On an extension problem on the moment curve

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Outline for section 1

Moment curve γ_d and cyclic polytopes

- The *moment curve* in $\mathbb{R}^d$ is $\gamma_d(t) = (t, t^2, \ldots, t^d)$.
- Any (finite) points on γ_d are in convex general position.
- The convex hull of n points, *the* cyclic polytope $C(n, d)$, is *neighborly* and attains the maximum number of faces (by McMullen for polytopes and Stanley for simplicial spheres).

An extension problem on the moment curve

Question

Does every non-overlappning collection of simplices on $A \subseteq \gamma_d$ extend to a triangulation of $\mathrm{conv}(A)$ *without adding new vertices?*

Here we only consider geometric triangulations.

An extension problem on the moment curve

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The answer to this question is negative in general from $\mathbb{R}^3$.

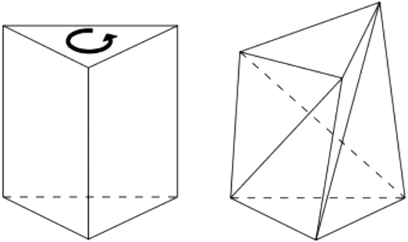


Figure: Schoenhardt polyhedron.

Main result

Question

Does every non-overlappning collection of simplices on $A \subseteq \gamma_d$ extend to a triangulation of $\operatorname{conv}(A)$ without adding new vertices?

Theorem (L., Nevo, 2025+)

For $d \leq 4$, the answer to the question is YES.
For $d \geq 5$, the answer to the question is NO.

Outline for section 2

Moment curve γ_d

- Affine properties of point sets on γ_d can be described *combinatorially* (*alternating oriented matroids*).
- $<$: The natural order for the points on γ_d.

$\sigma, \tau \subset \gamma_d$ are *k-interlacing* if $\exists v_1 < v_2 < \cdots < v_k$ from $\sigma \cup \tau$ which satisfies one of the following alternating conditions:

1. $v_1 \in \sigma, v_2 \in \tau, v_3 \in \sigma \ldots$, or
2. $v_1 \in \tau, v_2 \in \sigma, v_3 \in \tau \ldots$.

Proposition (Breen, 1973)

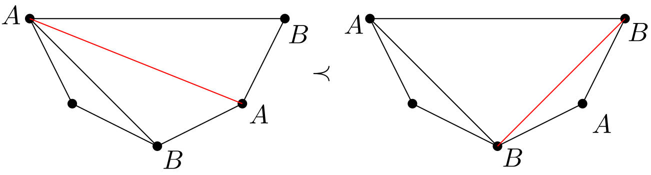
For (abstract) simplices $\sigma, \tau \subseteq \gamma_d$, σ and τ overlap iff σ and τ are $(d+2)$-interlacing.

σ and τ *overlap* if $\mathrm{conv}(\sigma) \cap \mathrm{conv}(\tau) \supsetneq \mathrm{conv}(\sigma \cap \tau)$.

Trianulations of a cyclic polytope

Edelman and Reiner (1996) introduced the *higher Stasheff-Tamari posets* (HST in short) on triangulations of $C(n, d)$ in two ways.

- $\leq_1$ by Pachner moves (of bistellar flips).



ABAB.

Trianulations of a cyclic polytope

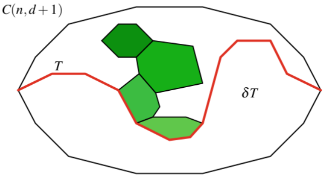


Figure 6.11: Finding an upflip by starting at a $(d+1)$ above $\mathscr{T}$ and searching "downwards".

- $\leq_2$ by using the heights after lifting into $\mathbb{R}^{d+1}$: $(t, t^2, \ldots, t^d) \mapsto (t, t^2, \ldots, t^d, t^{d+1})$.

Theorem (Nicholas Williams, 2024)

$\leq_1 = \leq_2$.

(a)

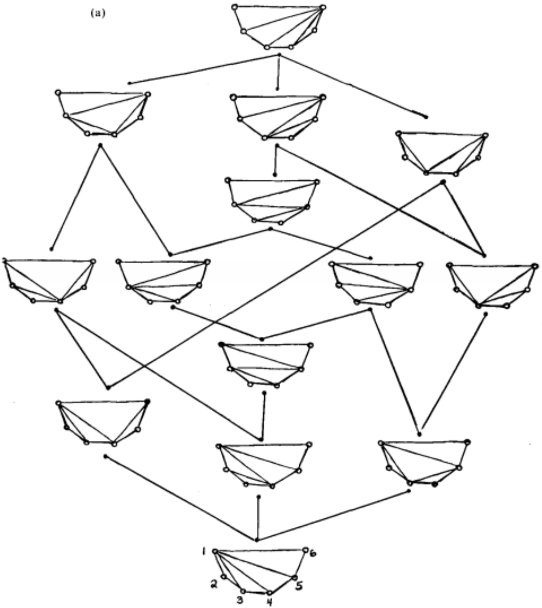
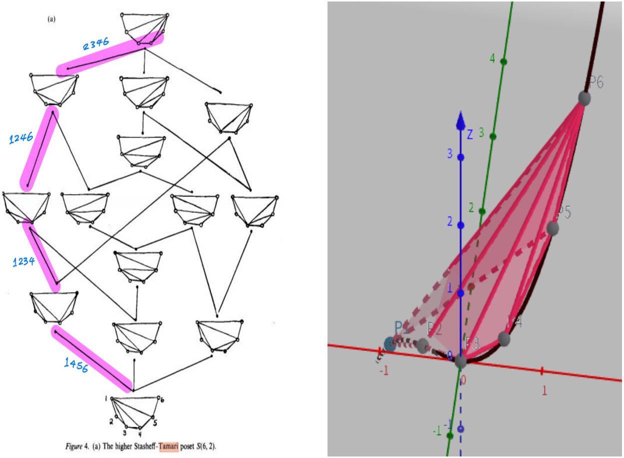


Figure 4. (a) The higher Stasheff-Tamari poset $S(6, 2)$.



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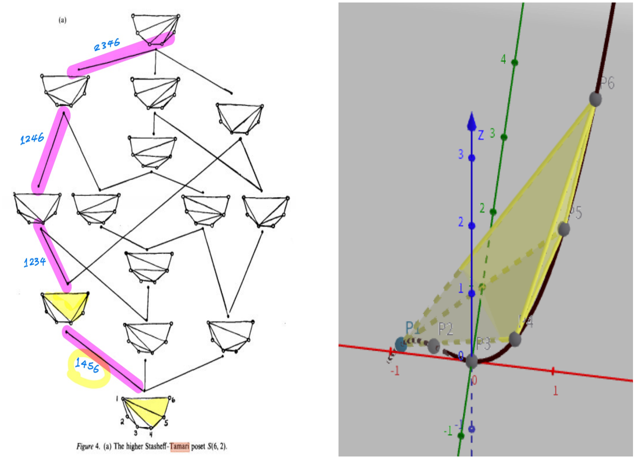

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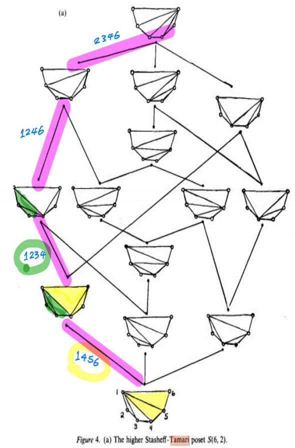

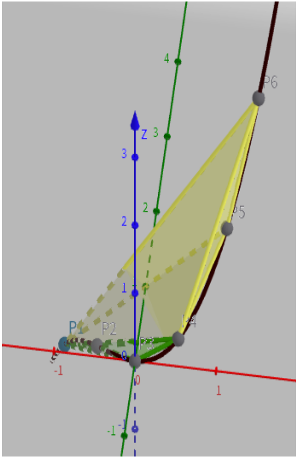

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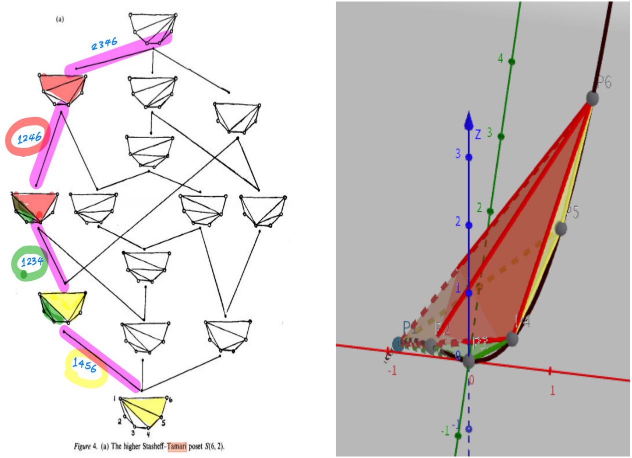

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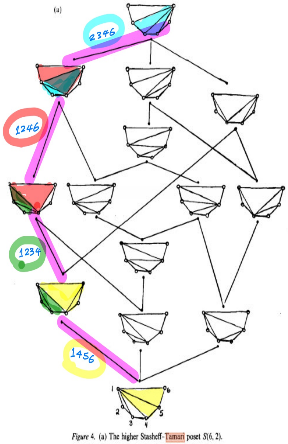

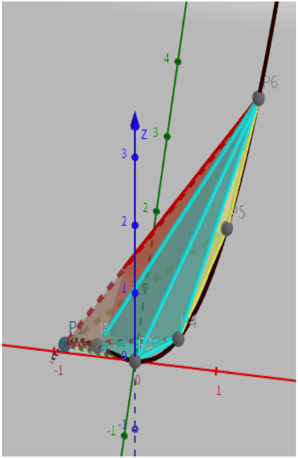

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Trianulations of a cyclic polytope

$S(n, d)$: the set of all triangulations of $C(n, d)$.

- Rambau (1997) proved that (the equivalence class of) a maximal chain in $(S(n, d-1), \leq_1)$ has a unique corresponding triangulation of $C(n, d)$.
- Our result can be also considered as an extension of Williams' result for $d = 3, 4$ saying: given the liftings of two triangulations T_1 and T_2 of $C(n, d-1)$ into $C(n, d)$, one higher than the other, we can find a sequence of flips between them.
- We rely on tools and known facts about the higher Stasheff-Tamari posets.
- The height order can be also described combinatorially: for two simplices $A, B \subset \gamma_d$, A and B overlap in $\mathbb{R}^d$ and the lifting of B is higher than that of A if and only if A and B are $(d+2)$-interlacing with
  - ABA... AB when d is even, and
  - BAB...AB when d is odd.

Connection to algebra (communicated with Nicholas Williams)

- Oppermann and Thomas (2012) gives a bijection from
  - from indecomposable objects in $\mathcal{O}_{A_n^\delta}$ to internal δ-simplices of $C(n+2\delta+1, 2\delta)$, which induces bijections
  - from basic cluster tilting objects in $\mathcal{O}_{A_n^\delta}$ to triangulations of $C(n+2\delta+1, 2\delta)$, and
  - from rigid objects in $\mathcal{O}_{A_n^\delta}$ to non-overlapping (internal) δ-simplices.

  Here, A_n^δ is the $(\delta-1)$-st higher Auslander algebras of linearly oriented A_n introduced by Iyama in the context of Auslander-Reiten theory.
- Oppermann and Thomas showed that in the classical $\delta = 1$ case every maximal object in $\mathcal{O}_{A_n^\delta}$ is cluster-tilting, while in $\delta \geq 3$, it is not. Our result for $d = 4$ shows that

Corollary

Every maximal rigid object in $\mathcal{O}_{A_n^2}$ is cluster tilting.

Outline for section 3

Useful facts (from earlier literature)

$S(n, d)$: the set of all triangulations of $C(n, d)$.

- For $d = 2$ or 3, it was already shown by Edelman and Reiner that $\leq_1 = \leq_2$: $\mathrm{HST}(n, d) := (S(n, d), \leq_1) = (S(n, d), \leq_2)$.
- (Rambau, 1997) A maximal chain in $HST(n, d)$ corresponds to a triangulation in $HST(n, d + 1)$ for $d = 2, 3$.
- $HST(n, 2)$ and $HST(n, 3)$ are lattices. There is a 1-1 correspondence $T \leftrightarrow sub(T)$, where $sub(T)$ is the set of $\lceil d/2 \rceil$-dimensional faces bounded above by T, and $T_1 \wedge T_2$ is given by $sub(T_1) \cap sub(T_2)$.
- For $T_1, T_2 \in HST(n, d)$ for $d = 2, 3$, $T_1 \leq T_2$ can be recognized by the interlacing pattern of length $d + 2$. It applies similarly for simplices on γ_d.

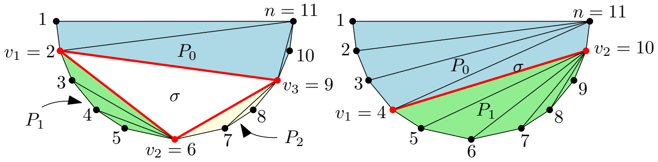
Technical lemma

D: the dimension for extension.
$d = D - 1$: one lower dimension where we use useful facts.
When $D = 3, 4$, we only need to consider 2-skeleton from a given collection of non-overlapping simplices. In fact, edges and triangles only for $D = 3$, and triangles only for $D = 4$.

Lemma

These lower dimensional faces can be ordered nicely as $\sigma_1, \sigma_2, \ldots, \sigma_m$ so that for every $i \in [m]$ there is $T_i \in HST(n, d)$ such that $\sigma_i \in T_i$ and $\sigma_j \leq T_i$ for $j < i$.

- We can define height order for simplices. It is a partial order.



Proof sketch for $D = 3, 4$

Lemma

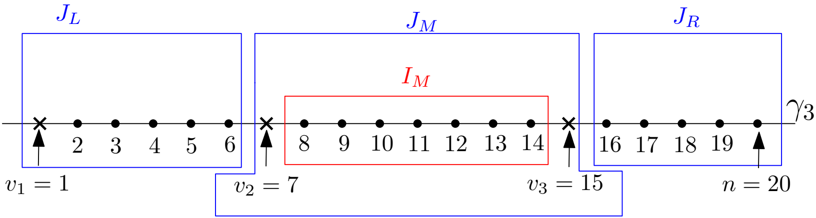
These lower dimensional faces can be ordered nicely as $\sigma_1, \sigma_2, \ldots, \sigma_m$ so that for every $i \in [m]$ there is $T_i \in HST(n, d)$ such that $\sigma_i \in T_i$ and $\sigma_j \leq T_i$ for $j < i$.

1. We order the 2 (or less)-dimensional faces $\sigma_1, \ldots, \sigma_m$ so that we can apply the lemma.
2. After applying the lemma, we obtain $T_1, \ldots, T_m$. Set $S_k = T_k \wedge T_{k+1} \wedge \cdots \wedge T_m$ for $k \in [m]$.
3. $S_1 \leq S_2 \leq \cdots \leq S_m$ is a chain in $HST(n, d)$ which corresponds to a triangulation in $HST(n, D)$.
4. Finally we show that $\sigma_i \in S_i$: $sub(S_k) = sub(\bigwedge_{i=k}^{m} T_i) = \bigcap_{i=k}^{m} sub(T_i)$ $\square$

Proof sketch of the lemma for $D = 4$

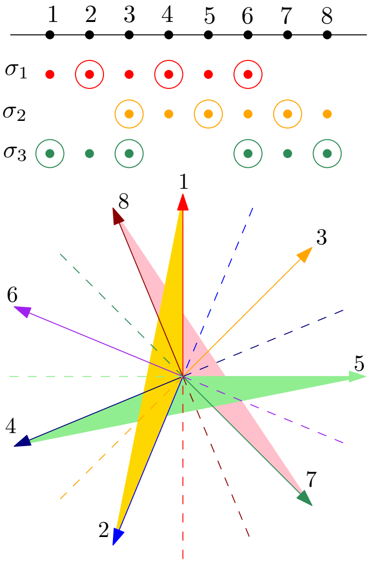
Lemma

Let $\tau_1, \dots, \tau_m, \sigma \subset \gamma_3$ be distinct triangles whose liftings are pairwise non-overlapping in $\mathbb{R}^4$. Furter assume $\min(\sigma) \leq \min(\tau_i)$ *and* $\min(\sigma) = \min(\tau_i)$ *implies* $\max(\sigma) \geq \max(\tau_i)$*. Then there is a triangulation $T \in HST(n, 3)$ such that $\sigma \in T$ and $\tau_i \leq T$.*



Interlacing between edges in T and triangles.

When $D \geq 5$



Outline for section 4

Transversal ratio of simplicial spheres (and polytopes, balls...)

- For a hypergraph $H = (V, E)$, $T \subseteq V$ is a transversal if for every $h \in E$ $h \cap V \neq \emptyset$.
- The minimum size of a transversal is $\tau(H)$. The transversal ratio, $\rho(H)$, is the ratio $\tau(H)/|V|$.

Question

What is the supremum of transveral ratios for d-dimensional polytopes (considering facets) for $d \geq 4$?

- $1/2$ for $d = 3$ by Briggs, Dobbins and Lee.
- $1/2$ by cyclic polytopes for even d.
- $2/5$ for odd $d \geq 5$ (polytopes) by Novik and Zheng.
- For 3-spheres $\rho \geq 4/7$, for 4-spheres $\rho \geq 1/2$, for 5-spheres $\rho \geq 6/11$ by Novik and Zheng.

Extension after embedding on the moment curve?

Strategy: First embed a hypergraph with large transversal ratio, and then extend it to a triangulation of a polytope, or to a sphere (for example using the Hales-Jewett theorem).

Theorem (L., Dobbins; 2026+)

For $d \geq 5$ and $0 \leq r < 1$, there is a simplicial d-polytope P with the transversal ratio at least r.

However, the $d = 4$ case (for polytopes, or for 3-spheres) is open (Best: $\rho \geq 4/7$ by Novik and Zheng).

Question

Is it possible to embed a d-dimensional simplicial complex with large transversal ratio onto γ_d?

Algorithmic aspects

`EMBEDDABILITY ON THE MOMENT CURVE:`
`INPUT: A d-dimensional simplicial complex K.`
`OUTPUT: YES if K is embeddable on` γ_d `(all vertices on` γ_d `without overlapping simplices); NO otherwise.`

Question

What is the computational complexity of EMBEDDABILITY ON THE MOMENT CURVE?

Damásdi, Keszegh, Pálvölgyi and Singh proved that detecting a similar property ($(AB)^k$-freeness) for hypergraphs is NP-complete, but this contrasts with a particular case of the above question when $d = 2$ (outerplanarity).

For higher dimensions $D \geq 5$

Question

Is $\mathrm{HST}(d+4, d)$ *a lattice for every* $d \geq 4$*?*

$\mathrm{HST}(d+3, d)$ is a lattice, but for $d = 4, 5$, $\mathrm{HST}(d+5, d)$ is not a lattice.

Question

Find the minimum number $m(d, n)$ *of new vertices that suffices for extension (to a triangulation) of a geometric complex embedded on* γ_d*.*

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Thank you for your attention!