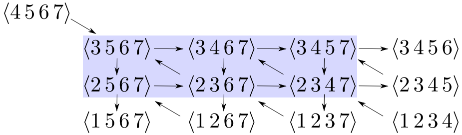

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E.g. also used in [Gorsky–Scroggin '24].

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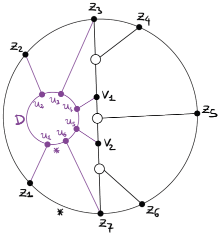

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A plabic tangle is *dominant* if φ has Zariski-dense image.

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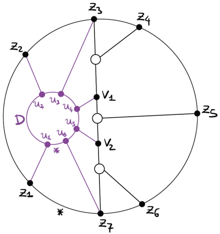

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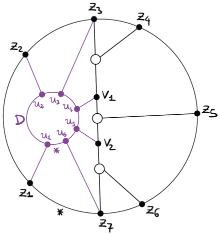

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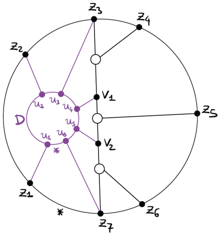

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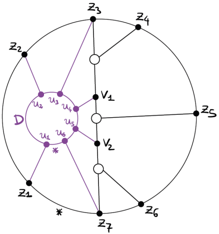

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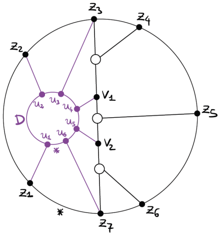

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$$\begin{tikzcd}
\langle 4567\rangle \arrow[dr] & & & & \\
& \langle 3567\rangle \arrow[r] \arrow[d] & \langle 3467\rangle \arrow[r] \arrow[d] & \langle 3457\rangle \arrow[r] \arrow[d] & \langle 3456\rangle \\
& \langle 2567\rangle \arrow[r] \arrow[d] & \langle 2367\rangle \arrow[r] \arrow[d] \arrow[ul] & \langle 2347\rangle \arrow[r] \arrow[d] \arrow[ul] & \langle 2345\rangle \arrow[ul] \\
& \langle 1567\rangle & \langle 1267\rangle \arrow[ul] & \langle 1237\rangle \arrow[ul] & \langle 1234\rangle \arrow[ul]
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is a cluster var for $\mathrm{Gr}_{4,7}$, obtained by mutating $\langle 2367\rangle$ in $\Sigma_{4,7}$.

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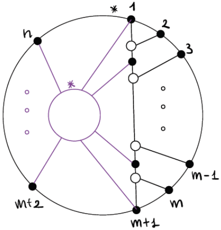

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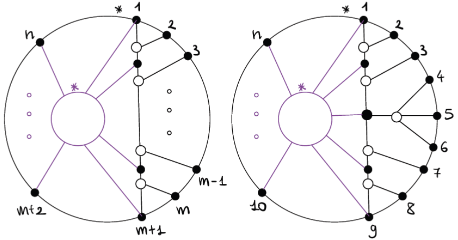

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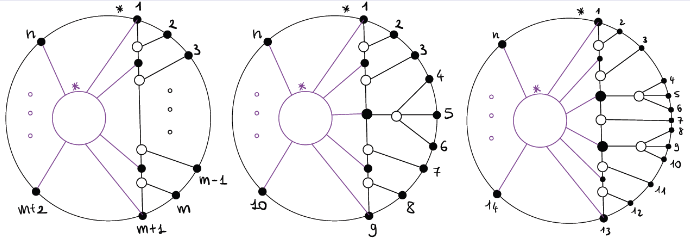

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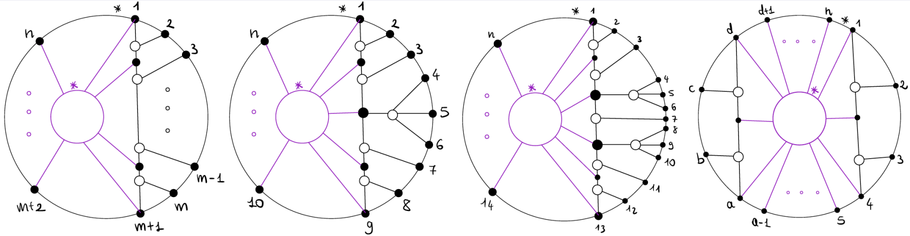

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  1 incoming edge at each white vertex and 1 outgoing edge at each internal black vertex.

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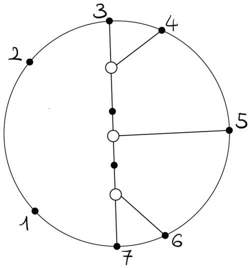

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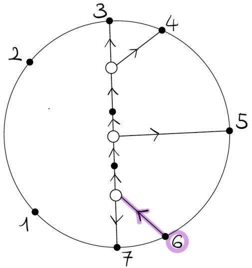

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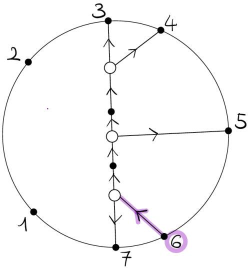

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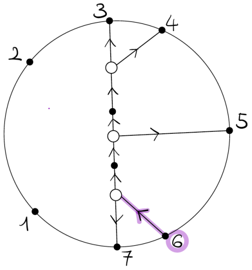

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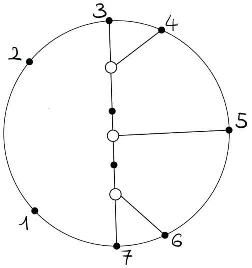

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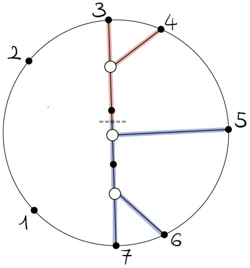

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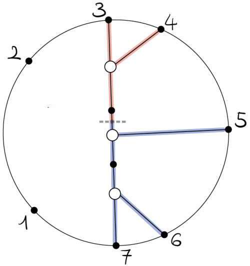

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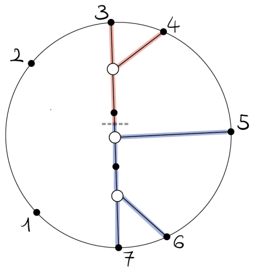
G_1 of type $(1, 3)$: $4 \cdot (1 - 1) < 3 - 1 \leq 4 \cdot 1$ ✓



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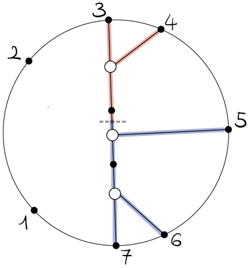

Counting Solvable Trees

Let $c_{k,m} := \#$ m-solvable trees of type $(k, km+1)/\sim$, $\qquad G \sim G'$ if $\mathcal{M}(G) = \mathcal{M}(G')$.

$m \backslash k$	1	2	3	4	5
1	1	1	1	1	1
2	1	5	35	285	2530
3	1	14	280	6565	160916
4	1	30	1274	63410	
5	1	55	4228	380310	
6	1	91	11438		

Theorem [EPSBTW '25]

When $k = 1$, we have $c_{1,m} = 1$; for $k = 2, 3$ the generating functions are:

$$\sum_{m \geq 0} c_{2,m} x^m = \frac{1+x}{(1-x)^4} \quad \text{and} \quad \sum_{m \geq 0} c_{3,m} x^m = \frac{1 + 28x + 56x^2 + 14x^3}{(1-x)^7},$$

$$\text{where } c_{2,m} = 1^2 + 2^2 + \cdots + m^2 = \frac{m(m+1)(2m+1)}{6}$$

is the sequence of *square pyramidal numbers* (cf. A000330 in *OEIS*).

For fixed k, we expect $c_{k,m}$ to satisfy a generating function of the form $p(x)/(1-x)^{3k-2}$.

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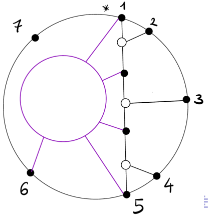

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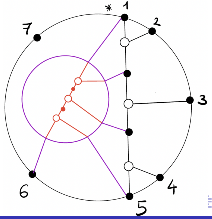

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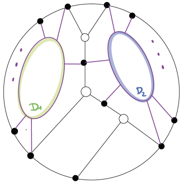

BCFW plabic tangle

Plabic tangles as recursive structure for:

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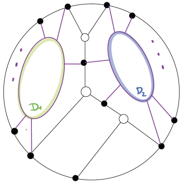

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Proved for *BCFW* tiles and promotions, see [FPSAC '24].

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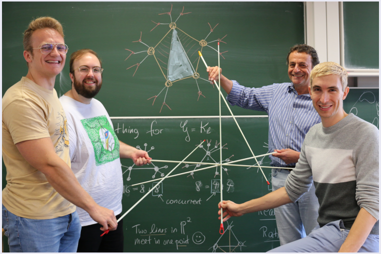

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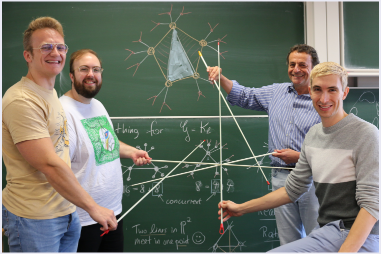

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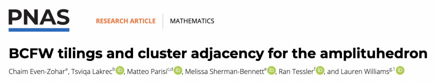

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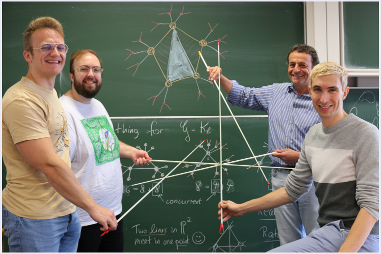

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['13 Golden, Goncharov, Spradlin, Vergu, Volovich; '17 Drummond, Foster, Gürdogan]

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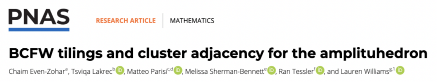

PNAS RESEARCH ARTICLE | MATHEMATICS

BCFW tilings and cluster adjacency for the amplituhedron

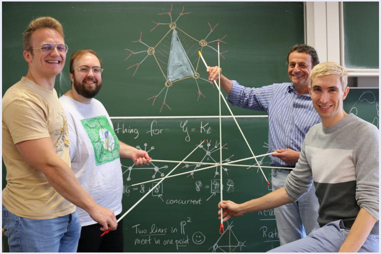
Chaim Even-Zohar[a], Tsviqa Lakrec[b], Matteo Parisi[c,d], Melissa Sherman-Bennett[e], Ran Tessler[f], and Lauren Williams[g,1]

- at tree-level, poles of amplitudes come from facets of amplituhedron tiles;
- for BCFW tiles, cluster promotion maps recursively produce the compatible cluster variables cutting out these facets.

Even-Zohar–Lakrec–P.–Sherman-Bennett–Tessler–Williams

Cluster promotion maps provide the mechanism for cluster variables appearing as singularities in scattering amplitudes.

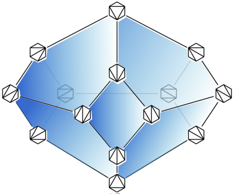
Even-Zohar, P., Sherman-Bennett, Tessler, Williams, *Plabic tangles and cluster promotion maps*, Journal of Algebra 695 (2026), 14–102.

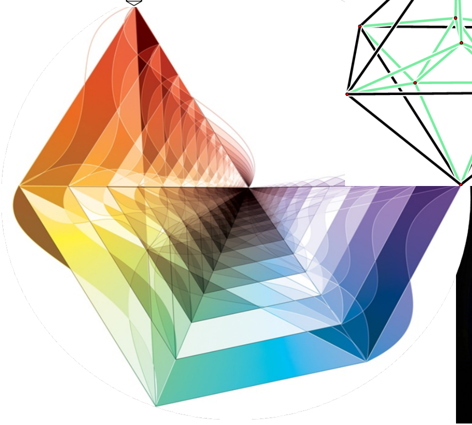


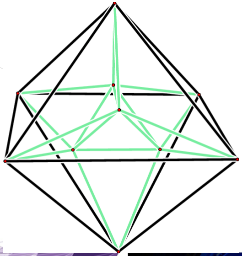
- beyond tree level, singularities are branch points of Feynman integrals/periods;
- in recent work, these arise as discriminants of projections from line-incidence varieties;
- the corresponding recursions involve cluster promotion maps and produce cluster vars.

Hollering–Mazzucchelli–P.–Sturmfels

Questions?

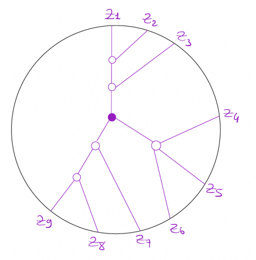






Extras: Spurion Promotion

Definition (Spurion Promotion) [EPSBTW '25]



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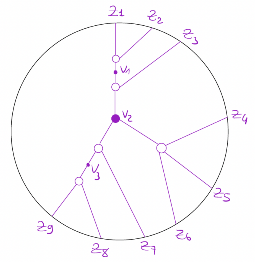

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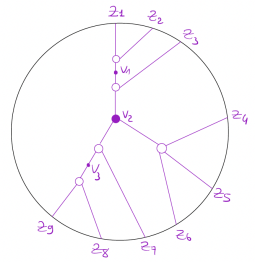

$v_2 \in \mathrm{Span}\{z_1, z_2, z_3\} \cap \mathrm{Span}\{z_4, z_5, z_6\} \cap \mathrm{Span}\{z_7, z_8, z_9\}$;

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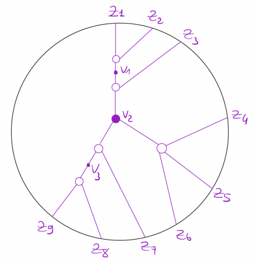

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$$v_2 := (123) \cap (456) \cap (789) := z_1\langle 23|456|789\rangle - z_2\langle 13|456|789\rangle + z_3\langle 12|456|789\rangle,$$

where $-\langle de|abc|fgh\rangle = \langle abc|de|fgh\rangle = \langle abcd\rangle\langle efgh\rangle - (d \leftrightarrow e)$

Extras: Spurion Promotion

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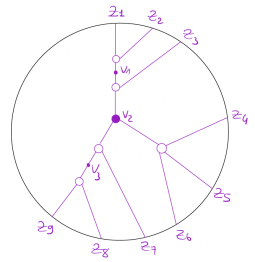

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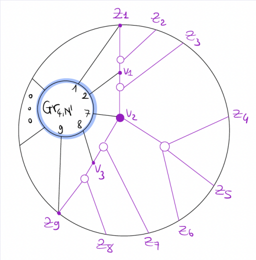
$v_1 \in \mathrm{Span}\{z_1, z_2\} \cap H$, where H is the 3-plane spanned by z_3 and the 2-plane $\mathrm{Span}\{z_4, z_5, z_6\} \cap \mathrm{Span}\{z_7, z_8, z_9\}$, denoted as $H \sim (3(456) \cap (789))$.

Extras: Spurion Promotion

Definition (Spurion Promotion) [EPSBTW '25]



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Extras: Spurion Promotion

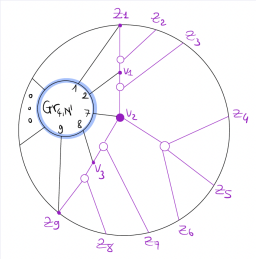
Definition (Spurion Promotion) [EPSBTW '25]

$$\Psi_{sp} : \mathcal{A}(\mathrm{Gr}_{4,N'}) \to \mathcal{A}(\mathrm{Gr}_{4,n})$$

$$2 \mapsto \frac{(21) \cap (3(456) \cap (789))}{\langle 13 \,|\, 456 \,|\, 789 \rangle} \sim v_1$$

$$7 \mapsto \frac{(789) \cap (123) \cap (456)}{\langle 89 \,|\, 123 \,|\, 456 \rangle} \sim v_2$$

$$8 \mapsto \frac{(89) \cap (7(123) \cap (456))}{\langle 97 \,|\, 123 \,|\, 456 \rangle} \sim v_3$$



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$$
\begin{aligned}
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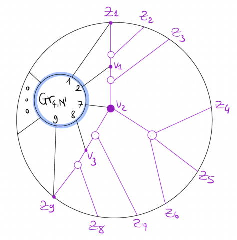
\end{aligned}
$$



Definition (Spurion Promotion) [EPSBTW '25]

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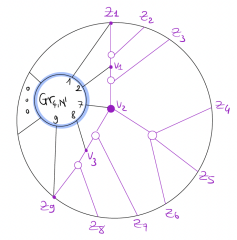

$$\Psi_{sp}(\langle 7ij\ell\rangle) = \frac{\langle 7ij\ell\rangle\langle 89|123|456\rangle - \langle 8ij\ell\rangle\langle 79|123|456\rangle + \langle 9ij\ell\rangle\langle 78|123|456\rangle}{\langle 89|123|456\rangle} =$$

$$= \frac{-\langle ij\ell|789|123|456\rangle}{\langle 89|123|456\rangle} \propto -\langle ij\ell|789|123|456\rangle \quad \text{cubic cluster var for } \mathrm{Gr}_{4,N'}$$

Extras: Spurion Promotion

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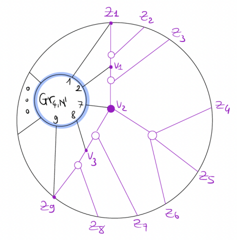

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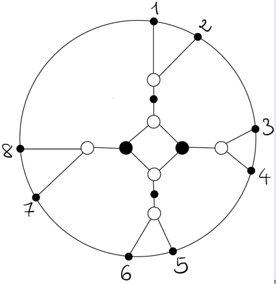
Theorem (Spurion Cluster Promotion Map) [EPSBTW '25]

Ψ_{sp} is a cluster map, i.e. it maps collections of compatible cluster variables of $\mathrm{Gr}_{4,N'}$ to collections of compatible cluster variables of $\mathrm{Gr}_{4,n}$ (up to frozens).

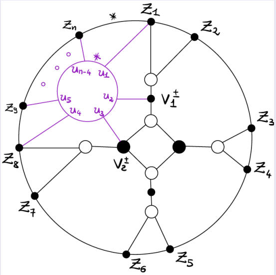
Beyond Rational Promotions

Definition (4-Mass Box Promotion) [EPSBTW '25]



Beyond Rational Promotions

Definition (4-Mass Box Promotion) [EPSBTW '25]



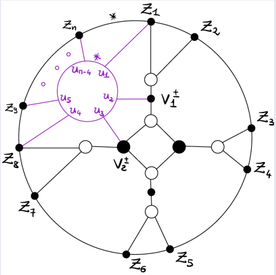
Beyond Rational Promotions

Definition (4-Mass Box Promotion) [EPSBTW '25]

$$\varphi_{\pm} : \mathrm{Gr}_{4,n} \dashrightarrow \mathrm{Gr}_{4,n-4}$$

$$\mathbf{z} = [z_1, \ldots, z_n] \mapsto [z_1, v_1^{\pm}, v_2^{\pm}, z_8, \ldots, z_n]$$

$$v_2^{\pm} = z_7 + \alpha_{\pm} z_8, \quad v_1^{\pm} = z_2 \langle 156 v_2^{\pm} \rangle - z_1 \langle 256 v_2^{\pm} \rangle$$



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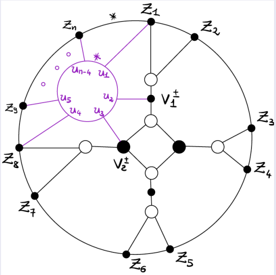
$$v_2^\pm = z_7 + \alpha_\pm z_8, \quad v_1^\pm = z_2\langle 156 v_2^\pm\rangle - z_1\langle 256 v_2^\pm\rangle$$

$$\alpha_\pm := \frac{-B \pm \sqrt{\Delta}}{2A}, \quad \Delta := B^2 - 4AC$$

$$A = Q_{88}, B = Q_{78} + Q_{87}, C = Q_{77},$$

$$Q_{ij} := \langle 12i3\rangle\langle 456j\rangle - \langle 12i4\rangle\langle 356j\rangle$$



Beyond Rational Promotions

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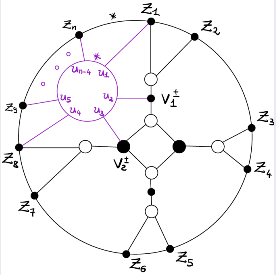

E.g. for $n = 9$: $\Psi_\pm(\langle 1235\rangle) = \langle 12 v_2^\pm 9\rangle = \langle 1279\rangle + \alpha_\pm\langle 1289\rangle$ algebraic function on $\mathrm{Gr}_{4,9}$

Definition (4-Mass Box Promotion) [EPSBTW '25]

$$\varphi_\pm : \mathrm{Gr}_{4,n} \dashrightarrow \mathrm{Gr}_{4,n-4}$$

$$\mathbf{z} = [z_1, \ldots, z_n] \mapsto [z_1, v_1^\pm, v_2^\pm, z_8, \ldots, z_n]$$

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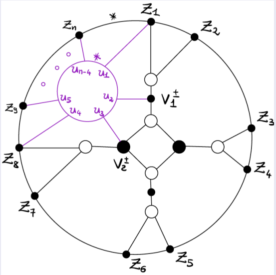

E.g. for $n = 9$: $\Psi_\pm(\langle 1235\rangle) = \langle 12v_2^\pm 9\rangle = \langle 1279\rangle + \alpha_\pm\langle 1289\rangle$ algebraic function on $\mathrm{Gr}_{4,9}$

Theorem (4-mass box Cluster positivity) [EPSBTW '25]

The map $\Psi_\pm$ preserves positivity of cluster vars for $\mathrm{Gr}_{4,n-4}$:
x is a cluster var for $\mathrm{Gr}_{4,n-4} \Rightarrow \Psi_\pm(x) > 0$ on $\mathrm{Gr}_{4,n}^{>0}$. Equivalently, $\varphi_\pm(\mathrm{Gr}_{4,n}^{>0}) \subseteq \mathrm{Gr}_{4,n-4}^{>0}$.

Beyond Rational Promotions

Definition (4-Mass Box Promotion) [EPSBTW '25]

$$\varphi_{\pm} : \mathrm{Gr}_{4,n} \dashrightarrow \mathrm{Gr}_{4,n-4}$$

$$\mathbf{z} = [z_1, \dots, z_n] \mapsto [z_1, v_1^{\pm}, v_2^{\pm}, z_8, \dots, z_n]$$

$$v_2^{\pm} = z_7 + \alpha_{\pm} z_8, \quad v_1^{\pm} = z_2 \langle 156 v_2^{\pm} \rangle - z_1 \langle 256 v_2^{\pm} \rangle$$

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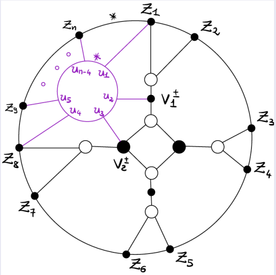

E.g. for $n = 9$: $\Psi_{\pm}(\langle 1235 \rangle) = \langle 12 v_2^{\pm} 9 \rangle = \langle 1279 \rangle + \alpha_{\pm} \langle 1289 \rangle$ algebraic function on $\mathrm{Gr}_{4,9}$

Theorem (4-mass box Cluster positivity) [EPSBTW '25]

The map $\Psi_{\pm}$ preserves positivity of cluster vars for $\mathrm{Gr}_{4,n-4}$:
x is a cluster var for $\mathrm{Gr}_{4,n-4} \Rightarrow \Psi_{\pm}(x) > 0$ on $\mathrm{Gr}_{4,n}^{>0}$. Equivalently, $\varphi_{\pm}(\mathrm{Gr}_{4,n}^{>0}) \subseteq \mathrm{Gr}_{4,n-4}^{>0}$.

We expect this positivity property holds for (non rational) promotion maps with $\mathrm{IN} > 1$.