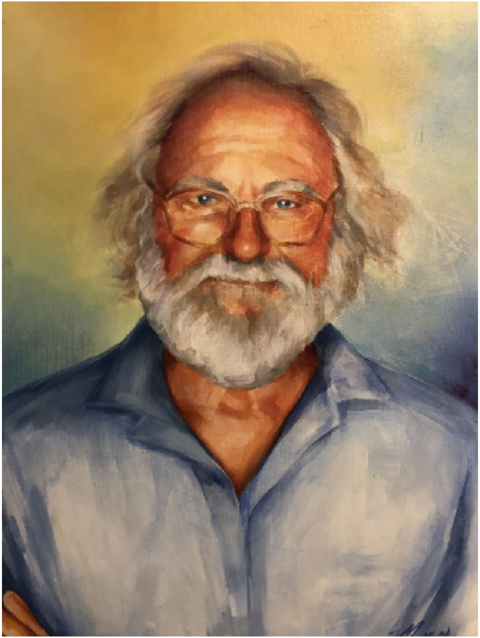
Adriano Garsia

(1928–2024)



Painting by Maya Santangelo

Adriano passed down great generosity

He fathered many FPSAC organizers

Nantel Bergeron – Toronto, Helene Barcelo – Arizona,
Luisa Carini – Taormina, Mike Zabrocki – San Diego,
Luc Lapointe – Valparaiso, Sara Billey – Seattle

In a nutshell

Adriano believed that there was no boundary between

family friendship mathematics life.

every visitor could become a collaborator,

every collaborator could become a friend,

every math exchange could be had at the beach and could become a shared dinner.

more tributes to Adriano: https://arxiv.org/abs/2607.14184

Adriano had a colorful career

Probability, Ergodic Theor Analysis

Transition to Combinatorics

Macdonald Theory and q,t-Combinatorics

1955 — 1975 — 1988 — 2024

- Bernoulli convolutions
- Garsia entropy
- Garsia–Rodemich–Rumsey inequality
- Hopf's inequality

- q-Lagrange inversion
- Garsia–Milne involution principle
- Cohen–Macaulay posets
- symmetric functions

- modified Macdonald polynomials
- Garsia–Procesi modules
- $n!$ conjecture
- diagonal harmonics
- ∇ operator
- q,t-Catalans and Dyck paths

more than three decades

Throughout, he distilled complicated math into simple truths

We'll talk about

Involution principle

- influential
- hallmark Adriano presentation
- computer exploration
- elements of drama

The beginnings of modified Macdonald theory


COHEN
MACAULAY
RINGS

The Rogers-Ramanujan identities

$$1+\sum_{n\geq 0}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{k\geq 1}\frac{1}{(1-q^{5k-4})(1-q^{5k-1})}$$

$$1+\sum_{n\geq 0}\frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{k\geq 1}\frac{1}{(1-q^{5k-2})(1-q^{5k-3})}$$

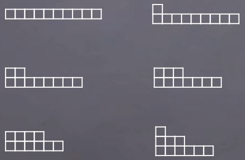
$$1 + \sum_{n \geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{k \geq 1} \frac{1}{(1-q^{5k-4})(1-q^{5k-1})}$$

partitions of n with parts differing by at least 2	$\leftrightarrow$	partitions of n with parts equivalent to 1, 4 (mod 5)

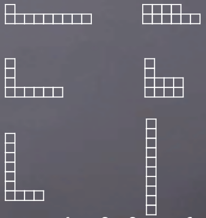
$$1 + \sum_{n \geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{k \geq 1} \frac{1}{(1-q^{5k-4})(1-q^{5k-1})}$$

partitions of n with parts differing by at least 2 $\leftrightarrow$ partitions of n with parts equivalent to 1, 4 (mod 5)

$$n = 10$$



(10), (9, 1), (8, 2), (7, 3), (6, 4), (6, 3, 1)



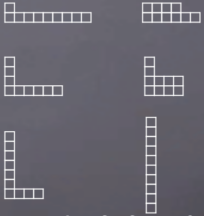
(9, 1), (6, 4), (6, 1^4), (4^2, 1^2), (4, 1^6), (1^{10})

$$1+\sum_{n\geq 0}\frac{q^{1+3+5+\cdots+(2n-1)}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{k\geq 1}\frac{1}{(1-q^{5k-4})(1-q^{5k-1})}$$

partitions of n with parts differing by at least 2 $\leftrightarrow$ partitions of n with parts equivalent to 1, 4 (mod 5)

$n = 10$

(**10**), (**9**, **1**), (**8**, **2**), (**7**, **3**), (**6**, **4**), (**6**, **3**, **1**)



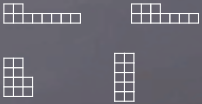
(**9**, **1**), (**6**, **4**), (**6**, $\mathbf{1^4}$), ($\mathbf{4^2}$, $\mathbf{1^2}$), (**4**, $\mathbf{1^6}$), ($\mathbf{1^{10}}$)

$$1+\sum_{n\geq 0}\frac{q^{2+4+\cdots+2n}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{k\geq 1}\frac{1}{(1-q^{5k-2})(1-q^{5k-3})}$$

partitions of n with parts > 1 differing by at least 2 $\leftrightarrow$ partitions of n with parts equivalent to 2, 3 (mod 5)

$$n = 10$$

$(\mathbf{10}), (\mathbf{8}, \mathbf{2}), (\mathbf{7}, \mathbf{3}), (\mathbf{6}, \mathbf{4})$



$(\mathbf{8}, \mathbf{2}), (\mathbf{7}, \mathbf{3}), (\mathbf{3}, \mathbf{3}, \mathbf{2}, \mathbf{2}), (\mathbf{2}^{\mathbf{5}})$

Jacobi Triple Product identity (Jacobi, 1829)

$$\prod_{m=1}^{\infty} \left(1 - x^{2m}\right)\left(1 + x^{2m-1}y^2\right)\left(1 + \frac{x^{2m-1}}{y^2}\right) = \sum_{n=-\infty}^{\infty} x^{n^2} y^{2n}$$

Euler's pentagonal number theorem (combinatorial proof: Franklin 1881)

$$\prod_{n=1}^{\infty} (1 - x^n) = \sum_{k=-\infty}^{\infty} (-1)^k \, x^{k(3k-1)/2}$$

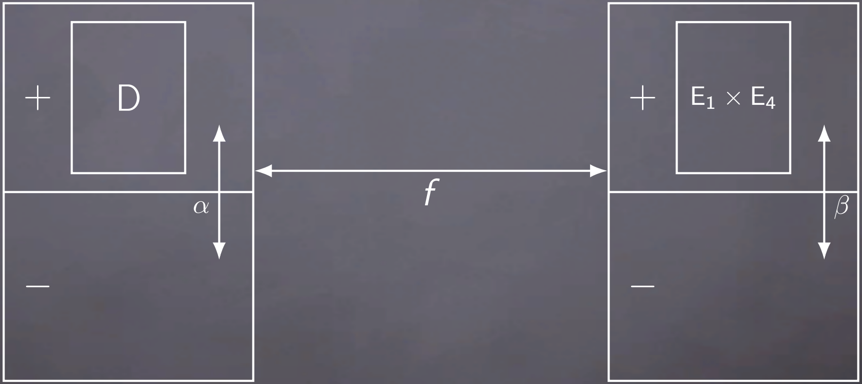
Doron Zeilberger (Experimenting with the Garsia-Milne Involution Principle, 2025):

> Like Christopher Columbus [Wik1] and Alexander the Great [Wik2] before them, Garsia and Milne had a brilliant idea: think outside the box!

D partitions of n whose parts differ by at least 2.
$E_1 \times E_2$ partitions of n with parts 1 and 4 mod 5.

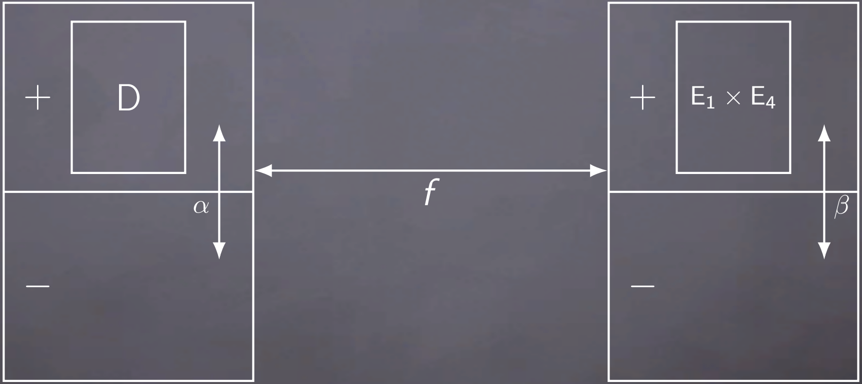
$$\boxed{D} \longleftrightarrow \boxed{E_1 \times E_4}$$

Sign reversing involution α with fixed points D,
Sign reversing involution β with fixed points $E_1 \times E_2$,
sign preserving bijection f



$$\alpha^* := f \circ \alpha \qquad \beta^* := f^{-1} \circ \beta$$

$$a_1 \in \mathsf{D}, b_1 := \alpha^*(a_1), a_2 := \beta^*(b_1), b_2 := \alpha^*(a_2), \ldots, b_n := \alpha^*(a_n) \in \mathsf{E}_1 \times \mathsf{E}_2$$



JCT A, **31**, 289–339 (1981). November

A Rogers–Ramanujan Bijection

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AND

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Communicated by the Managing Editors

Received November 1, 1980

In this paper we give a bijection between the partitions of n with parts congruent to 1 or 4 (mod 5) and the partitions of n with parts differing by at least 2. This bijection is obtained by a cut-and-paste procedure which starts with a partition in one class and ends with a partition in the other class. The whole construction is a combination of a bijection discovered quite early by Schur and two bijections of our own. A basic principle concerning pairs of involutions provides the key for connecting all these bijections. It appears that our methods lead to an algorithm for constructing bijections for other identities of Rogers–Ramanujan type such as the Gordon identities.

A distinctly 'Adriano' presentation of the paper

The actual final bijection itself is given in Section 6 as a completely debugged APL program.

...The APL language is one of the easiest computer languages to learn and one which most closely resembles the mathematical language of common use, but is infinitely more precise. The reader should have no difficulty learning what is needed of APL here.

```
     ∇ ALPHA
[1]    MIN←⌊/EZ,E1,E2,E3,E4,E5
[2]    →G×⍳~(MIN∊EZ)
[3]    EZ←EZ[⍳¯1+⍴EZ]
[4]    →GATE1+2×5|MIN
[5]   GATE1:E5←E5,MIN
[6]    →END
[7]    E1←E1,MIN
[8]    →END
[9]    E2←E2,MIN
[10]   →END
[11]   E3←E3,MIN
[12]   →END
[13]   E4←E4,MIN
[14]   →END
[15] G:EZ←EZ,MIN
[16]   →GATE2+2×5|MIN
[17] GATE2:E5←E5[⍳¯1+⍴E5]
[18]   →END
[19]   E1←E1[⍳¯1+⍴E1]
[20]   →END
[21]   E2←E2[⍳¯1+⍴E2]
[22]   →END
[23]   E3←E3[⍳¯1+⍴E3]
[24]   →END
[25]   E4←E4[⍳¯1+⍴E4]
[26] END:
     ∇
```

```
     ∇ BETA
[1]    →OK×⍳(0=(⍴EZ)+(⍴D))
[2]    →G2×⍳(0=⍴EZ)∧(0≠⍴D)
[3]    →G3×⍳(0≠⍴EZ)∧(0=⍴D)
[4]    →G4×⍳(EZ[1]≥D[1]+2)
[5]    →G5×⍳(D[1]≥EZ[1]+1)
[6]   OK:CLASSDET
[7]    →GATEBETA
[8]   G2:EZ←(⍳0),D[1]
[9]    D←(0,(¯1+⍴D)⍴1)/D
[10]   →GATEBETA
[11] G3:D←EZ[1]
[12]   EZ←(0,(¯1+⍴EZ)⍴1)/EZ
[13]   →GATEBETA
[14] G4:D←EZ[1],D
[15]   EZ←(0,(¯1+⍴EZ)⍴1)/EZ
[16]   →GATEBETA
[17] G5:EZ←D[1],EZ
[18]   D←(0,(¯1+⍴D)⍴1)/D
[19] GATEBETA:
     ∇
```

Discrete Mathematics **38** (1982) 313–315

COMMUNICATION

A SHORT ROGERS–RAMANUJAN BIJECTION

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Communicated by D.M. Jackson
Received 17 September 1981

Recently Garsia and Milne [2, 3] proved this result by presenting a bijection between $A(n)$ and $C(n)$. We are going to give another bijective proof which is very similar in form to that of Garsia and Milne in that it involves an iteration of two involutions .... However, our second involution is considerably simpler than that provided by Garsia and Milne.

Doron Zeilberger, Enumerative and Algebraic Combinatorics, in: Princeton Companion to Mathematics, (Timothy Gowers, ed.), Princeton University Press, 550–561, 2008.

While the so-called Wilf–Zeilberger (WZ) method can handle many such problems, there are many other cases where one still needs a human proof. In either case such proofs involve (algebraic, and sometimes analytic) manipulations. The great combinatorialist Adriano Garsia derogatorily calls such proofs "manipulatorics," and real enumerators do not manipulate, or at least try to avoid it whenever possible. The preferred method of proof is by bijection.

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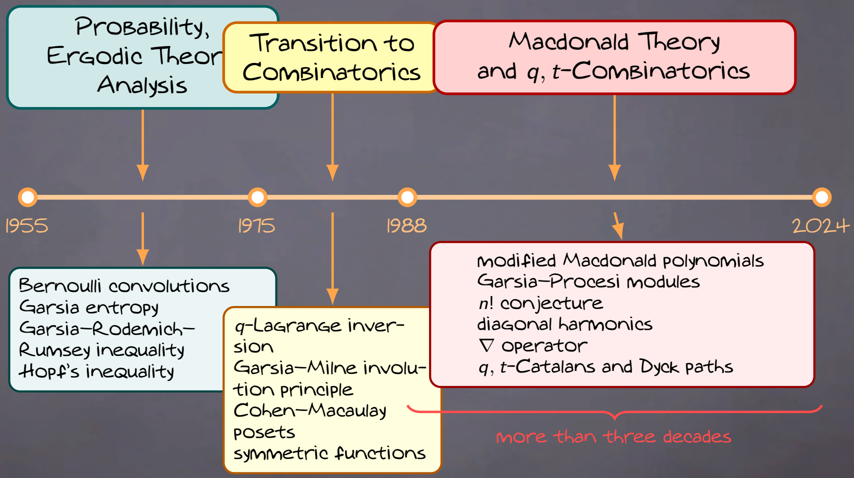
Experimenting with the Garsia-Milne Involution Principle

Shalosh B. EKHAD and Doron ZEILBERGER

In memory of Adriano M. Garsia (1928-2024) and in honor of Stephen C. Milne on his 75th birthday

Abstract: In 1981, Adriano Garsia and Stephen Milne found the first bijective proof of the celebrated Rogers-Ramanujan identities. To achieve this feat, they invented a versatile tool that they called the Involution Principle. In this note we revisit this useful principle from a very general perspective, independent of its application to specific combinatorial identities, and will explore its complexity.

Adriano had a colorful career



Throughout, he distilled complicated math into simple truths

The 80's

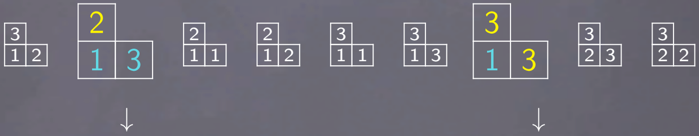
Adriano gets interested in symmetric functions

Ian Macdonald was thinking about symmetric functions, too!

The 80's

Adriano gets interested in symmetric functions

Lessons of Young



$$s_{\yng(2,1)} = x_1x_2x_3 + x_1x_2x_3 + x_1x_1x_2 + x_1x_2x_2 + x_1x_1x_3 + x_1x_3x_3 + x_2x_3x_3 + x_2x_2x_3$$

Ian Macdonald was thinking about symmetric functions, too!

Orthogonal Bases

Macdonald's Brain

Definition of Schur functions

- an orthonormal basis
- unitriangularly related to $\{m_\lambda\}$

Inner product on Λ

$$\langle p_\lambda, p_\mu \rangle = \begin{cases} 0 & \text{if } \lambda \neq \mu \\ \text{some number } z_\lambda & \text{otherwise} \end{cases}$$

Compute $s_{2,1}$

triangularity $\implies$ $s_{1,1,1} = m_{1,1,1}$

$s_{2,1} = m_{2,1} + ??m_{1,1,1} \implies s_{2,1} = m_{2,1} + ??s_{1,1,1}$

Macdonald's Brain

Definition of Schur functions

- an orthonormal basis
- unitriangularly related to $\{m_\lambda\}$

Inner product on Λ

$$\langle p_\lambda, p_\mu \rangle = \begin{cases} 0 & \text{if } \lambda \neq \mu \\ \text{some number } z_\lambda & \text{otherwise} \end{cases}$$

Compute $s_{2,1}$

triangularity $\Longrightarrow$ $s_{1,1,1} = m_{1,1,1}$

$$s_{2,1} = m_{2,1} + ??m_{1,1,1} \Longrightarrow s_{2,1} = m_{2,1} + ??s_{1,1,1}$$

orthogonality $\Longrightarrow$

$$0 = \langle s_{2,1}, s_{1,1,1} \rangle = \langle m_{2,1} + ??s_{1,1,1}, s_{1,1,1} \rangle$$

$$= \langle m_{2,1}, s_{1,1,1} \rangle + ??$$

$$0 = \langle p_{2,1} - p_3, \tfrac{1}{6}p_{1,1,1} - \tfrac{1}{2}p_{2,1} + \tfrac{1}{3}p_3 \rangle + ??$$

1988: Macdonald goes wild

Study symmetric functions over $\mathbb{Q}(q,t)$!

Inner product on Λ

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \begin{cases} 0 & \text{if } \lambda \neq \mu \\ z_\lambda \prod_i \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}} & \text{otherwise} \end{cases}$$

A basis $\{P_\lambda\}$ is uniquely defined by

- unitriangularity wrt $\{m_\lambda\}$
- orthogonality

$$P_{(2,2)} = m_{(2,2)} + \frac{(qt-q+t-1)}{(qt-1)} m_{(3,1)} + \frac{(2q^2t^3-3q^2t^2+3qt^3-3qt^2+t^3+q^2-3qt+3q-3t+2)}{(q^2t^3-qt^2-qt+1)} m_{(1,1,1,1)}$$

Macdonald's bold conjecture

A basis $\{P_\lambda\}$ is uniquely defined by

- unitriangularity wrt $\{m_\lambda\}$
- orthogonality

$$P_{22} = m_{22} + \frac{(qt-q+t-1)}{(qt-1)} m_{31} + \frac{(2q^2t^3 - 3q^2t^2 + 3qt^3 - 3qt^2 + t^3 + q^2 - 3qt + 3q - 3t + 2)}{(q^2t^3 - qt^2 - qt + 1)} m_{1111}$$

Conjecture $\star P_\lambda$ has a q,t-positive "funny" basis expansion

one q,t monomial for each standard tableaux

$$\star\ P_{22} = t^2 f_{4} + (qt^2 + qt + t)\, f_{31} + (q^2t^2 + 1)\, f_{22} + (q^2t + qt + q)\, f_{211} + q^2 f_{1111}$$

$\begin{array}{cccc}1&2&3&4\end{array}$ $\begin{array}{ccc}2&&\\1&3&4\end{array}$ $\begin{array}{ccc}3&&\\1&2&4\end{array}$ $\begin{array}{ccc}4&&\\1&2&3\end{array}$ $\begin{array}{cc}3&4\\1&2\end{array}$ $\begin{array}{cc}2&4\\1&3\end{array}$ $\begin{array}{cc}4&\\2&\\1&3\end{array}$ $\begin{array}{cc}4&\\3&\\1&2\end{array}$ $\begin{array}{cc}3&\\2&\\1&4\end{array}$ $\begin{array}{c}4\\3\\2\\1\end{array}$

Garsia's reaction

Macdonald polynomials, P_λ

- unitriangular wrt $\{m_\lambda\}$
- orthogonal

Conjecture: q,t-positive "funny" basis expansion

$$\star\, P_{22} = t^2 f_{4} + (qt^2 + qt + t) f_{31} + (q^2t^2 + 1) f_{22} + (q^2t + qt + q) f_{211} + q^2 f_{1111}$$

$$\mathrm{pleth} : f_\lambda \mapsto s_\lambda$$

$$H_{22} = t^2 s_{4} + (qt^2 + qt + t) s_{31} + (q^2t^2 + 1) s_{22} + (q^2t + qt + q) s_{211} + q^2 s_{1111}$$

Def: Modified Macdonald polynomials

$$H_\lambda(\mathbf{x}; q, t) = \star\, \mathrm{pleth}(P_\lambda(\mathbf{x}; q, t))$$

Conjecture: modified Macdonalds are q,t-Schur positive

Young explored interplay of modules, symmetric functions, combinatorics

Harmonic polynomials in $x_1, \ldots, x_n$

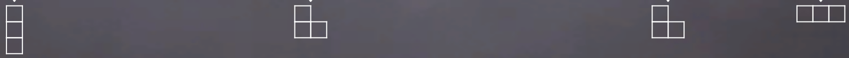
$$\mathcal{M}_n = \{\, f(x) : (\partial^a_{x_1} + \cdots + \partial^a_{x_n})\, f(x) = 0 \;\forall a > 0 \,\}$$

$$\Delta, \quad 2x_1(x_2 - x_3) - x_2^2 + x_3^2, \qquad x_3 - x_1, \qquad x_2 - x_3, \qquad 1$$

a vector space of dimension $n!$

closed under permutations of the variables

$$\underbrace{\mathrm{sp}\{\Delta\}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2 - x_3) - x_2^2 + x_3^2,\, x_1^2 - 2x_2(x_1 - x_3) - x_3^2\}} \oplus \underbrace{\mathrm{sp}\{x_3 - x_1,\, x_2 - x_3\}} \oplus \underbrace{\mathrm{sp}\{1\}}$$



Building blocks (irreducibles) indexed by partitions

modules to symmetric functions

Harmonic polynomials in $x_1, \ldots, x_n$

$$\mathcal{M}_n = \{\, f(x) : \left(\partial_{x_1}^a + \cdots + \partial_{x_n}^a\right) f(x) = 0 \ \forall a > 0 \,\}$$

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2,\, x_1^2-2x_2(x_1-x_3)-x_3^2\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1,\, x_2-x_3\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

Frobenius $\mathcal{F}$

irreducible submodule $V^\lambda \mapsto s_\lambda$

some symmetric function $= s_{\ydiagram{1,1,1}} + s_{\ydiagram{1,2}} + s_{\ydiagram{1,2}} + s_{\ydiagram{3}}$

$\mathcal{F}(\mathcal{M})$ = Schur positive symmetric function

coefficient of s_λ in $\mathcal{F}(\mathcal{M})$ = # irreducibles V^λ in $\mathcal{M}$

modules to combinatorics

Harmonic polynomials in $x_1, \ldots, x_n$

$$\mathcal{M}_n = \{\, f(x) : \left(\partial_{x_1}^a + \cdots + \partial_{x_n}^a\right) f(x) = 0 \;\forall a > 0 \,\}$$

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2,\, x_1^2-2x_2(x_1-x_3)-x_3^2\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1,\, x_2-x_3\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

Frobenius $\mathcal{F}$

irreducible $V^\lambda \mapsto s_\lambda$

$$\mathcal{F}(\mathcal{M}) = (z_1+z_2+z_2)^3 = s_{\ytableaushort{3,2,1}} + s_{\ytableaushort{2,13}} + s_{\ytableaushort{3,12}} + s_{\ytableaushort{123}}$$

Study Schur expansion of $\mathcal{F}(\mathcal{M})$ to understand irreducible decomposition of the module, $\mathcal{M}$

Garsia's idea

Harmonic polynomials in $x_1, \ldots, x_n$

$$\mathcal{M} = \{\, f(x) : (\partial_{x_1}^a + \cdots + \partial_{x_n}^a)\, f(x) = 0 \ \forall a > 0 \,\}$$

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2,\, x_1^2-2x_2(x_1-x_3)-x_3^2\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1,\, x_2-x_3\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

Frobenius $\mathcal{F}$

irreducible $V^\lambda \mapsto s_\lambda$

$$\mathcal{F}(\mathcal{M}) = (z_1 + z_2 + z_2)^3 = s_{\ytableaushort{3,2,1}} + s_{\ytableaushort{2,13}} + s_{\ytableaushort{3,12}} + s_{\ytableaushort{123}}$$

a modified Macdonald when $q = t = 1$, for any partition μ

Garsia's idea

~~Harmonic~~ polynomials in $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$

$\mathcal{M}_\mu = ???$ $\cancel{\mathcal{M} = \{ f(x) : (\partial^a_{x_1} + \cdots + \partial^a_{x_n}) f(x) = 0 \; \forall a > 0 \}}$

$$\underbrace{\mathrm{sp}\{1\}}_{(0,0)} \oplus \underbrace{\mathrm{sp}\{x_3 - x_1, x_1 - x_2\}} \oplus \underbrace{\mathrm{sp}\{y_3 - y_1, y_1 - y_2\}}_{(1,0)} \oplus \underbrace{\mathrm{sp}\{x_1 y_1 - x_2 y_2\}}_{(1,1)}$$

degree 0 in y degree 1 in x

V_λ with bidegree $(r, d) \mapsto t^r q^d s_\lambda$

$$\widetilde{\mathrm{H}}_{2,1} = t^0 q^0 s_{\boxed{1}\boxed{2}\boxed{3}} + t^0 q^1 s_{\substack{3\\12}} + t^1 q^0 s_{\substack{2\\13}} + t^1 q^1 s_{\substack{3\\2\\1}}$$

Find bigraded modules, $\mathcal{M}_\mu$, each of dimension $n!$

whose Frobenius images are the modified Macdonalds, $\widetilde{\mathrm{H}}_\mu$.

It can it be done when $q = 0$!

Garsia-Procesi gave an elegant proof

For each partition μ, the modified Macdonald, $\tilde{H}_\mu(x; 0, t)$, is the Frobenius image of a graded module in $x_1, \ldots, x_n$,

Garsia-Procesi Modules (the $q = 0$ solution)

A result of reworking scattered and new ideas until the perfect solution was brought forth

algebraic geometry (Hotta-Springer and DiConcini-Procesi), rep theory (Luzstig), algebra (Tanisaki), and combinatorics (Lascoux-Schützenberger)

But what about Macdonald's conjecture, and the q parameter?

Garsia's idea

~~Harmonic~~ polynomials in $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$

$\mathcal{M}_\mu = ???$ $\cancel{\mathcal{M} = \{ f(x) : (\partial^a_{x_1} + \cdots + \partial^a_{x_n}) f(x) = 0 \ \forall a > 0 \}}$

$$\underbrace{\mathrm{sp}\{1\}}_{(0,0)} \oplus \underbrace{\mathrm{sp}\{x_3 - x_1, x_1 - x_2\}}_{} \oplus \underbrace{\mathrm{sp}\{y_3 - y_1, y_1 - y_2\}}_{(1,0)} \oplus \underbrace{\mathrm{sp}\{x_1 y_1 - x_2 y_2\}}_{(1,1)}$$

degree 0 in y degree 1 in x

V_λ with bidegree $(r, d) \mapsto t^r q^d s_\lambda$

$$\widetilde{\mathrm{H}}_{2,1} = t^0 q^0 s_{\boxed{1|2|3}} + t^0 q^1 s_{\substack{3\\12}} + t^1 q^0 s_{\substack{2\\13}} + t^1 q^1 s_{\substack{3\\2\\1}}$$

Find bigraded modules, $\mathcal{M}_\mu$, of dimension $n!$

whose Frobenius images are the modified Macdonalds, $\widetilde{\mathrm{H}}_\mu$.

Garsia-Haiman seek bigraded modules

harmonic polynomials in $\boldsymbol{x}$

$f(\boldsymbol{x})$ killed by $\sum_i \partial_{x_i}^a$ for all $a > 0$

diagonal harmonics in $\boldsymbol{x}, \boldsymbol{y}$

$f(\boldsymbol{x}, \boldsymbol{y})$ killed by $\sum_i \partial_{x_i}^a \partial_{y_i}^b \; \forall a + b > 0$

$\downarrow$

not a Macdonald polynomial

Garsia-Haiman seek bigraded modules

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span of all partials of the Vandermonde

Garsia-Haiman modules

Vandermondes in $\mathbf{x}$ and $\mathbf{y}$

$\downarrow$

modified Macdonald

diagonal harmonics in $\mathbf{x}, \mathbf{y}$

$f(\mathbf{x}, \mathbf{y})$ killed by $\sum_i \partial_{x_i}^a \partial_{y_i}^b \; \forall a + b > 0$

$\downarrow$

not a Macdonald polynomial

What about that other module?

harmonic polynomials in $\boldsymbol{x}$

$f(\boldsymbol{x})$ killed by $\sum_i \partial_{x_i}^a$ for all $a > 0$
span of all partials of the Vandermonde

Garsia-Haiman modules

Vandermondes in $\boldsymbol{x}$ and $\boldsymbol{y}$

$\downarrow$

modified Macdonald

diagonal harmonics in $\boldsymbol{x}, \boldsymbol{y}$

$f(\boldsymbol{x}, \boldsymbol{y})$ killed by $\sum_i \partial_{x_i}^a \partial_{y_i}^b \ \forall a + b > 0$

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$\downarrow$

∇e_n

Modules, Symmetric functions, and combinatorics

Harmonics modules generalized to bigraded modules in $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$

Garsia-Haiman modules, $\mathcal{M}_\mu$ diagonal harmonics, $\mathcal{DH}_n$

$$\underbrace{\mathrm{sp}\{1\}}_{\substack{\square\square\square \\ (0,0)}} \oplus \underbrace{\mathrm{sp}\{x_3 - x_1, x_1 - x_2\}}_{\boxplus} \oplus \underbrace{\mathrm{sp}\{y_3 - y_1, y_1 - y_2\}}_{\substack{\boxplus \\ (1,0)}} \oplus \underbrace{\mathrm{sp}\{x_1y_1 - x_2y_2\}}_{\substack{\boxminus \\ (1,1)}}$$

degree 0 in y, degree 1 in x

Frobenius map

$$\widetilde{\mathrm{H}}_{2,1} = t^0 q^0 s_{\boxed{1|2|3}} + t^0 q^1 s_{\substack{3 \\ 1\,2}} + t^1 q^0 s_{\substack{2 \\ 1\,3}} + t^1 q^1 s_{\substack{3 \\ 2 \\ 1}}$$

Prove: the modules, $\mathcal{M}_\mu$ and $\mathcal{DH}_n$, have Frobenius images given by the modified Macdonalds, $\widetilde{\mathrm{H}}_\mu$ and ∇e_n.

This only brings us to the early-90's

Adriano inspired 3 more decades (and counting) of research bringing together algebra, combinatorics, and geometry

• geometry of Hilbert schemes • Dyck path and tableaux tuple combinatorics • generalized diagonal harmonics/coinvariant frameworks • affine Schubert calculus • ∇ and the elliptic Hall algebra • torus knots • Dyck path algebra • $\cdots$

And still remaining from the earliest questions

- Find explicit bases for the Garsia–Haiman modules and for diagonal harmonics.
- Find a tableau formula for the q, t-Schur coefficients.
- Prove Science Fiction

Adriano inspired the q, t-world combining algebra, combinatorics, and geometry

Every deep result hides a simple gem, and the fun is digging for it.

- compute until the pattern becomes visible,
- rewrite until an idea becomes inevitable,
- explain until the joy becomes shareable.

He taught us not only math,
but about having a meaningful life.

mathematics that became clearer because he rewrote it

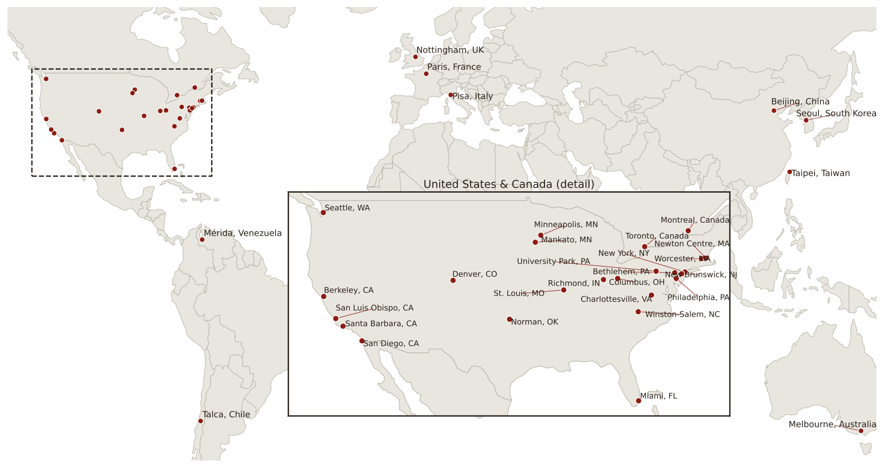
students and colleagues who became family

Adriano’s legacy

joy, humor, food, stories, and generosity

fields that continue to grow from his questions

Adriano's students, around the world



mathematics friendship family life.

Adriano, Thank you for giving us rich lives.