Hyperbinary partitions and q-deformed rationals

- Based on joint work with

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Overview

q-deformed rationals

$$\left(\frac{a}{b}\right)_q$$

[Morier-Genoud–Ovsienko '20]

Hyperbinary partitions

$$h(n)$$

[Reznick '90]

+ many antecedents

This talk [M–Propp–Scyss '26]

We give several interpretations for $\left(\frac{a}{b}\right)_q$ using a q-analogue of $h(n)$.

Outline

(I) Sequences

(II) Lattices

(III) Matrices

(I) Stern's diatomic sequence (or obfuscating sequence)

Defined by recurrence:

$$f_{usc}(0) = 0,$$
$$f_{usc}(1) = 1,$$

and for $n \geq 1$,

$$f_{usc}(2n) = f_{usc}(n),$$

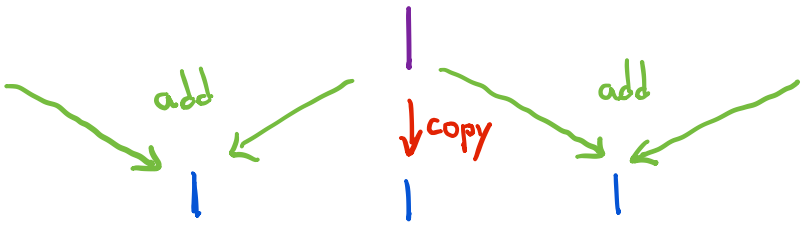
$$f_{usc}(2n+1) = f_{usc}(n+1) + f_{usc}(n).$$

$(0, 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, \ldots)$

n=	0	1	2	3	4	5	6	7	8	9	10	11	12	13

Stern's diatomic sequence (or obfuscating sequence)

Alternatively, we can construct Stern's sequence from an array.



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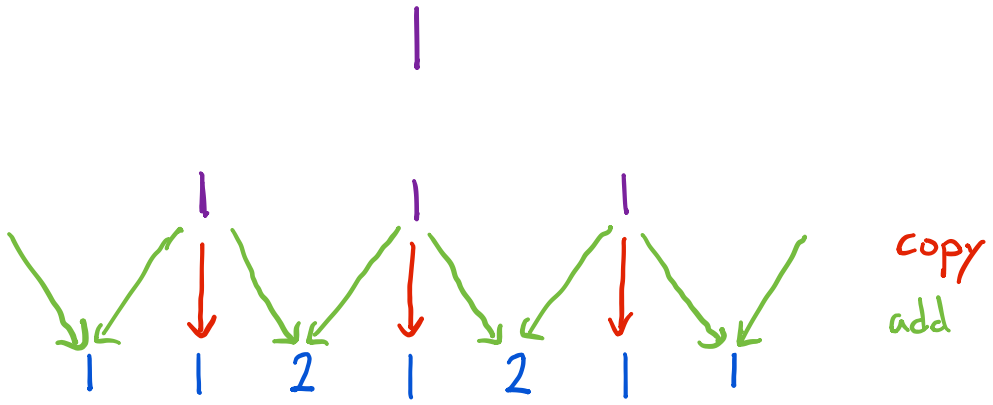

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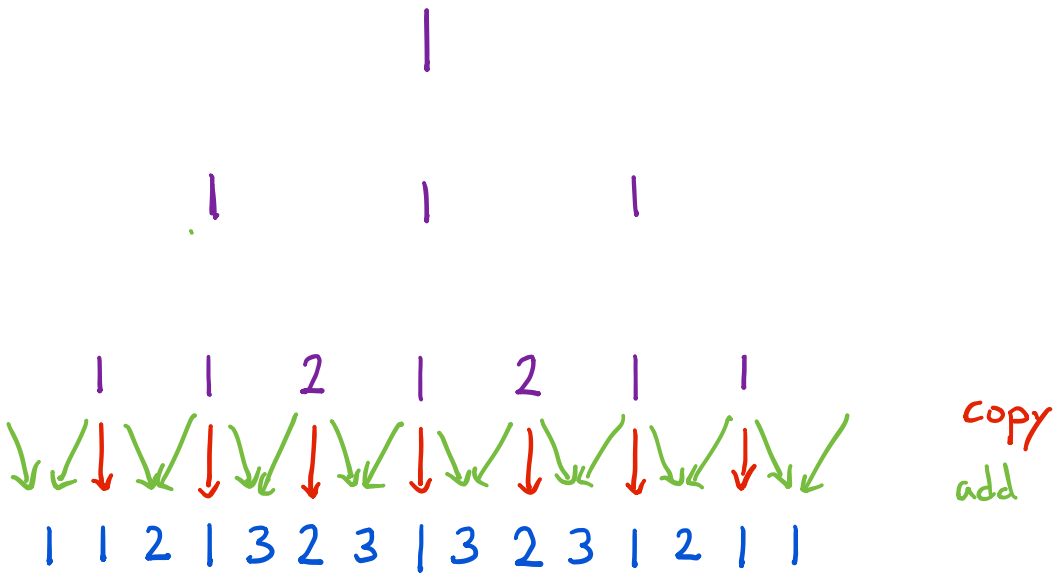

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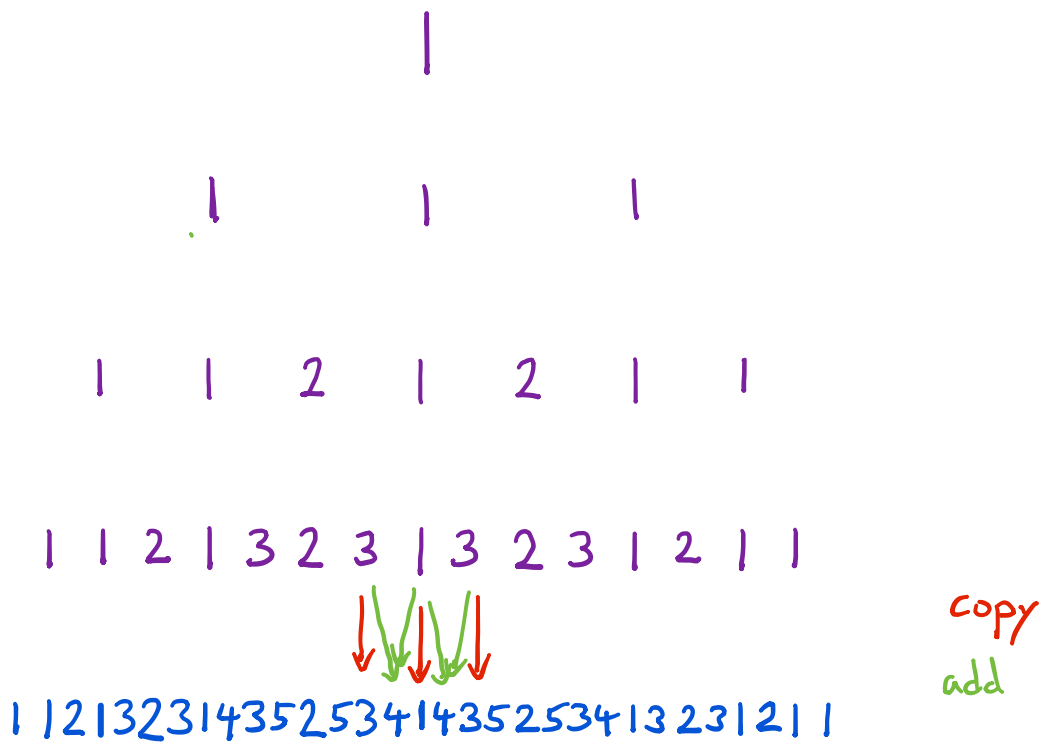


Stern's diatomic sequence (or obfuscating sequence)

Alternatively, we can construct Stern's sequence from an array.

1

1 1

1 1 2 1 2 1 1

1 1 2 1 3 2 3 1 3 2 3 1 2 1 1

1 1 2 1 3 2 3 1 4 3 5 2 5 3 4 1 4 3 5 2 5 3 4 1 3 2 3 1 2 1 1

copy

add

1 1 2 1 3 2 3 1 4 3 5 2 5 3 4 1 4 3 5 2 5 3 4 1 3 2 3 1 2 1 1

Stern's diatomic sequence (or obfuscating sequence)

Alternatively, we can construct Stern's sequence from an array.

1

1 1 1

1 1 2 1 2 1 1

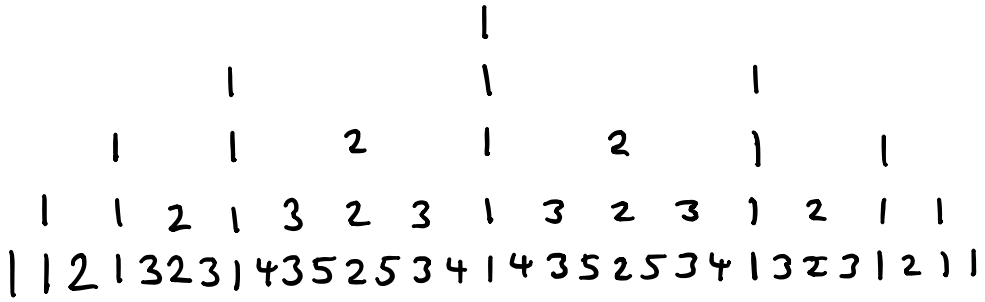
1 1 2 1 3 2 3 1 3 2 3 1 2 1 1

1 1213231435253414352534132312 1 1

copy

add

Stern's triangle



Notation: $\left\langle {n \atop k} \right\rangle$ = k^{th} term of n^{th} row of the triangle

Theorem see [Stanley '19, Northshield '10]

Let $H_r(x) = \prod_{k=0}^{r-1} \left(1 + x^{2^k} + x^{2 \cdot 2^k}\right)$,

and $H(x) = \lim_{r \to \infty} H_r(x) = \prod_{k=0}^{\infty} \left(1 + x^{2^k} + x^{2 \cdot 2^k}\right)$

① $\left\langle {r \atop n} \right\rangle = [x^n] H_r(x)$

$$H_{r+1}(x) = \underbrace{\left(1 + x + x^2\right)}_{\text{add}} \underbrace{H_r(x^2)}_{\text{copy}}$$

② $\text{fusc}(n) = [x^{n+1}] H(x)$

$$H(x) = \left(1 + x + x^2\right) H(x^2)$$

Hyperbinary partitions

An integer partition of n is hyperbinary if it only has parts of size 2^k for some k, and each part has multiplicity 0, 1, or 2.

Example: $n = 43$

Hyperbinary partitions: (expressed as weakly decreasing sequences of positive integers)

(32, 8, 2, 1)

(16, 16, 8, 2, 1)

(32, 4, 4, 2, 1)

(16, 16, 4, 4, 2, 1)

(16, 8, 8, 4, 4, 2, 1)

Hyperbinary partitions

Let $h(n) = \#\{\text{hyperbinary partitions of } n\}$.

Its generating function is

$$H(x) = \sum_{n=0}^{\infty} h(n)\, x^n = \prod_{k=0}^{\infty} \left(1 + x^{2^k} + x^{2 \cdot 2^k}\right).$$

Theorem [Reznick '90]

For all $n \geq 0$,

$$h(n) = fusc(n+1).$$

$fusc(0) = 0, \quad fusc(1) = 1$

$fusc(2n) = fusc(n)$

$fusc(2n+1) = fusc(n+1) + fusc(n)$

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Theorem [Reznick '90]

For all $n \geq 0$,

$$h(n) = \text{fusc}(n+1).$$

$\text{fusc}(0) = 0, \quad \text{fusc}(1) = 1$

$\text{fusc}(2n) = \text{fusc}(n)$

$\text{fusc}(2n+1) = \text{fusc}(n+1) + \text{fusc}(n)$

Example: $n = 43$

43	21	10	5	2
(32, 8, 2, 1)	(16, 4, 1)	(8, 2)	(4, 1)	(2)
(16, 16, 8, 2, 1)	(8, 8, 4, 1)	(4, 4, 2)	(2, 2, 1)	(1, 1)
(32, 4, 4, 2, 1)	(16, 2, 2, 1)	(8, 1, 1)	4	2
(16, 16, 4, 4, 2, 1)	(8, 8, 2, 2, 1)	(4, 4, 1, 1)	(4)	(2)
(16, 8, 8, 4, 4, 2, 1)	(8, 4, 4, 2, 2, 1)	(4, 2, 2, 1, 1)	(2, 2)	(1, 1)
			(2, 1, 1)	1
				(1)

Calkin-Wilf sequence

For $n \geq 0$, let $CW(n) = \dfrac{fusc(n)}{fusc(n+1)} = \dfrac{h(n-1)}{h(n)}$.

n	fusc(n)	CW(n)
0	0	0/1
1	1	1/1
2	1	1/2
3	2	2/1
4	1	1/3
5	3	3/2
6	2	2/3
7	3	3/1
8	1	1/4
9	4	4/3
10	3	3/5

Calkin-Wilf sequence

For $n \geq 0$, let $CW(n) = \frac{fusc(n)}{fusc(n+1)} = \frac{h(n-1)}{h(n)}$.

Theorem [Calkin-Wilf '00]

For any nonnegative rational number r/s, there is a unique n such that

$$CW(n) = r/s .$$

$CW(0), CW(1), CW(2), CW(3), \ldots$ lists all nonnegative rationals (in reduced form)

Explicitly, we can find $n \in \mathbb{N}$ such that $CW(n) = r/s$ from the continued fraction expansion of r/s.

Continued fractions

For $\frac{r}{s} \geq 0$, there is a unique sequence of integers $a_1, a_2, \ldots, a_{2m-1}$ such that $a_1 \geq 0$, $a_2, \ldots a_{2m-1} \geq 1$ and

$$\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_{2m-1}}}}}$$

Example

$$\frac{5}{12} = 0 + \frac{1}{\frac{12}{5}} = 0 + \cfrac{1}{2 + \cfrac{1}{\frac{5}{2}}} = 0 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \frac{1}{1}}}}$$

Convert a-sequence 02211 to binary.

0	2	2	1	1
	00	11	0	1

Reverse the binary sequence.

101100 is the binary expansion of $n=44$.

$$CW(44) = \frac{h(43)}{h(44)} = \frac{5}{12}$$

q-deformed rationals

For $n \in \mathbb{N}$, set $[n]_q = 1 + q + \cdots + q^{n-1}$.

For rational $\frac{r}{s} \geq 0$ with continued fraction expansion

$$\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_{2m-1}}}},$$

Morier-Genoud and Ovsienko defined the q-analogue

$$\left[\frac{r}{s}\right]_q = [a_1]_q + \cfrac{q^{a_1}}{[a_2]_{q^{-1}} + \cfrac{q^{-a_2}}{[a_3]_q + \cfrac{q^{a_3}}{\ddots \cfrac{}{[a_{2m-2}]_{q^{-1}} + \cfrac{q^{-2m-2}}{[a_{2m-1}]_q}}}}}.$$

Example: $\left[\frac{5}{12}\right]_q = [0]_q + \cfrac{q^0}{[2]_{q^{-1}} + \cfrac{q^{-2}}{[2]_q + \cfrac{q^2}{[1]_{q^{-1}} + \cfrac{q^{-1}}{[1]_q}}}}$

$$= \frac{q^{-1} + 2 + q + q^2}{q^{-3} + 2q^{-2} + 3q^{-1} + 3 + 2q + q^2}$$

A "good" q-analogue?

The modular group $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm Id\}$ acts faithfully and transitively on $\mathbb{Q} \cup \{\frac{1}{0}\}$ via fractional-linear transformations.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x) = \frac{ax+b}{cx+d}$$

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The group also acts on $\mathbb{Z}(q)$.

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The group also acts on $\mathbb{Z}(q)$.

Theorem: [Morier-Genoud - Ovsienko '20]

There is a unique map $\mathbb{Q} \cup \{\frac{1}{0}\} \to \mathbb{Z}(q)$

that commutes with the action of $PSL(2,\mathbb{Z})$

and $\left[\frac{0}{1}\right]_q = 0$.

Length generating function

Let $h_q(n) = \sum_{\lambda} q^{\ell(\lambda)}$

where the sum ranges over hyperbinary partitions and $\ell(\lambda)$ is the length of λ.

Example

$n=43$

$(32, 8, 2, 1)$ q^4

$(16, 16, 8, 2, 1)$ q^5

$(32, 4, 4, 2, 1)$ q^5

$(16, 16, 4, 4, 2, 1)$ q^6

$(16, 8, 8, 4, 4, 2, 1)$ q^7

$h_q(43) = q^4 + 2q^5 + q^6 + q^7$

q-Calkin-Wilf

Let $CW_q(n) = \dfrac{h_q(n-1)}{h_q(n)}$.

Example: $CW_q(44) = \dfrac{h_q(43)}{h_q(44)} = \dfrac{q^4+2q^5+q^6+q^7}{q^3+2q^4+3q^5+3q^6+2q^7+q^8}$

Compare to $[CW(44)]_q = \dfrac{q^{-1}+2+q+q^2}{q^{-3}+2q^{-2}+3q^{-1}+3+2q+q^2}$.

q-Calkin-Wilf

Let $CW_q(n) = \dfrac{h_q(n-1)}{h_q(n)}$.

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Compare to $\left[CW(44)\right]_q = \dfrac{q^{-1}+2+q+q^2}{q^{-3}+2q^{-2}+3q^{-1}+3+2q+q^2}$.

Theorem A: [MPS '25]

For all $n \geq 1$,

$$CW_q(n) = q\left[CW(n)\right]_q.$$

See also: [Bates-Mansour '11; Mansour-Shattuck '11]

Proof idea: There is a q-analogue for the recurrence for $h(n)$. Compare with continued fraction recurrence.

(II) A poset of hyperbinary partitions

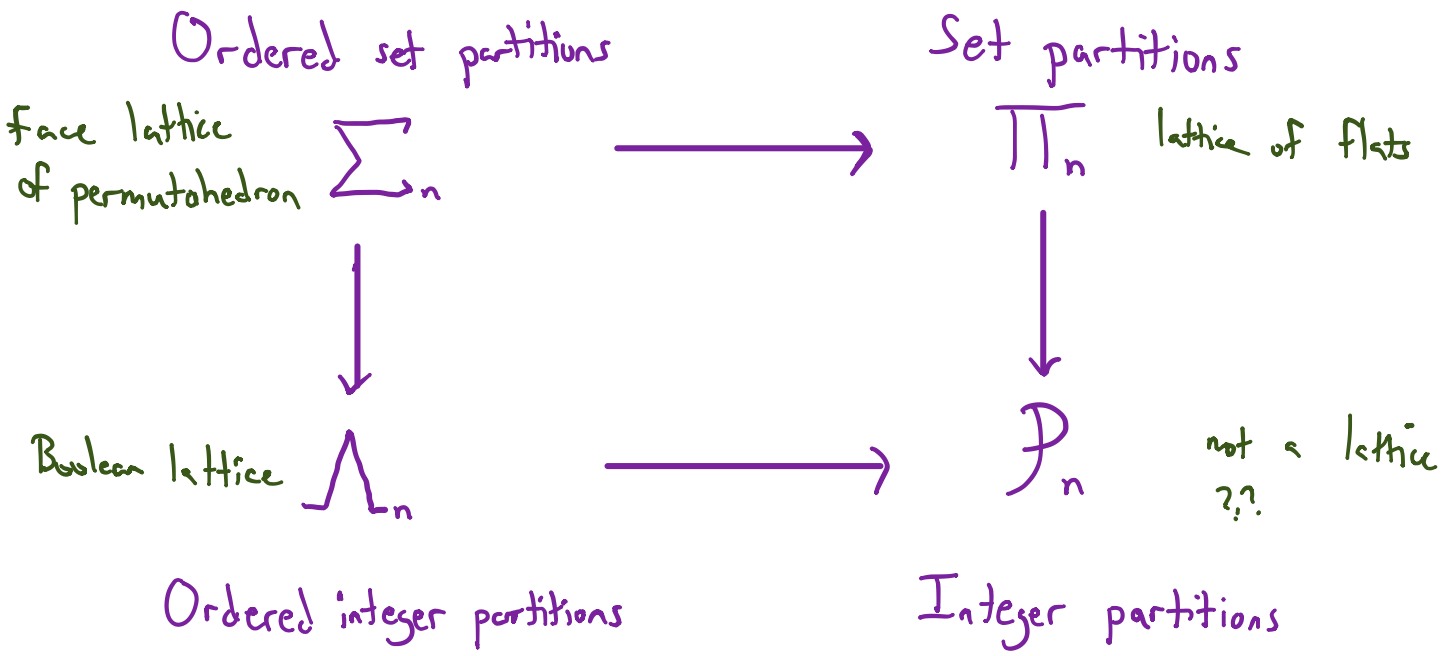
We partially order partitions of n by **refinement**.

* $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ **refines** $\mu = (\mu_1, \ldots, \mu_m)$ if there is an ordered set partition $B = (B_1, \ldots, B_m)$ of $[\ell]$ such that $\mu_i = \sum_{j \in B_i} \lambda_j$.

Example:

$\lambda = (3, 3, 2, 2, 2, 1, 1)$ refines $\mu = (7, 4, 3)$ since

$\mu_1 = \lambda_1 + \lambda_3 + \lambda_4$, $\mu_2 = \lambda_5 + \lambda_6 + \lambda_7$, $\mu_3 = \lambda_2$.

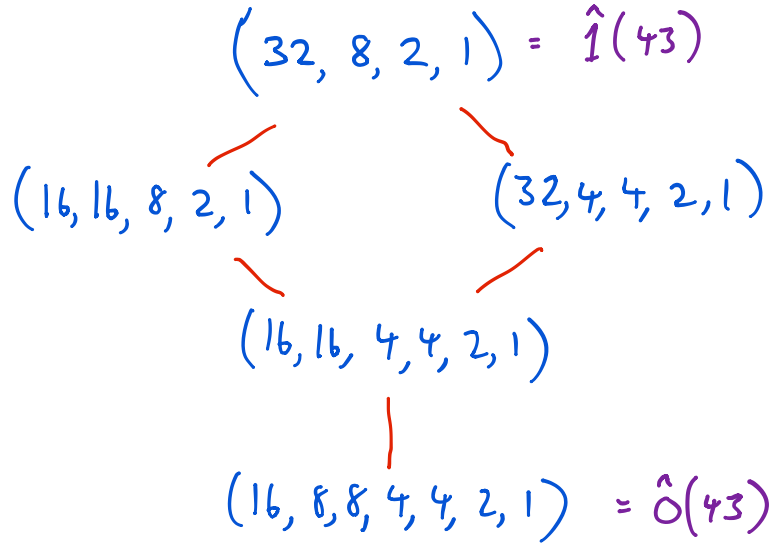


A poset of hyperbinary partitions

Let $\mathcal{H}(n)$ denote the subposet of hyperbinary partitions of n.

Example:

$\mathcal{H}(43)$



Theorem B: [MPS '25] (extending [Brunetti, D'Aniello '19])

(i) $\mathcal{H}(n)$ is a distributive lattice.

(ii) There is a unique minimum $\hat{0}(n)$ and maximum $\hat{1}(n)$.

(iii) The rank of λ is $\ell(\hat{0}(n)) - \ell(\lambda)$.

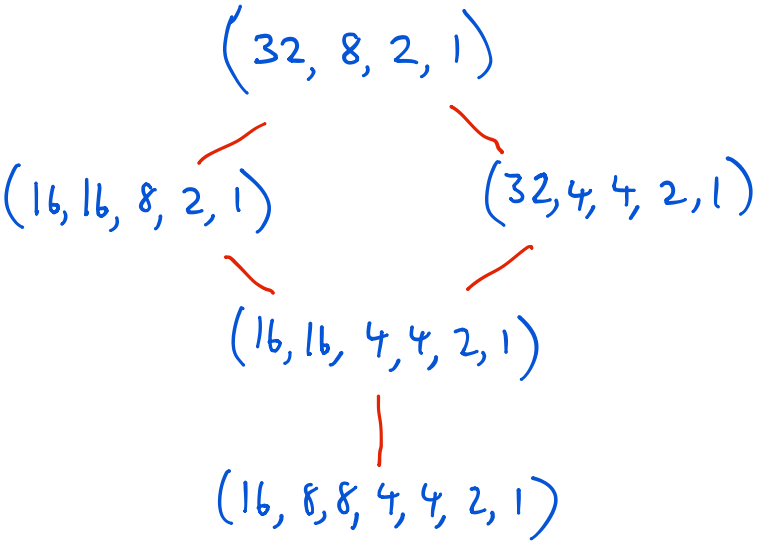
(iv) $\mathcal{H}(n)$ is isomorphic to the lattice of order ideals of a fence poset.

Hyperbinary expansions

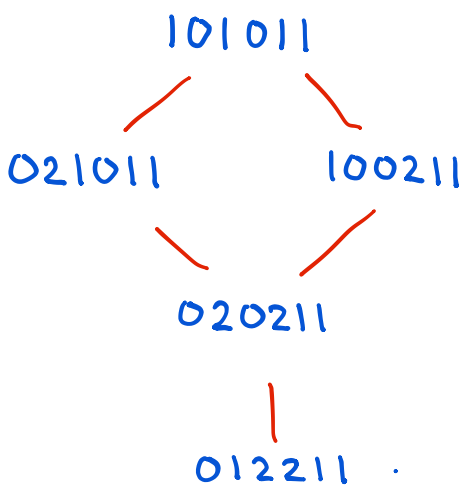
The multiplicity vector of a hyperbinary partition is called a hyperbinary expansion.

Let $\mathcal{D}(n)$ be the set of hyperbinary expansions of n.

$\mathcal{H}(43)$



$\mathcal{D}(43)$



Lemma: [MPS '25]

The covers in $\mathcal{D}(n)$ are of the form

$$u10v \;\text{—}\; u02v \quad \text{or} \quad u20v \;\text{—}\; u12v \,.$$

Extrema

The principal prefix of $\beta(n)$ is $b_1 \cdots b_r$ if b_{r+1} is the rightmost 0 in $\beta(n)$.

Example

The principal prefix of $\beta(43) = 101011$ is 101.

Proposition [MPS '25, BD '19]

(1) The maximum element of $\mathcal{D}(n)$ is the binary expansion.

$$\hat{1}(n) = \beta(n) = b_1 b_2 \cdots b_k$$

(2) The minimum element of $\mathcal{D}(n)$ is the unique expansion whose zeroes form a prefix.

$$\hat{0}(n) = \begin{cases} 1^k & \text{if } n = 2^k - 1 \\ 0(b_2+1)\cdots(b_r+1)21^{k-r-1} & \text{if } n \neq 2^k - 1 \end{cases}$$

Fence posets

A fence (or zigzag poset) is a partial order on a set $\{x_1, \ldots, x_r\}$ whose covers are of the form $x_i \lessdot x_{i+1}$ or $x_i \gtrdot x_{i+1}$ for all i.

If $b_1 b_2 \cdots b_r$ is the principal prefix of $\beta(n)$, we define $\mathcal{F}(n)$ to be the fence on $\{x_1, \ldots, x_r\}$ such that for $2 \le i \le r$,

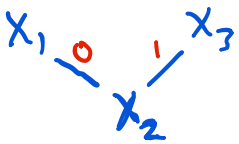
- if $b_i = 0$, then $x_{i-1} > x_i$
- if $b_i = 1$, then $x_{i-1} < x_i$.

Example: $n = 43$

$\beta(43) = 101011$

$r = 3$

$\mathcal{F}(43)$



Isomorphism

Theorem B: [MPS '25]

(i) $\mathcal{H}(n)$ is a distributive lattice.

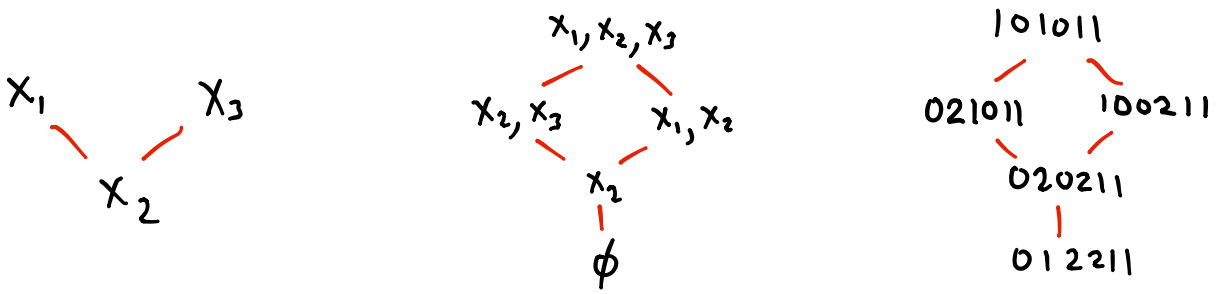
(ii) There is a unique minimum $\hat{0}(n)$ and maximum $\hat{1}(n)$.

(iii) The rank of λ is $\ell(\hat{0}(n)) - \ell(\lambda)$.

(iv) $\mathcal{H}(n)$ is isomorphic to the lattice of order ideals of a fence poset.

Example

$\mathcal{F}(43)$ $\quad$ $\mathcal{J}(\mathcal{F}(43))$ $\quad$ $\mathcal{Q}(43)$



Rank generating function

The rank generating function of $\mathcal{J}(F(n))$ is

$$rgf_n(t) = \sum_I t^{|I|}.$$

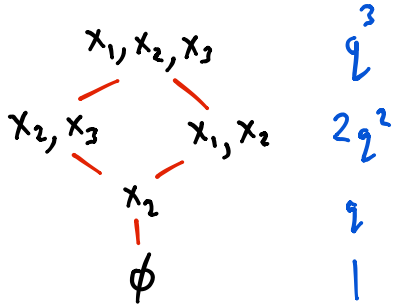
Theorem: [MPS '25]

For all $n \geq 1$, if the principal prefix of $\beta(n)$ has length r and $\beta(n)$ has s ones, then

$$h_q(n) = q^{r+s} \cdot rgf_n(q^{-1}).$$

Example:

$\mathcal{J}(F(43))$



$r = 3, s = 4$

$$rgf_{43}(q) = 1 + q + 2q^2 + q^3$$

$$q^{r+s} \cdot rgf_{43}(q^{-1}) = q^4 + 2q^5 + q^6 + q^7$$

q-deformed rationals from fences

Corollary: [MPS '25]

For all $n \geq 1$, there exists $a \in \mathbb{Z}$ such that

$$\left[CW(n)\right]_q = q^a \frac{rgf_{n-1}(q^{-1})}{rgf_n(q)}.$$

Remark: This result recovers another method of constructing q-deformed rational numbers given in [Morier-Genoud – Ovsienko '20]. Their construction is expressed in terms of "closure sets" of linear type A quivers, which are essentially equivalent to order ideals of fence posets.

(III) Matrices

Let $L = \begin{bmatrix} 1 & 0 \\ 1 & q^{-1} \end{bmatrix}$ and $R = \begin{bmatrix} q & 1 \\ 0 & 1 \end{bmatrix}$.

We define a matrix $M(n)$ as a product of L's and R's.

Let $B(n) = b_1 b_2 \cdots b_k$ be the binary expansion of n.

Reverse the sequence and remove b_1 to get

$$b_k b_{k-1} \cdots b_2 .$$

Replace 0 with L and 1 with R in this sequence,

and let $M(n)$ be the product of these matrices.

Example:

$B(43) = 101011$

reverse: $11010\,1$

$M(43) = RRLRL$

$$= \begin{bmatrix} q & 1 \\ 0 & 1 \end{bmatrix}^2 \begin{bmatrix} 1 & 0 \\ 1 & q^{-1} \end{bmatrix} \begin{bmatrix} q & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & q^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} q^{-1} + 2 + 2q + 2q^2 + q^3 & q^{-2} + 2q^{-1} + 1 + q \\ q^{-1} + 1 + q & q^{-1} + 1 \end{bmatrix}$$

Matrices

Theorem C: [MPS '25]

Let $2^k \le n < 2^{k+1} - 1$, $\beta(n) = b_1 b_2 \cdots b_{k+1}$.

Let j be the largest index such that

$$b_1 = b_2 = \cdots = b_j = 1,$$

and let $n' \ge 0$ such that $\beta(n') = 1 b_{j+2} b_{j+3} \cdots b_{k+1}$.

Then
$$M(n) = \begin{bmatrix} q^{-k+2j-1} h_q(n'-1) & q^{-k+1} h_q(n-2^k-1) \\ q^{-k+2j-2} h_q(n') & q^{-k} h_q(n-2^k) \end{bmatrix}.$$

If $n = 2^{k+1} - 1$, then

$$M(n) = \begin{bmatrix} q^k & q^{-k+1} h_q(n-2^k-1) \\ 0 & q^{-k} h_q(n-2^k) \end{bmatrix}.$$

Row sums

Theorem: [MPS '25]

If $2^k \le n < 2^{k+1}$, then

$$M(n) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} q^{-k}\, h_q(n-1) \\ q^{-k-1}\, h_q(n) \end{bmatrix}.$$

Corollary:

For all $n \ge 1$,

$\left[CW(n)\right]_q$ is the ratio of the two entries

of $M(n) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(IV) Future questions

① Other statistics on hyperbinary partitions

$\ell(\lambda) = p_1 + 2p_2$ where

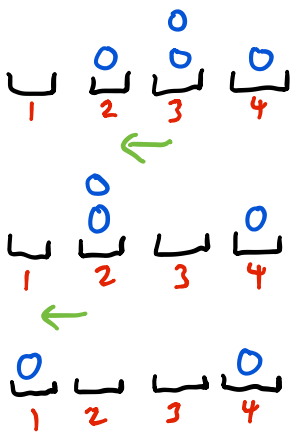
p_1 = # parts of λ of multiplicity 1

and

p_2 = # parts of λ of multiplicity 2.

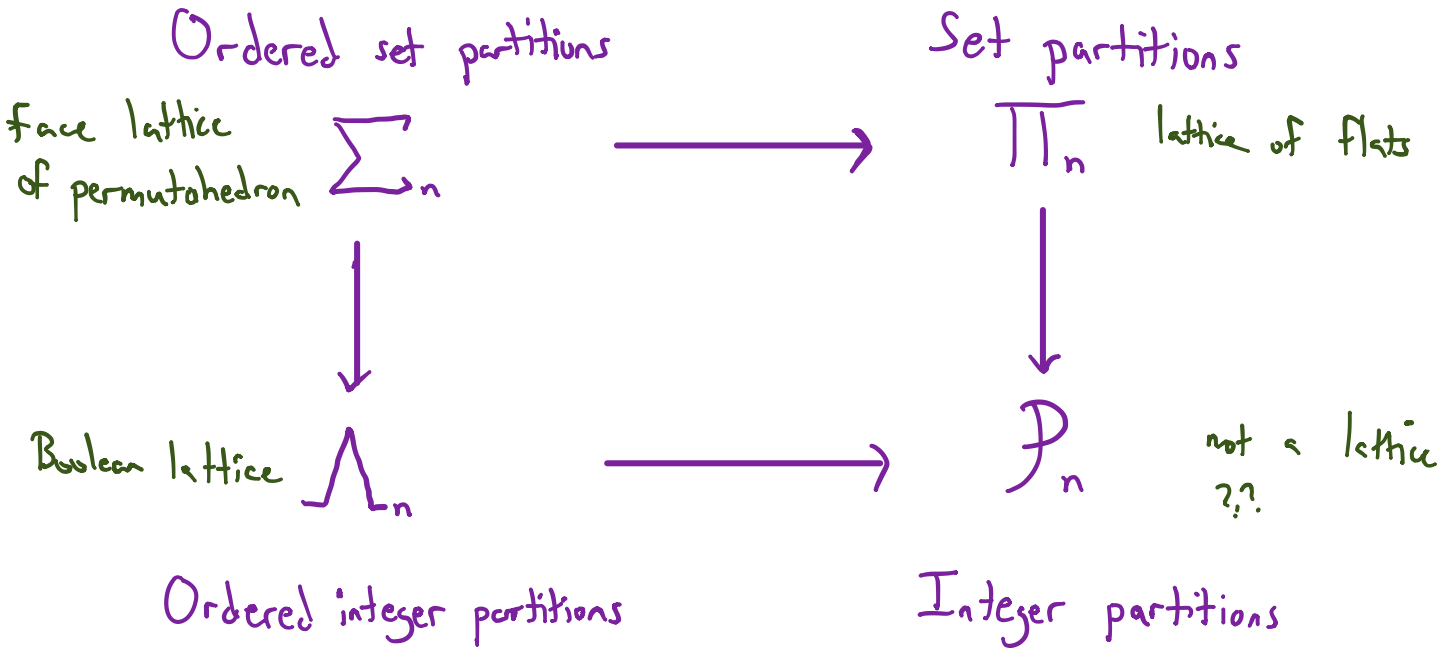
See [Klavžar - Milutinović - Petr '07,
Bates - Mansour '11,
Mansour - Shattuck '11, '15]

② Chip firing ~ other rules/statistics?



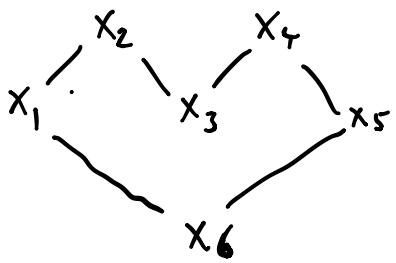
Future questions

③ Other subposets of integer partitions?



④ Circular fences and hyperbinary expansions?

[Oğuz-Ravichandran '23]



THANKS!