On the Combinatorics of Schubert Calculus in Elliptic Cohomology

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Mostly concerned with $H^*(G/P)$ and $K(G/P)$, including their T-equivariant versions, where T is the corresponding maximal torus.

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There are many related combinatorial formulas: for the classes, their multiplication, etc.

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More on these later.

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- Initial work of topologists (1980s). Modern geometric theory of equivariant elliptic cohomology: Grojnowski (1994), Ginzburg-Kapranov-Vasserot (1995).

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- Rephrasing of the Rimányi-Weber construction in the Demazure algebra setup: L.-Zhao-Zhong (2023).

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- Billey-type formula – generalizes: Billey formula for cohomology (*Duke Math. J.*, 1999), Graham-Willems formula for K-theory, others for $\hbar$-deformed classes;
- pipe dream formula in type A – generalizes classical formulas for double Grothendieck polynomials (Fomin-Kirillov, 1994) and double Schubert polynomials (Bergeron-Billey, Billey-Jockush-Stanley, 1993).

Theta function

A crucial role is played by the Jacobi theta function:

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The same happens for the algebraic groups corresponding to $H^*(\,\cdot\,)$ and $K(\,\cdot\,)$, where the f.g.l.'s are, respectively:

$$F_a(x,y) = x + y \,, \qquad F_m(x,y) = x + y - xy \,.$$

The Demazure algebra setup (cf. Rimányi-Weber, L.-Zhao-Zhong)

Root system: simple roots α_i, simple reflections $s_i = s_{\alpha_i}$, Weyl group W; a formal parameter $\hbar$; elliptic curve E.

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Dynamical parameters (new!): $\lambda_{\beta^\vee}$ for $\beta^\vee$ a coroot.

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Remark. The elliptic Schubert classes E_w considered here are dual to those in Rimányi-Weber, being supported on $u \geq w$ (i.e., $\mathrm{E}_w(u) = 0$ unless $u \geq w$ in Bruhat order).

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Remark. The elliptic Schubert classes degenerate to SMC classes in $K_T(G/B)$ via the K-theory limit.

Elliptic Billey-type formula

We define the following two functions:

$$\mathsf{P}(x,y) = \frac{\theta(x-y)\,\theta(\hbar)}{\theta(y+\hbar)\,\theta(x)}, \qquad \mathsf{Q}(x,y) = \frac{\theta(x+\hbar)\,\theta(y)}{\theta(y+\hbar)\,\theta(x)}\,.$$

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Theorem (L.-Xiong-Zhong)

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We express $E_w(u)$ combinatorially as a sum over $J \subseteq \{1, \ldots, n\}$ such that $w = s_{i_1}^{\epsilon_1} \cdots s_{i_\ell}^{\epsilon_\ell}$, where $\epsilon_j = \delta^{\mathrm{Kr}}_{j \in J}$:

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$$E_w(u) = \sum_J \prod_{j=1}^{\ell} \begin{cases} \mathrm{Q}(\lambda_{\check{\gamma}_j^J}, z_{\beta_j}), & j \in J, \\ \mathrm{P}(\lambda_{\check{\gamma}_j^J}, z_{\beta_j}), & j \notin J, \end{cases} \qquad \begin{cases} \beta_j = s_{i_1} \cdots s_{i_{j-1}} \alpha_{i_j}, \\ \check{\gamma}_j^J = s_{i_\ell}^{\epsilon_\ell} \cdots s_{i_{j+1}}^{\epsilon_{j+1}} \alpha_{i_j}^\vee. \end{cases}$$

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We also have a parabolic version of this formula.

About the proof

The twisted dynamical group algebra:

$$\mathcal{Q}_{W^{\mathsf{d}}} := \mathcal{Q} \rtimes \mathbb{Q}[W^{\mathsf{d}}]\,,$$

spanned over $\mathcal{Q}$ by δ_w^{d}, for $w \in W$.

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Define the dynamical R-matrix (where $\delta_i^{\mathsf{d}} := \delta_{s_i}^{\mathsf{d}}$):

$$h_i(\beta) := \mathsf{P}(\lambda_{\alpha_i^\vee}, z_\beta) + \delta_i^{\mathsf{d}} \cdot \mathsf{Q}(\lambda_{\alpha_i^\vee}, z_\beta) \in \mathcal{Q}_{W^{\mathsf{d}}}\,.$$

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Proof idea: the elliptic Billey-type formula follows from the factorization of the generating function below, where u is fixed (cf. the above notation and Billey's proof in ordinary cohomology).

$$\sum_{w\in W} \delta_w^{\mathrm{d}} \cdot \mathrm{E}_w(u) = h_{i_1}(\beta_1) \cdots h_{i_\ell}(\beta_\ell) \; = \text{root polynomial [Billey]}\,.$$

Comparison with Billey's proof in ordinary cohomology

Fact. The R-matrix property of $h_i(x)$, $x \in X^*(T)$, for $H^*(\cdot)$, $K(\cdot)$ (Fomin-Kirillov, type by type) and $Ell(\cdot)$:

- Unitary property $h_i(x)h_i(-x) = 1$.

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$$\begin{aligned} h_i(x)h_j(y) &= h_j(y)h_i(x), && \text{if } (s_i s_j)^2 = \mathrm{id}; \\ h_i(x)h_j(x+y)h_i(y) &= h_j(y)h_i(x+y)h_j(x), && \text{if } (s_i s_j)^3 = \mathrm{id}. \end{aligned}$$

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- for us, the R-matrix property is a corollary (uniform proof), as opposed to an input for the main proof (in Billey's approach);
- instead, our proof of the factorization is deduced directly from the recursive generation of the elliptic Schubert classes.

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gives a polynomial representative of E_w, where $u_0 \in S_{2n}$ is

$$u_0 : i \xleftrightarrow{\text{transposition}} n + i, \qquad i = 1, \ldots, n,$$

and $w \times \mathrm{id}$ is viewed as an element of S_{2n}.

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gives a polynomial representative of E_w, *where* $u_0 \in S_{2n}$ *is*

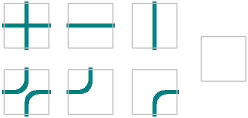
$$u_0 : i \xleftrightarrow{\text{transposition}} n+i, \qquad i = 1, \ldots, n\,,$$

and $w \times \mathrm{id}$ *is viewed as an element of* S_{2n}.

By our Billey-type formula before, $\mathcal{E}_w$ *is expressed combinatorially, and then this expression is translated using the generic pipe dreams of Knutson and Zinn-Justin.*

Generic pipe dreams

Tilings of the following $n \times n$ square grid by the following tiles:



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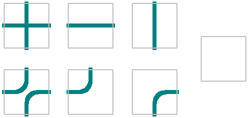

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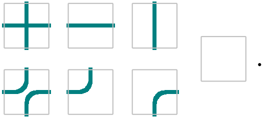

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Theorem (L.-Xiong-Zhong)

For any $w \in S_n$, we have

$$\mathcal{E}_w = \sum_{\pi \in \mathrm{GPD}(w)} \mathrm{weight}(\pi)\,.$$

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	$Q(\lambda_a - \lambda_b, y_j - x_i)$		$Q(\lambda_a - c\hbar, y_j - x_i)$		$Q(c\hbar - \lambda_b, y_j - x_i)$
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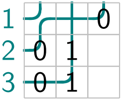
Example for $w = 132 \in S_3$.

weight (

)

$$= \prod \begin{cases} P(\lambda_1 - \lambda_2, y_1 - x_1) & Q(\lambda_2 - \lambda_3, y_2 - x_1) & P(\lambda_2, y_3 - x_1) \\ P(\lambda_2, y_1 - x_2) & Q(\hbar - \lambda_3, y_2 - x_2) & 1 \\ Q(\lambda_3, y_1 - x_3) & P(\lambda_3 - \hbar, y_2 - x_3) & 1 \end{cases}$$

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This recovers the pipe dream model of double Schubert polynomials.

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In type A, by applying this process to $\mathcal{E}_w$, and by further setting $y = 0$, we recover the polynomial representatives for Schubert classes in equivariant K-theory (double Grothendieck polynomials; Fomin-Kirillov, 1994) and equivariant cohomology (double Schubert polynomials; Bergeron-Billey, Billey-Jockush-Stanley, 1993).