An Eventown Result for Permutations

FPSAC 2026

38th Int'l Conference on Formal Power Series and Algebraic Combinatorics

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July 16, 2026

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2. no two clubs have exactly the same members;
3. any two clubs must share an even number of members.

To maximize the $\#$ clubs, the residents first wish to formalize this problem.

Formal Eventown

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Goal: maximize m, i.e., find the largest # of clubs that satisfy the Eventown rules.

A Lower Bound

Partner off the n residents of Eventown into $\lfloor n/2 \rfloor$ many pairs.

Take all of the $2^{\lfloor n/2 \rfloor}$ different subsets of these $\lfloor n/2 \rfloor$ pairs.

Any subset must have an even number of members (they join the club in pairs).

Any two subsets have even intersection (if x is in both clubs, then so is x's partner).

This gives $2^{\lfloor n/2 \rfloor}$ many clubs satisfying Eventown's rules.

An Upper Bound

Berlekamp '69

To each club $C_i \subseteq [n]$, define its *characteristic vector* $v_i \in \mathbb{F}_2^n$ defined such that

$$v_i(x) := \begin{cases} 1 & \text{if } x \in C_i; \\ 0 & \text{if } x \notin C_i. \end{cases}$$

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Let $V := \mathrm{Span}\{v_1, \ldots, v_m\}$. Since $\langle v_i, v_j \rangle = 0$ for all i, j, we have

$$V \leq V^\perp = \{w \in \mathbb{F}_2^n : \langle w, v \rangle = 0 \quad \forall v \in V\}.$$

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Since $\dim V + \dim V^\perp = n$, we have

$$2 \dim V \leq \dim V + \dim V^\perp = n$$

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Bijections & Characters

Proof Sketch

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$$2 \dim V \leq \dim V + \dim V^\perp = n \implies \dim V \leq \lfloor n/2 \rfloor \implies m \leq |V| \leq 2^{\lfloor n/2 \rfloor}. \square$$

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Eventown Analogues

Eventown for subsets:

$$\max_{\mathcal{F} \subseteq 2^{[n]}} |\mathcal{F}| \quad \text{subject to}$$

- $|\pi| \equiv 0 \mod 2$ for all $\pi \in \mathcal{F}$;
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Eventown for finite groups G:

$$\max_{\mathcal{F}\subseteq G} |\mathcal{F}| \quad \text{subject to}$$

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We want to solve the following optimization problem for all symmetric groups S_n:

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A family $\mathcal{F} \subseteq S_n$ satisfying Property 2 is called an **even-cycle-intersecting** family.

Reversing Families of Permutations

Körner '10 / Füredi, Kantor, Monti, Sinaimeri '10

We say $\sigma, \pi \in S_n$ are *reversing* if their difference $\sigma\pi^{-1}$ has a transposition.

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Füredi et al. (2010) construct a reversing family of permutations $\mathcal{F} \subseteq S_n$ of size

$$2^{\lfloor n/2 \rfloor} < 1.515^n \approx 8^{\lfloor n/5 \rfloor} = |\mathcal{F}|.$$

Improved to $1.587^n \approx 4^{\lfloor n/3 \rfloor}$ by Harcos & Soltész (2020).

Körner's Conjecture

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A reversing family of S_n has size no greater than C^n for some constant $C > 1$.

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This conjecture has since spawned other interesting lines of combinatorial inquiry:

- Füredi, Kantor, Monti, Sinaimeri (2010, 2011)
- Körner, Messuti, Simonyi (2012)
- Cibulka (2013)
- Soltész (2018)
- Harcos, Soltész (2020)
- Byrne, Tait (2024)
- $\vdots$

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Recall $\mathcal{F} \subseteq S_n$ is even-cycle-intersecting if $\sigma\pi^{-1}$ has an even cycle for all $\sigma, \pi \in \mathcal{F}$.

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Theorem (L. '25)

If $\mathcal{F} \subseteq S_n$ is even-cycle-intersecting, then for all $n \geq 2$, we have

$$|\mathcal{F}| \leq 2^{n-1}.$$

Equality holds if $\mathcal{F} \leq S_n$ is a Sylow 2-subgroup and $n = 2^\ell, 2^\ell + 1$ for some $\ell \in \mathbb{N}$.

Conjecture (L. '25)

The Sylow 2-subgroups are the extremal even-cycle-intersecting families $\forall n \geq 6$.

Main Results

Corollary (L. '25)

The maximum size of an eventown family of S_n is 2^{n-1}.

Open Question: Is there a dimension argument that achieves this bound?

Theorem (L. '25)

Körner's Conjecture holds for $C = 2$.

Open Question: Can C be improved?

Bijections & Characters

The Odd Cycles Bijection

Bóna, Hopkins (https://mathoverflow.net/questions/412727)

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Let's call this *standard cycle form*. Assume all permutations are in this form.

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Proceed to the next bit b_{i+1}.

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Example 2:

$$\pi = (321)(746)(8)(950) \in S_{10} \qquad b = b_1 b_2 b_3 = 101$$

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The Odd Cycles Bijection

This procedure maps 'gap-signed' odd cycle only permutations to the set S_n:

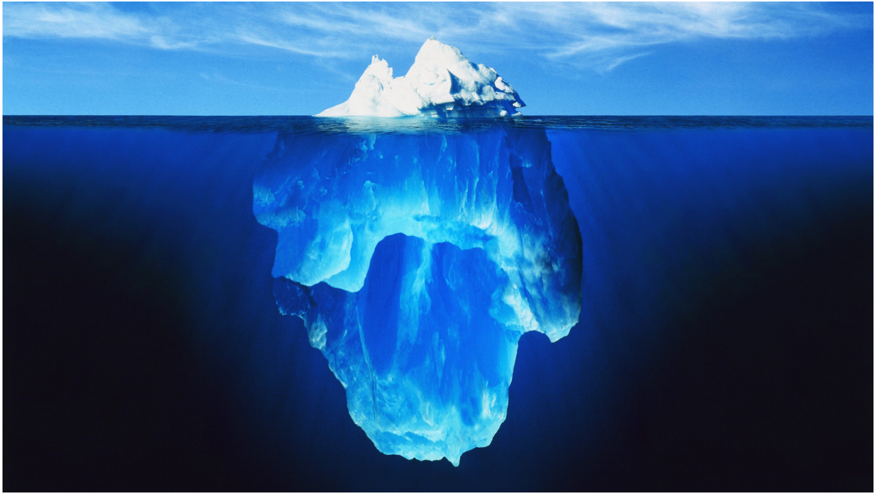
$$\{(\sigma, b) : \sigma_1 \cdots \sigma_m = \sigma \in S_n,\ |\sigma_i| \text{ odd } \forall i \quad \text{and} \quad b \in \{0,1\}^m\} \longrightarrow S_n.$$

In fact, it's a bijection: given $\pi \in S_n$ in standard cycle form, we can recover (σ, b).

We obtain the following identity:

$$n! = \sum_{\substack{\sigma \in S_n \\ \sigma \text{ has only odd cycles}}} 2^{\#\text{cycles}(\sigma)-1}.$$

Bijections of the Symmetric Group



Hooks

A *hook* is an integer partition of the form $(n-k, 1^k) \vdash n$ for some k.

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Let

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Theorem (Regev '13, Taylor '17)

For all $\sigma \in S_n$, we have

$$H(\sigma) = \begin{cases} 2^{\#\textit{cycles}(\sigma)-1} & \textit{if } \sigma \textit{ has only odd cycles} \\ 0 & \textit{if } \sigma \textit{ has an even cycle.} \end{cases}$$

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Projectors

For any irreducible character χ of a finite group G, let E_χ be the $|G| \times |G|$ matrix that is the orthogonal projection onto the χ-isotypic component of $\mathbb{C}G$:

$$(E_\chi)_{h,g} = \frac{\chi(1)}{|G|}\chi(gh^{-1}) \quad \text{for all } g, h \in G.$$

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It is well known that

$$\chi^\lambda(1) = \frac{n!}{h_\lambda}$$

where h_λ denotes the *hook product* of λ.

Proof Sketch

The Odd Cycle Matrix

Define the following $n! \times n!$ matrix such that for all $\pi, \sigma \in S_n$, we have

$$\mathrm{OC}_{\pi,\sigma} := \begin{cases} 2^{\#\text{cycles}(\sigma\pi^{-1})-1} & \text{if } \sigma\pi^{-1} \text{ has only odd-length cycles;} \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\mathrm{OC} = \sum_{k=0}^{n-1} h_{(n-k,1^k)} E_{(n-k,1^k)},$$

i.e., OC is a low-rank positive semi-definite matrix with maximum eigenvalue $n!$.

The diagonal entries of OC are all 2^{n-1}.

$\mathrm{OC}' := \mathrm{OC} - 2^{n-1}I$ has $\underline{\text{max eigenvalue } n! - 2^{n-1}}$ and $\underline{\text{least eigenvalue } -2^{n-1}}$.

The Delsarte–Hoffman Bound

OC' is a weighted adj. matrix for the Cayley graph of S_n generated by permutations with only odd-length cycles. Independent sets $\longleftrightarrow$ even-cycle-intersecting families.

Theorem (Delsarte–Hoffman)

Let A be a weighted adjacency matrix of a regular graph $G = (V, E)$ with largest eigenvalue $\eta_{\max}$ and least eigenvalue $\eta_{\min}$. If $S \subseteq V$ is an independent set, then

$$|S| \leq |V| \frac{-\eta_{\min}}{\eta_{\max} - \eta_{\min}}.$$

Moreover, if equality holds, then the characteristic vector $1_S \in \mathbb{R}^V$ lies in the direct sum of the $\eta_{\max}$-eigenspace and $\eta_{\min}$-eigenspace.

Thus, if $S \subseteq S_n$ is even-cycle-intersecting, then $|S| \leq n! \frac{2^{n-1}}{(n! - 2^{n-1}) + 2^{n-1}} = 2^{n-1}$.

Eventown for Permutations

Nathan Lindzey

Eventown

Main Results

Bijections & Characters

Proof Sketch

Sylow 2-Subgroups P_n of S_n

$|P_n| =$ highest power of 2 dividing $n!$, i.e., $2^{\sum_{k \geq 1} \lfloor n/2^k \rfloor}$.

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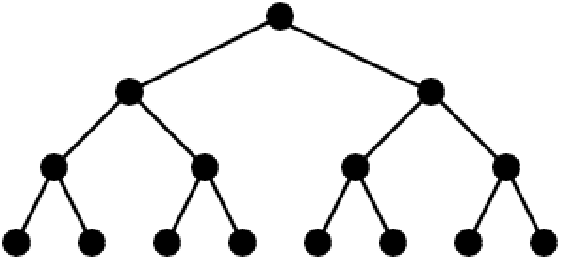
The Sylow 2-subgroups are automorphism groups of perfect rooted binary trees:



Equality in the Delsarte–Hoffman Bound

Delsarte–Hoffman is met with equality, so extremal even-cycle-intersecting families live in the direct sum of the greatest and least eigenspaces of OC'...

... so the characteristic vector of any Sylow 2-subgroup lives there.

An application of Frobenius reciprocity gives us the following result of Gianelli.

Corollary (Gianelli '16)

The permutation representation of S_{2^ℓ} acting on cosets S_{2^ℓ}/P_{2^ℓ} does not have a non-trivial hook as an irreducible constituent.

Conclusion + Future Work

What about (mod p)-cycle-intersection?

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Theorem (L. '26+)

If $\mathcal{F} \subseteq S_n$ is (mod p)-cycle-intersecting, then $|\mathcal{F}| \leq \binom{2n}{n} < 4^n$.

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That's all. Thanks.

If you have any Paul Erdős memorabilia that you would like to donate to the Special Collections Library at the University of Memphis, email me (Nathan.Lindzey@memphis.edu)