FPSAC 2026

Lusztig's q-weight multiplicities and their refinements

Seung Jin Lee

Seoul National University

July 14, 2026

Type C_n: joint with Hyeonjae Choi and Donghyun Kim ▪ refinements: ongoing with Hyeonjae Choi and Mark Shimozono

Kostka numbers

For a partition λ, the Schur function expands in the monomial symmetric functions as

$$s_\lambda(x) = \sum_\mu K_{\lambda,\mu}\, m_\mu(x),$$

or, equivalently,

$$h_\mu(x) = \sum_\lambda K_{\lambda,\mu}\, s_\lambda(x),$$

where h_μ is the complete homogeneous symmetric function.

Definition

$$K_{\lambda,\mu} = \#\,\mathrm{SSYT}(\lambda,\mu),$$

where $\mathrm{SSYT}(\lambda,\mu)$ is the set of semistandard Young tableaux of shape λ and weight μ.

Kostka–Foulkes polynomials

The q-analogue appears by replacing monomial symmetric functions with Hall–Littlewood polynomials:

$$s_\lambda(x) = \sum_\mu K_{\lambda,\mu}(q)\, P_\mu(x;q),$$

where $P_\mu(x;q)$ is the Hall–Littlewood polynomial. Equivalently,

$$H_\mu(x;q) = \sum_\lambda K_{\lambda,\mu}(q)\, s_\lambda(x),$$

where $H_\mu(x;q)$ is the modified Hall–Littlewood polynomial.

Specialization

With the usual normalization,

$$P_\mu(x;1) = m_\mu(x), \qquad K_{\lambda,\mu}(1) = K_{\lambda,\mu}.$$

COUNT $K_{\lambda,\mu}$ $\Longrightarrow$ **GRADED COUNT** $K_{\lambda,\mu}(q)$

The Lascoux–Schützenberger theorem

Theorem (Lascoux–Schützenberger)

$$K_{\lambda,\mu}(q) = \sum_{T \in \mathrm{SSYT}(\lambda,\mu)} q^{\mathrm{charge}(T)},$$

where $\mathrm{charge}(T)$ is a non-negative integer associated to T.

Corollary

$$K_{\lambda,\mu}(q) \in \mathbb{Z}_{\geq 0}[q].$$

Remark

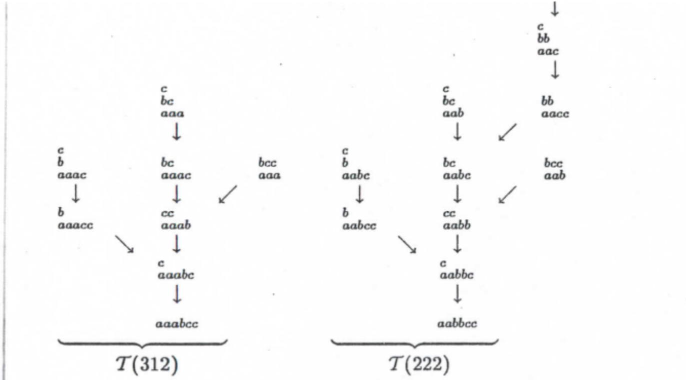
The same polynomials occur in several settings: Lusztig's q-weight multiplicities in type A, Springer fibers, Green polynomials, Hall–Littlewood/Macdonald theory, and affine-crystal one-dimensional sums.

A canonical embedding

Lascoux's paper gives the following explicit cocharge-preserving embedding of rank posets:

$$T(312) \hookrightarrow T(222),$$

where $T(\alpha) = \mathrm{SSYT}(\text{weight} = \alpha)$.



Dominance injections and standardization

For $\mu, \nu \vdash a$, write $\mu \succeq \nu$ if

$$\sum_{i=1}^{k} \mu_i \geq \sum_{i=1}^{k} \nu_i \qquad \text{for every } k \geq 1.$$

Cocharge-preserving injection

For every fixed shape λ and $\mu \succeq \nu$, there is an injection

$$\iota_{\mu\to\nu} : \mathrm{SSYT}(\lambda,\mu) \hookrightarrow \mathrm{SSYT}(\lambda,\nu), \qquad \mathrm{cocharge}(\iota_{\mu\to\nu}(T)) = \mathrm{cocharge}(T).$$

Thus passing downward in dominance order can be realized without changing the cocharge.

Standardization $=$ the type A splitting map

For $\nu = (1^a)$,

$$\mathrm{SSYT}(\lambda,\mu) \hookrightarrow \mathrm{SSYT}\big(\lambda,(1^a)\big) = \mathrm{SYT}(\lambda).$$

This special injection is called **standardization**, or the **splitting map in type A**: repeated letters are split into distinct letters, while cocharge is preserved.

Three generalizations of $K_{\lambda,\mu}(q)$

1. Macdonald

Macdonald–Kostka coefficients in

$$\widetilde{H}_\mu(x;q,t) = \sum_\lambda K_{\lambda,\mu}(q,t)s_\lambda(x)$$

2. Catalan functions

Root-ideal deformations appearing in the study of k-Schur functions and flag variety.

3. Lusztig

q-weight multiplicities for other Lie types, with the present focus on type C.

Goal of this talk

This talk defines and investigates generalizations of both Catalan functions and Lusztig's q-weight multiplicities for type C.

Goal of this talk

This talk defines and investigates generalizations of both Catalan functions and Lusztig's q-weight multiplicities for type C.

1. Give a combinatorial formula for Lusztig's q-weight multiplicity in type C using semistandard oscillating tableaux. (Choi–Kim–Lee)
2. Briefly review the Schur positivity of Catalan functions (Blasiak–Morse–Pun).
3. Introduce a root-ideal (Catalan-type) version of the above q-weight multiplicity.

PART 2

Oscillating tableaux and KR energy

Lusztig's q-weight multiplicity

For dominant weights λ, μ, the ordinary weight multiplicity is

$$\mathrm{KL}^{\mathfrak{g}}_{\lambda,\mu}(1) = \sum_{w \in W} (-1)^{\ell(w)} [e^{w(\lambda+\rho)-(\mu+\rho)}] \prod_{\alpha \in R^+} \frac{1}{1-e^{\alpha}},$$

known as Weyl dimension formula. When $\mathfrak{g} = \mathfrak{gl}_n$ (i.e., for type A), the $\mathrm{KL}^{\mathfrak{g}}_{\lambda,\mu}(1) = K_{\lambda,\mu}$.

Lusztig's q-weight multiplicity

Definition [Lusztig]

$$\mathrm{KL}^{\mathfrak{g}}_{\lambda,\mu}(q) = \sum_{w\in W} (-1)^{\ell(w)} [e^{w(\lambda+\rho)-(\mu+\rho)}] \prod_{\alpha\in R^+} \frac{1}{1-qe^{\alpha}}.$$

In type A, this is the Kostka–Foulkes polynomial.

Theorem [Lusztig]

For dominant weights λ, μ, we have

$$\mathrm{KL}^{\mathfrak{g}}_{\lambda,\mu}(q) \in \mathbb{Z}_{\geq 0}[q].$$

The proof uses affine Kazhdan-Lusztig polynomials.

What replaces SSYT in type C?

First we need a positive graded model for $\mathrm{KL}^{C_n}_{\lambda,\mu}(q)$.

$$\mathrm{KL}^{C_n}_{\lambda,\mu}(q) \quad = \quad \sum_T q^{D(T)}.$$

$$\boxed{\begin{array}{c}\textbf{Semistandard oscillating tableaux}\\ \text{the type } C \text{ replacement for SSYT}\end{array}} \overset{\phi_c}{\longleftrightarrow} \boxed{\begin{array}{c}\textbf{Classical highest-weight elements}\\ \text{in } B^{r_m,1}\otimes\cdots\otimes B^{r_1,1}\\ \text{column KR crystals of affine type } D_N^{(1)},\ N \gg 0\end{array}}$$

The dictionary

$$\underbrace{\text{SSOTs}}_{\text{objects}} \;+\; \underbrace{\text{KR energy } D}_{\text{grading}} \;=\; \text{the type } C \text{ analogue of SSYT} + \text{charge.}$$

What is a KR crystal $B^{r,s}$?

For a non-affine node r and $s \geq 1$, the Kirillov–Reshetikhin module $W_s^{(r)}$ has an affine crystal $B^{r,s}$. In affine type A its classical restriction is

$$B^{r,s}\Big|_{A_N} \cong V(s\omega_r),$$

where $V(\nu)$ denotes the classical highest-weight crystal; deleting the 0-arrows gives this restriction.

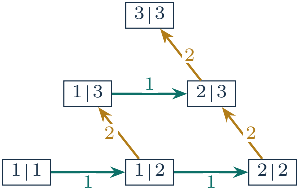
What is a KR crystal $B^{r,s}$?

For a non-affine node r and $s \geq 1$, the Kirillov–Reshetikhin module $W_s^{(r)}$ has an affine crystal $B^{r,s}$. In affine type A its classical restriction is

$$B^{r,s}\Big|_{A_N} \cong V(s\omega_r),$$

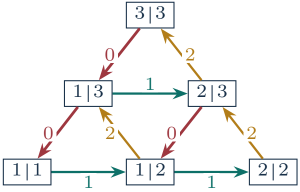
where $V(\nu)$ denotes the classical highest-weight crystal; deleting the 0-arrows gives this restriction.

CLASSICAL $V(2\omega_1)$ of type A_2



Only colors $1, 2$: the usual highest-weight crystal.

AFFINE $B^{1,2}$ of type $A_2^{(1)}$



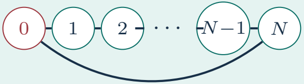
Same six tableaux, now with the affine 0-arrows.

$B^{1,2}\Big|_{A_2} \cong V(2\omega_1)$; the 0-arrows are the affine data from which energy is built.

Column KR crystals: $A_N^{(1)}$ and $D_N^{(1)}$

We now specialize to $s = 1$, the **column** KR crystals $B^{r,1}$. The extra crystal arrows are indexed by the affine node 0.

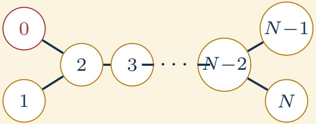
Dynkin diagram for type $A_N^{(1)}$



$$B^{r,1}\big|_{A_N} \cong V(\omega_r).$$

One classical component; on tensor products, the energy statistic recovers charge.

Dynkin diagram for type $D_N^{(1)}$



$$B^{r,1}\big|_{D_N} \cong V(\omega_r) \oplus V(\omega_{r-2}) \oplus \cdots .$$

We take N sufficiently large and use only non-spin columns $1 \leq r \leq N-2$.

KR energy: the column setting

Fix N sufficiently large and a tensor product of non-spin column KR crystals of affine type $D_N^{(1)}$:

$$B = B^{r_m,1} \otimes \cdots \otimes B^{r_1,1}, \qquad 1 \leq r_i \leq N-2.$$

Let $\mathrm{HW}_\lambda(B)$ denote its classical highest-weight elements of weight λ.

One-dimensional sum

$$X_{\lambda,B}(q) = \sum_{b \in \mathrm{HW}_\lambda(B)} q^{D(b)}.$$

KR energy: the column setting

Fix N sufficiently large and a tensor product of non-spin column KR crystals of affine type $D_N^{(1)}$:

$$B = B^{r_m,1} \otimes \cdots \otimes B^{r_1,1}, \qquad 1 \leq r_i \leq N-2.$$

Let $\mathrm{HW}_\lambda(B)$ denote its classical highest-weight elements of weight λ.

One-dimensional sum

$$X_{\lambda,B}(q) = \sum_{b \in \mathrm{HW}_\lambda(B)} q^{D(b)}.$$

The energy function D is constant when applying e_i, f_i for $i \neq 0$, and varies by ± 1 or 0 when applying e_0. (We will not use this definition)

Type $A_N^{(1)}$: energy recovers charge

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}$$

$$\mathrm{Des}(T) = \{2,4\}$$

$$[n-1] \setminus \mathrm{Des}(T) = \{1,3\}$$

$$\mathrm{charge}(T) = (5-1) + (5-3) = 6.$$

Translate to a KR path

Record the column containing i, read from right to left:

$$b_T = 2 \otimes 3 \otimes 1 \otimes 2 \otimes 1 \in (B^{1,1})^{\otimes 5}.$$

Compute the grading

Note that we have $3 > 1$ at second position and $2 > 1$ at fourth position. Therefore

$$\mathrm{charge}(T) = 4 + 2 = 6.$$

This is the energy–charge correspondence of Nakayashiki–Yamada.

Recall that there is cocharge preserving injection from $T(\lambda)$ to $T(1^a)$ for any partition λ of a.

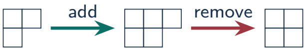
Oscillating horizontal strips

Definition (L., 2019)

An oscillating horizontal strip is a sequence $S = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$ such that $\lambda^{(2)}$ contains both endpoints and $\lambda^{(2)}/\lambda^{(i)}$ is a horizontal strip for $i = 1, 3$.

$$\mathrm{size}(S) = |\lambda^{(2)}/\lambda^{(1)}| + |\lambda^{(2)}/\lambda^{(3)}|,$$

$$\mathrm{inside}(S) = \lambda^{(1)}, \qquad \mathrm{outside}(S) = \lambda^{(3)}.$$



Semistandard oscillating tableaux

Definition

A semistandard oscillating tableau is a sequence $T = (S_1, S_2, \ldots)$ of oscillating horizontal strips such that

$$\mathrm{inside}(S_1) = \varnothing, \qquad \mathrm{outside}(S_i) = \mathrm{inside}(S_{i+1}),$$

and $\mathrm{size}(S_k) = 0$ for all sufficiently large k.

The shape is $\mathrm{outside}(S_k)$ for k sufficiently large, and

$$\mathrm{wt}(T) = (\mathrm{size}(S_1), \mathrm{size}(S_2), \ldots).$$

Let $c(T)$ be the maximum number of columns of the partitions appearing in T. Then

$$\mathrm{SSOT}_g(\lambda, \mu) = \{T \in \mathrm{SSOT}(\mathrm{sh} = \lambda, \mathrm{wt} = \mu) : c(T) \leq g\}.$$

From an SSOT to a column-crystal path

For one oscillating strip $Y = (\alpha, \beta, \gamma)$, record the columns that were added and removed:

$$\mathrm{cind}(Y) = \{i : \alpha_i^{\mathsf{t}} + 1 = \beta_i^{\mathsf{t}}\} \cup \{\bar{i} : \gamma_i^{\mathsf{t}} + 1 = \beta_i^{\mathsf{t}}\}.$$

Unbarred letters encode additions; barred letters encode removals.

The embedding

For $T = (T_1, \ldots, T_m)$, define

$$\phi_c(T) = (\mathrm{cind}(T_m)) \otimes \cdots \otimes (\mathrm{cind}(T_1)),$$

a classical highest-weight element in a tensor product of type $D_N^{(1)}$ column KR crystals, with N sufficiently large.

Type $D_N^{(1)}$: the SOT grading

For a standard oscillating tableau with row record

$$r(T) = (\bar{1})(1)(2)(1)(1),$$

associate the highest-weight path

$$v_T = \bar{1} \otimes 1 \otimes 2 \otimes 1 \otimes 1 \in (B^{1,1})^{\otimes 5}.$$

Local energy

$$H(\bar{1}, 1) = 2, \qquad H(2, 1) = 1, \qquad H(1, 1) = H(1, 2) = 0.$$

The two nonzero contributions have position weights 1 and 3, so

$$D(v_T) = 1 \cdot 2 + 3 \cdot 1 = 5.$$

Namely, we count the adjacent pair $(\bar{1}, 1)$ as a double descent. (counted twice)

The splitting map makes energy computable

A column tensor can be split into single boxes:

$$S:\ B^{r_m,1} \otimes \cdots \otimes B^{r_1,1} \longrightarrow (B^{1,1})^{\otimes N}.$$

The splitting map preserves the classical crystal structure and the relevant (co)energy, up to an explicit normalization, which generalize the standardization

$$T(\text{weight} = \lambda) \mapsto T(1^N)$$

we saw in type A.

PART 3

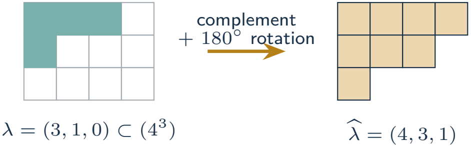
Lusztig's q-weight multiplicity for type C

The orthogonal-complement dictionary

For $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $g \geq \lambda_1$, write

$$\widehat{\lambda} = \mathrm{oc}(\lambda, g) = (g - \lambda_n, \ldots, g - \lambda_1).$$

This is the 180°-rotated complement of λ in the $n \times g$ rectangle.



Type C: the main formula

Let $\lambda, \mu \in \mathrm{Par}_n$ and choose $g \geq \max(\lambda_1, \mu_1)$. Set

$$\widehat{\lambda} = \mathrm{oc}(\lambda, g), \qquad \widehat{\mu} = \mathrm{oc}(\mu, g).$$

Theorem (Choi–Kim–Lee)

$$\boxed{\mathrm{KL}^{C_n}_{\lambda,\mu}(q) = \sum_{T \in \mathrm{SSOT}_g(\widehat{\lambda},\widehat{\mu})} q^{D(\phi_c(T))}}$$

Corollaries

Corollary 1: Positivity

$$\mathrm{KL}^{C_n}_{\lambda,\mu}(q) \in \mathbb{Z}_{\geq 0}[q],$$

Corollary 2: Monotonicity

$$\mathrm{KL}^{C_n}_{\lambda+(1^n),\mu+(1^n)}(q) - \mathrm{KL}^{C_n}_{\lambda,\mu}(q) \in \mathbb{Z}_{\geq 0}[q].$$

The second statement naturally follows from the inclusion

$$\mathrm{SSOT}_g \subseteq \mathrm{SSOT}_{g+1}\,.$$

PART 4

Catalan functions and Schur positivity

Root ideals and type A Catalan functions

On the positive roots R^+, use the root-poset order

$$\alpha \preceq \beta \quad \Longleftrightarrow \quad \beta - \alpha \in \mathbb{Z}_{\geq 0}\Pi,$$

where Π is the set of simple roots.

Definition

A subset $\Psi \subseteq R^+$ is a *root ideal*, equivalently an *upper order ideal*, if

$$\alpha \in \Psi,\ \alpha \preceq \beta \quad \Longrightarrow \quad \beta \in \Psi.$$

For type $A_{\ell-1}$, write $\alpha_{ij} = \varepsilon_i - \varepsilon_j$ for $i < j$. Then the condition is equivalently

$$(i,j) \in \Psi, \quad a \leq i < j \leq b \quad \Longrightarrow \quad (a,b) \in \Psi.$$

12	13	14	15
	23	24	25
		34	35
			45

an upper ideal Ψ

The Schur-positivity problem

For a root ideal $\Psi \subseteq R_\ell^+$ and a partition $\mu = (\mu_1 \geq \cdots \geq \mu_\ell \geq 0)$, write

$$H(\Psi;\mu)(X;q) = \sum_\lambda K_{\lambda,\mu}^{\Psi}(q)\, s_\lambda(X).$$

One possible definition: $K_{\lambda,\mu}^{\Psi}(q)$ is defined by

$$\boxed{\mathrm{KL}_{\lambda,\mu}^{A_n}(\Psi;q) := \sum_{w\in W(A_n)} (-1)^{\ell(w)} \left[e^{w(\lambda+\rho)-(\mu+\rho)}\right] \prod_{\alpha\in\Psi} \frac{1}{1-qe^{\alpha}}}$$

Schur positivity of Catalan functions

Conjectured by [Chen-Haiman]. Proved by [Blasiak–Morse–Pun, 2024]

For every root ideal $\Psi \subseteq R_\ell^+$ and every partition μ,

$$K^{\Psi}_{\lambda,\mu}(q) \in \mathbb{Z}_{\geq 0}[q].$$

The proof passes through a larger nonsymmetric family $H(\Psi;\mu;w)$ and generalized Demazure crystals.

A manifestly positive tableau formula

Let $\mathbf{n}(\Psi)$ be the boundary sequence determined by the root ideal Ψ. [BMPS] obtain

Katabolism formula

$$\boxed{H(\Psi;\mu)(X;q) = \sum_{\substack{U \in \mathrm{SSYT}_\ell(\mu) \\ U \text{ is } \mathbf{n}(\Psi)\text{-katabolizable}}} q^{\mathrm{charge}(U)}\, s_{\mathrm{shape}(U)}(X)}.$$

Here $\mathrm{SSYT}_\ell(\mu)$ consists of tableaux of weight μ. The katabolism condition recursively filters these tableaux according to the boundary of Ψ.

Full-root specialization

When $\Psi = R^+$, every tableau is allowed, and the formula becomes the Lascoux–Schützenberger charge formula for $H_\mu(X;q)$.

Root ideals come with a recursion

Write $\varepsilon_{(i,j)} = \varepsilon_i - \varepsilon_j$.
If β is addable to Ψ, then

$$H(\Psi;\mu) = H(\Psi \cup \{\beta\};\mu) - q\, H(\Psi \cup \{\beta\};\mu + \varepsilon_\beta).$$

For example,

$$H(R^+ \setminus \{\varepsilon_i - \varepsilon_{i+1}\};\mu) = H(R^+;\mu) - q\, H(R^+;\mu + \varepsilon_i - \varepsilon_{i+1}).$$

This is basically classifying the complement of the image $T(\mu + \varepsilon_i - \varepsilon_{i+1}) \hookrightarrow T(\mu)$.

Useful case

When both μ and $\mu + \varepsilon_\beta$ are partitions, the above recursion is very useful to deduce the right subset of SSYT recursively.

PART 5

Catalan-type refinements of $\mathrm{KL}^{C_n}_{\lambda,\mu}(q)$

The positive roots of type C_3

The positive roots of C_3 can be arranged in the following shifted diagram:

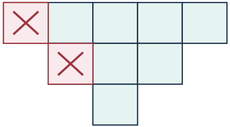
1	$\varepsilon_1 - \varepsilon_2$	$\varepsilon_1 - \varepsilon_3$	$\varepsilon_1 + \varepsilon_3$	$\varepsilon_1 + \varepsilon_2$	$2\varepsilon_1$
	2	$\varepsilon_2 - \varepsilon_3$	$\varepsilon_2 + \varepsilon_3$	$2\varepsilon_2$	
		3	$2\varepsilon_3$		

Root ideals

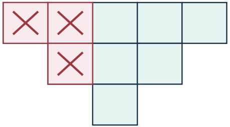
A root ideal is an upper order ideal in this diagram. Equivalently, its complement is a lower order ideal.

Three sample root ideals in $R^+(C_3)$

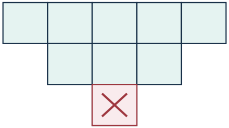
In the pictures below, a red cross marks a root omitted from the root ideal Ψ.



$\Psi_1 = R^+ \setminus \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\}$.



$\Psi_2 = R^+(C_3) \setminus R^+(A_2)$.



$\Psi_3 = R^+(C_3) \setminus \{2\varepsilon_3\}$.

Ψ_1 and Ψ_2 contain every non-A positive root; Ψ_3 deletes one long root.

Root-ideal q-weight multiplicities in type C

Let $\Psi \subseteq R^+(C_n)$ be a root ideal and let λ, μ be dominant weights.

Definition

$$\boxed{\mathrm{KL}^{C_n}_{\lambda,\mu}(\Psi;q) := \sum_{w\in W(C_n)} (-1)^{\ell(w)} \left[e^{w(\lambda+\rho)-(\mu+\rho)}\right] \prod_{\alpha\in\Psi} \frac{1}{1-qe^{\alpha}}}$$

One can show that $\mathrm{KL}^{C_n}_{\lambda,\mu}(\Psi;q)$ both generalizes Schur coefficients of Catalan functions for $\Psi_A \subset R^+(A_n)$ (and by setting $|\lambda| = |\mu|$) and Lusztig's q-weight multiplicities (by setting $\Psi = R^+(C_n)$). In this talk, we only consider *stable* version of $\mathrm{KL}^{C_n}_{\lambda,\mu}(\Psi;q)$, denoted by

$$^{\infty}\mathrm{KL}^{\mathfrak{g}_n}_{\lambda,\mu}(\Psi;q) := \lim_{a\to\infty} \mathrm{KL}^{\mathfrak{g}_n}_{\lambda+(a^n),\mu+(a^n)}(\Psi;q).$$

Questions

$$^{\infty}\mathrm{KL}^{\mathfrak{g}_n}_{\lambda,\mu}(\Psi;q) \in \mathbb{Z}_{\geq 0}[q]? \quad \text{Any monotonicity?}$$

The first local injection

Consider the root ideal obtained by removing one type A root:

$$\Psi = R^+(C_n) \setminus \{\varepsilon_{n-1} - \varepsilon_n\}.$$

In this case there is a natural injection of column KR crystals

$$\boxed{B^{\mu_2-1,1} \otimes B^{\mu_1+1,1} \hookrightarrow B^{\mu_2,1} \otimes B^{\mu_1,1}}$$

whenever the two sides are defined.

$$\begin{tikzcd}
T(\mu_2 - 1, \mu_1 + 1) \arrow[r, "\text{type } A \text{ splitting}"] \arrow[d, dashed] & T(\mu_2, \mu_1) \arrow[d, dashed] \\
B^{\mu_2-1,1} \otimes B^{\mu_1+1,1} \arrow[r, "\text{KR-crystal injection}"] & B^{\mu_2,1} \otimes B^{\mu_1,1}
\end{tikzcd}$$

This is the KR-crystal analogue of the type A map from the previous part, $T(\mu_2 - 1, \mu_1 + 1) \hookrightarrow T(\mu_2, \mu_1)$.

Refined version of $X = K$

For a nonexceptional stable family and a kind $\boxminus$, the K-polynomial is

$$K^{\boxminus}_{\lambda,\mu}(q) = q^{\frac{|\mu|-|\lambda|}{2}} \sum_{\nu \in P_m} K_{\nu,\mu}(q) \sum_{\gamma \in P^{\boxminus}_m} c^{\nu}_{\gamma,\lambda}.$$

Theorem [Choi-L.-Shimozono, 2026+]

Let $\Psi \subset R^+(C_n)$ be a root ideal for type C, containing all positive roots of the form $e_i + e_j$. Then we have

$${}^{\infty}\mathrm{KL}^{C_n}_{\widehat{\lambda},\widehat{\mu}}(\Psi; q) = q^{\frac{|\mu|-|\lambda|}{2}} \sum_{\nu \in P_m} K^{\Psi_A}_{\nu,\mu}(q) \sum_{\gamma \in P^{\boxminus}_m} c^{\nu}_{\gamma,\lambda}.$$

where $\Psi_A = \Psi \cap R^+(A_{n-1})$.

Here $P^{\boxminus}_m$ denotes partitions tiled by vertical dominoes, and $c^{\nu}_{\gamma,\lambda}$ is a Littlewood–Richardson coefficient.

Why similar to original $X = K$?

$$
\begin{tikzcd}
\mathrm{SSOT}(\lambda,\mu) \arrow[r, "\eta_\mu"] \arrow[d, "S_\mu^{D(1)}"'] & \displaystyle\bigcup_{\gamma\in P_m^{\square}} \bigcup_{\nu} \mathrm{SSYT}(\nu,\mu)\times LR^t(\nu/\lambda;\gamma^t) \arrow[d, "S_\mu^{A(1)}\times \mathrm{id}"] \\
\mathrm{SSOT}(\lambda,(1^n)) \arrow[r, "\eta_{(1^n)}"'] & \displaystyle\bigcup_{\gamma\in P_m^{\square}} \bigcup_{\nu} \mathrm{SSYT}(\nu,(1^n))\times LR^t(\nu/\lambda;\gamma^t)
\end{tikzcd}
$$

where S's are splitting maps.

Note that $\eta_{(1^n)}$ is the bijection studied by Lecouvey-Okado-Shimozono which combinatorially proves $X = K$.

The non-A endpoint

Take $\Psi_C = R^+(C_n) \setminus R^+(A_{n-1})$, so that $\Psi_A = \varnothing$ and $K^{\varnothing}_{\nu,\mu}(q) = \delta_{\nu,\mu}$. Therefore, we have

Corollary

$$\boxed{{}^{\infty}\mathrm{KL}^{C_n}_{\widehat{\lambda},\widehat{\mu}}(\Psi_C; q) = q^{\frac{|\mu|-|\lambda|}{2}} \sum_{\delta \in P^{\boxminus}_m} c^{\mu}_{\delta,\lambda}}$$

What happens when one removes a non-A root?

First test: delete the long root $2\varepsilon_n$

Take

$$\Psi_{\text{long}} = R^+(C_n) \setminus \{2\varepsilon_n\}.$$

The affected column factor is the first part of μ; all other factors are unchanged. The relevant map is

$$B^{\mu_1-2,1} \hookrightarrow B^{\mu_1,1}.$$

As classical crystals,

The common components

$$B^{\mu_1-2,1} \cong \bigoplus_{j\geq 1} V(\omega_{\mu_1-2j}) \hookrightarrow V(\omega_{\mu_1}) \oplus \bigoplus_{j\geq 1} V(\omega_{\mu_1-2j}) \cong B^{\mu_1,1}.$$

The complement of the image $V(\omega_{\mu_1})$, in terms of SSOT language, is oscillating horizontal strip with no removed cells, which is (classical) horizontal strip.

SSOT formulas

Let $\Psi^{(n)} = R^+(C_n) \setminus \{2\varepsilon_n\}$ and write $D(T) := D(\phi_c(T))$.

Theorem

If $\mu_2 \le \mu_1 - 2$, then

$$ {}^{\infty}\mathrm{KL}^{C_n}_{\widehat{\lambda},\widehat{\mu}}(\Psi^{(n)}; q) = \sum_{\substack{T \in \mathrm{SSOT}(\lambda,\mu) \\ T_1 \text{ does not delete a box}}} q^{D(T)}, $$

SSOT formulas

Let $\Psi^{(n)} = R^+(C_n) \setminus \{2\varepsilon_n\}$ and write $D(T) := D(\phi_c(T))$.

Theorem

If $\mu_2 \leq \mu_1 - 2$, then

$$ {}^{\infty}\mathrm{KL}^{C_n}_{\widehat{\lambda},\widehat{\mu}}(\Psi^{(n)}; q) = \sum_{\substack{T \in \mathrm{SSOT}(\lambda,\mu) \\ T_1 \text{ does not delete a box}}} q^{D(T)}, $$

Remark 1

Under mild condition on μ, one can similarly compute ${}^{\infty}\mathrm{KL}^{C_n}_{\widehat{\lambda},\widehat{\mu}}(\Psi^{(i)}; q)$ where $\Psi^{(i)} = R^+(C_n) \setminus \{\varepsilon_a \pm \varepsilon_b \mid i \leq a \leq b \leq n\}$.

Restricted monotonicity conjecture

Caution

The preceding proof is straightforward only when $\mu_2 \le \mu_1 - 2$. If $\mu_2 = \mu_1$, the formula is different, and the naive monotonicity statement fails in general.

This suggests keeping the long-root content fixed.

Conjecture

Suppose $\Psi^{(1)} \supseteq \Psi^{(2)}$ and

$$\Psi^{(1)} \cap \{2\varepsilon_1, \dots, 2\varepsilon_n\} = \Psi^{(2)} \cap \{2\varepsilon_1, \dots, 2\varepsilon_n\}.$$

Then

$$^{\infty}\mathrm{KL}^{C_n}_{\widehat{\lambda},\widehat{\mu}}(\Psi^{(1)}; q) - {}^{\infty}\mathrm{KL}^{C_n}_{\widehat{\lambda},\widehat{\mu}}(\Psi^{(2)}; q) \in \mathbb{Z}_{\ge 0}[q].$$

What remains open

1. Construct a direct catabolism or local recursion on SSOTs that changes energy in an expected way.

1-a. Describe the image of the splitting map from $\mathrm{SSOT}(\mathrm{weight} = \lambda)$ to $\mathrm{SSOT}(\mathrm{weight} = (1^{\mathrm{n}}))$.

2. Find a geometric or representation-theoretic realization of the refined coefficients.
3. Extend the Catalan/SSOT picture to type D and other column-crystal settings.
4. Relation with affine Kazhdan-Lusztig polynomials.

Thank you

Questions and comments

Lusztig q-weight multiplicities • KR energy • Catalan-type refinements

PART 6

Appendix

Two positivity landmarks

k-Schur functions are a special subfamily of Catalan functions.

Theorem (Kato, 2025)

Modified Macdonald polynomials are k-Schur positive. (not combinatorial)

Theorem (Blasiak–Morse–Pun, 2024)

Catalan functions indexed by a partition weight are Schur positive. (combinatorial)

Catalan functions provide the common root-ideal language.

We return to Catalan functions and their Schur positivity in **Part 4**.

Energy calculation: $\widehat{\lambda} = \varnothing$, $\widehat{\mu} = (1^4)$

After splitting, the three SSOTs become paths in $(B^{1,1})^{\otimes 4}$. For $b = b_4 \otimes b_3 \otimes b_2 \otimes b_1$,

$$D(b) = 3H(b_4, b_3) + 2H(b_3, b_2) + H(b_2, b_1).$$

The local values needed in this example are

$$\begin{aligned} &H(\bar{1}, \bar{1}) = H(1, 1) = H(1, \bar{1}) = 0, \qquad &&H(\bar{1}, 1) = 2, \\ &H(\bar{1}, \bar{2}) = H(\bar{2}, 2) = H(2, 1) = 1. \end{aligned}$$

KR path $\phi_c(T)$	weighted local contributions	energy
$\bar{1} \otimes \bar{1} \otimes 1 \otimes 1$	$3(0) + 2(2) + 1(0)$	4
$\bar{1} \otimes \bar{2} \otimes 2 \otimes 1$	$3(1) + 2(1) + 1(1)$	6
$\bar{1} \otimes 1 \otimes \bar{1} \otimes 1$	$3(2) + 2(0) + 1(2)$	8

g-BOUND The energy-4 and energy-8 paths lie in SSOT_1; the energy-6 path first appears in SSOT_2.

A monotonicity example: $g = 1$ versus $g = 2$

Take

$$(\lambda, \mu, g) = ((1^4), (0), 1) \qquad \text{and} \qquad ((2^4), (1^4), 2).$$

For both pairs,

$$\widehat{\lambda} = \varnothing, \qquad \widehat{\mu} = (1,1,1,1).$$

$g = 1$

$$\mathrm{SSOT}_1(\varnothing, (1^4)) = \left\{ \begin{matrix} \bar{1} \otimes \bar{1} \otimes 1 \otimes 1 & (D = 4), \\ \bar{1} \otimes 1 \otimes \bar{1} \otimes 1 & (D = 8) \end{matrix} \right\}.$$

Therefore

$$\mathrm{KL}^{C_4}_{(1^4),(0)}(q) = q^8 + q^4.$$

$g = 2$

$$\mathrm{SSOT}_2(\varnothing, (1^4)) = \mathrm{SSOT}_1(\varnothing, (1^4)) \sqcup \left\{ \begin{matrix} \bar{1} \otimes \bar{2} \otimes 2 \otimes 1 \\ (D = 6) \end{matrix} \right\}.$$

Hence

$$\mathrm{KL}^{C_4}_{(2^4),(1^4)}(q) = q^8 + q^6 + q^4.$$

MONOTONICITY $\mathrm{SSOT}_1 \subset \mathrm{SSOT}_2$ is energy-preserving; only the q^6 term is added.