When is a Schubert variety spherical?

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How is $\mathbb{P}^1$ a toric variety?

- $\mathbb{P}^1 = (\mathbb{C}^2 \setminus \{(0,0)\})/\sim$.
- $H = (\mathbb{C}^*)^2$ acts on $\mathbb{P}^1$.
- **Def/Fact:** $\mathbb{P}^1$ is toric because it has a dense H-orbit.
- $\mathbb{P}^1$ has no non-constant polynomial functions. Instead, study

$$\text{global section ring} \cong \mathbb{C}[x,y].$$

- H acts on

$$\mathbb{C}[x,y] = \bigoplus_{a,b\geq 0} \langle x^a y^b \rangle.$$

- Conclusion: $\mathbb{P}^1$ is toric $\Leftrightarrow$ $\mathbb{C}[x,y]$ is **multiplicity-free**.

How is $\mathbb{P}^1$ a spherical variety?

- $H = GL_2$ acts on $\mathbb{P}^1$ by linear change of coordinates.
- H acts on
$$\mathbb{C}[x,y] = \bigoplus_{d\geq 0} \mathbb{C}[x,y]_d.$$
- Let $B \subseteq H$ be the set of upper triangular matrices.
- **Def/Fact:** $\mathbb{P}^1$ is H-**spherical**
  $\Leftrightarrow$ $\mathbb{C}[x,y]$ is multiplicity-free $\Leftrightarrow$ $\exists$ dense B-orbit.
- General question: Given X with H-action, decide if X is H-spherical.
- Difficulty: Find a dense B-orbit or check infinitely many multiplicities.
- Goal: Solve this problem for Schubert varieties.
- Answers: Permutation patterns, key polynomials.

Schubert basics

- Assume in this talk $G = GL_n$.

Def **a parabolic subgroup** P $\Leftrightarrow$ **a partial flag variety** G/P.

Def **a Borel subgroup** B $\Leftrightarrow$ minimal parabolic subgroup.

Ex. $B = \left\{ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \right\}$, $P = Q = \left\{ \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix} \right\}$, $H = \left\{ \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix} \right\}$.

Def **A Schubert variety** $X(w)$ is a B-orbit closure in G/P.

- If Q acts on $X(w)$, choose H as a maximal reductive subgroup.

Thm **[M26+]** $X(w)$ not H-spherical $\Rightarrow \exists v \leq w$ with one of the permutation patterns 321, 246135, 351624, 356124, 415263 and 451623.

cf. Generalizes works of Can, Gaetz, Gao, Hodges, Lakshmibai, Saha and Yong.

Key polynomials

- Give a composition $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, [Lascoux-Schützenberger90] defines **key polynomials** $\kappa_\alpha \in \mathbb{Z}_{\geq 0}[x_1, \cdots, x_n]$ from tableaux.
- Works of [Demazure74] and [Joseph85] show that $\kappa_{w\lambda}$ is a character of a Demazure module $V_\lambda(w)$, a piece of the global section ring of $X(w)$.
- $V_\lambda(w)$ is reducible as a H-module.

Fact [Hodges-Yong22] $X(w) \subseteq G/B$ is H-spherical $\Leftrightarrow$ $V_\lambda(w)$ is H-multiplicity-free for all partitions λ.

Thm [M26+], Conj [Hodges-Yong22] Fix a single partition λ strictly decreasing.

$$X(w) \subseteq G/B \text{ is } H\text{-spherical} \;\Leftrightarrow\; V_\lambda(w) \text{ is } H\text{-multiplicity-free}$$

$$\Leftrightarrow \kappa_{w\lambda} \text{ is expanded into multiplicity-free in products of Schur polynomials.}$$

Ex. $\kappa_{1,3,0,2}(x_1, x_2, y_1, y_2) = 1 \cdot s_{3,1}(x_1, x_2) \cdot s_{2,0}(y_1, y_2) + 1 \cdot s_{3,2}(x_1, x_2) \cdot s_{1,0}(y_1, y_2)$.

Proof idea: Littlewood-Richardson rule

- The LR rule expands products of Schur polynomials and is a branching rule.
- Products of key polynomials are not key positive.

Prop [M26+?] Branching of key polynomials is positive, with a generalized LR rule.

$$\kappa_\alpha(x_1, \cdots, x_m, y_1, \cdots, y_n) = \sum_{\beta,\gamma} c^\alpha_{\beta,\gamma} \cdot \kappa_\beta(x_1, \cdots, x_m) \cdot \kappa_\gamma(y_1, \cdots, y_n).$$

pf. Use [Mathieu89] + [Joseph85]. Works in general type.

Classification theorem

- Input: $w \in S_n$, $I, J \subset [n-1]$.
- Output: a sequence with entries $[n-1] \cup \{*\}$.

    1. $b_1 = 1$, $j \in J \Rightarrow b_{j+1} = b_j$, $j \notin J \Rightarrow b_{j+1} = 1 + b_j$. $c_j = b_{w^{-1}(j)}$.
    2. $j \notin I \Rightarrow$ insert $*$ between c_j and c_{j+1}. Reverse numerical entries between $*$.

- The sequence has **a pattern** if there is a subsequence such that

    1. the symbols $*$ are inserted in the same place.
    2. the remaining numerical entries have the same relative order.

Thm [M26+] $X(w) \subseteq G/P_J$ is L_I-spherical if and only if the sequence avoids

$$\begin{array}{cccccc} 321, & 3412, & 3*312, & 231*1, & 2*21*1, & 244113, \\ 233112, & 3331122, & 2233111, & 22211*11, & 22*22111, & 22**1*1, \\ 33**12, & 44123, & 3312*2, & 2*2**11, & 23**11, & 23411, \\ 2*2311, & 221211, & 331322, & 221311, & 331422. & \end{array}$$

cf. This generalizes [Karuppuchamy13], [Gaetz22], [Hodges-Lakshmibai22], [Can-Saha23], [Gao-Hodges-Yong23], [Hodges-Yong23], [Anderson-Fulton24].

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Thank you for your attention.