Monotone paths on polytopes: Positive and negative results

Martina Juhnke

joint work with Germain Poullot

FPSAC 2026, University of Washington, Seattle

July 16, 2026

UNIVERSITÄT OSNABRÜCK

Outline

- (Coherent) monotone paths on polytopes and a question
- Positive answers
- Negative answers
- An almost positive answer

Monotone paths on polytopes

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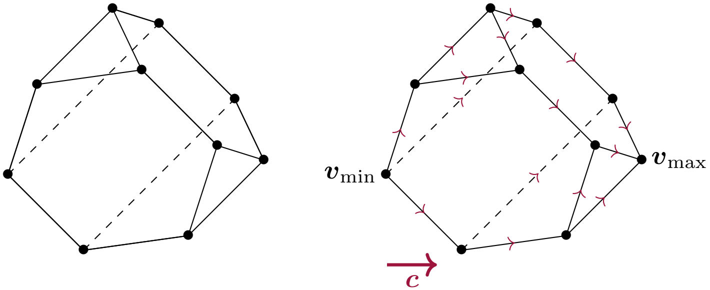

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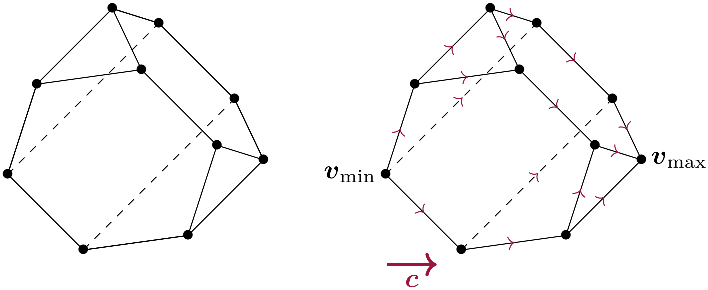

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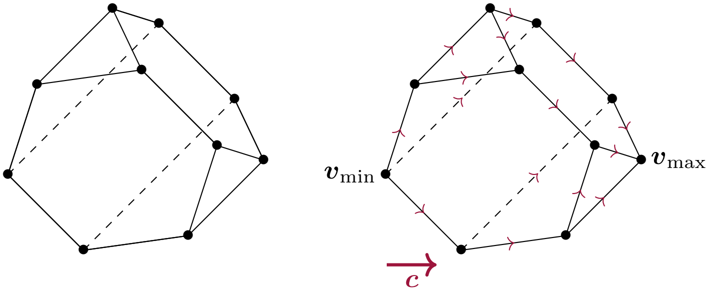

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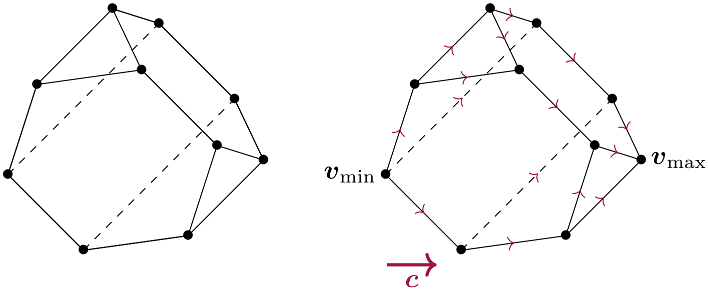

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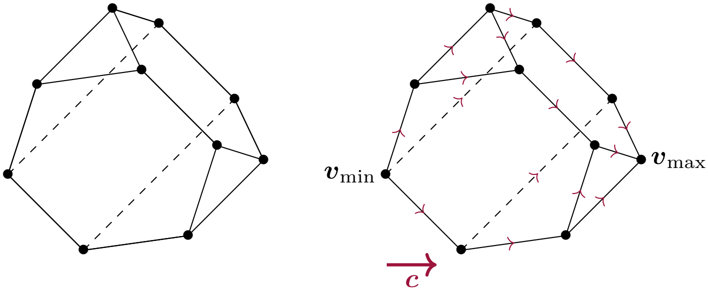

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When is a path coherent?

Given a polytope $P \subseteq \mathbb{R}^d$, $c \in \mathbb{R}^d$, we want to maximize $c^\intercal x$ for $x \in P$.

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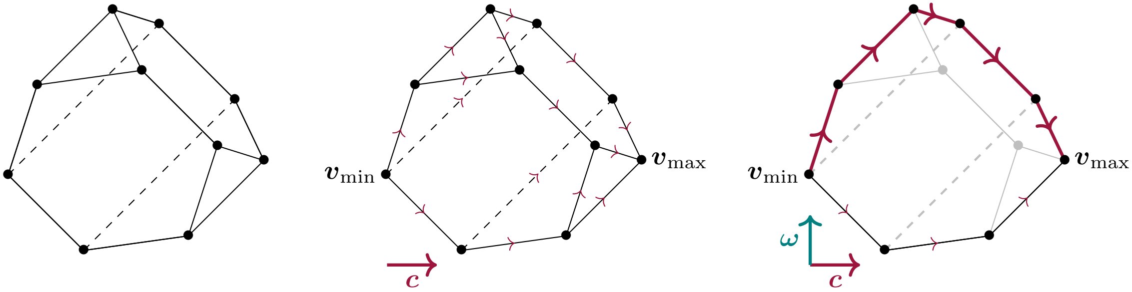

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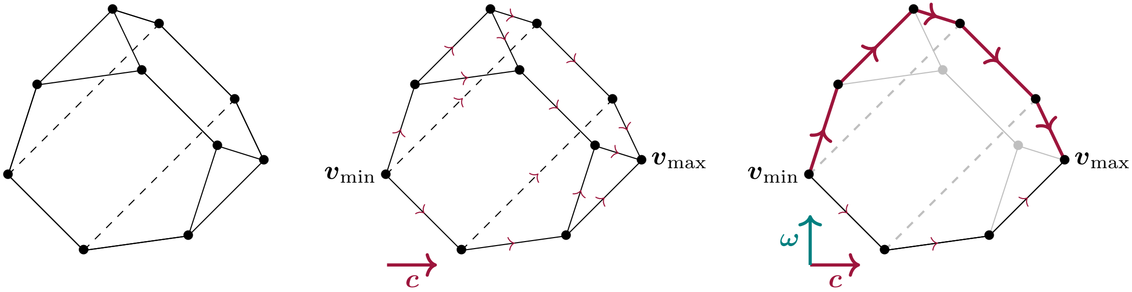

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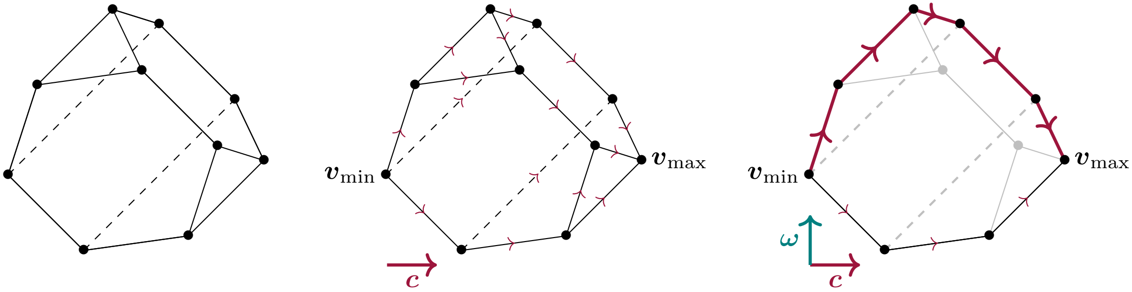

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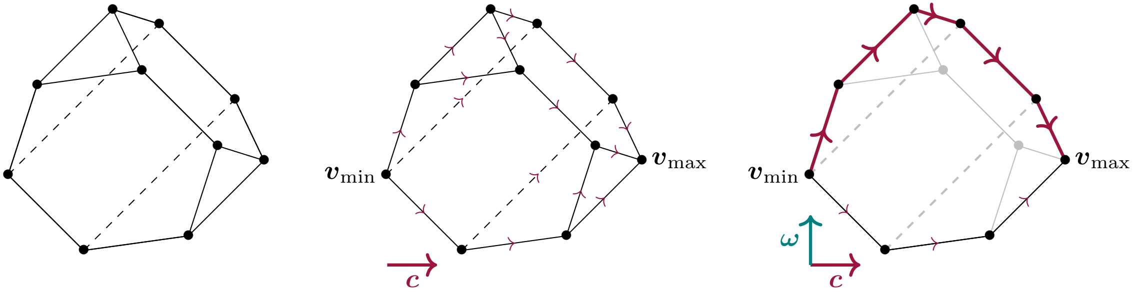

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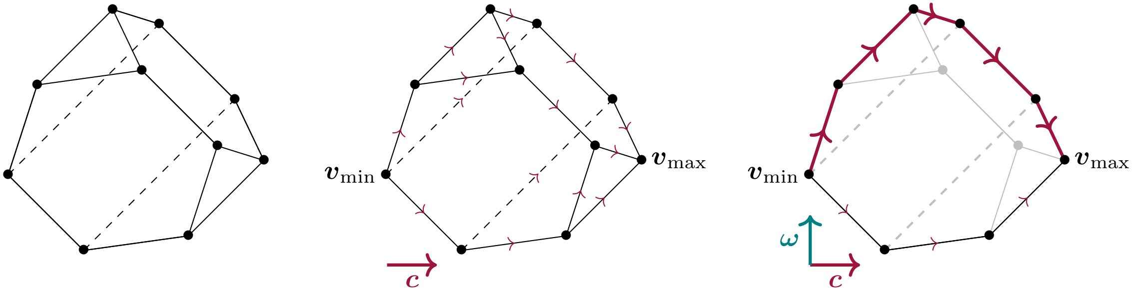

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A c-monotone path is coherent if it arises this way (for some ω).

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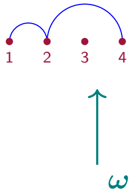

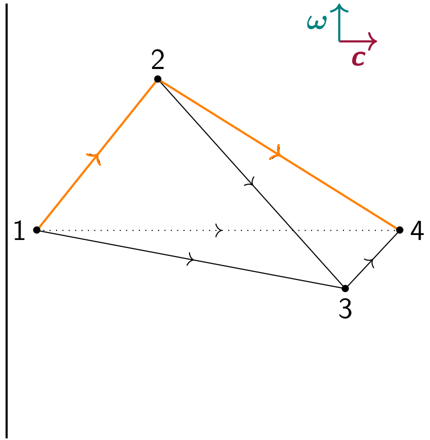

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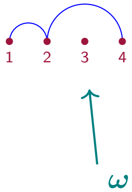

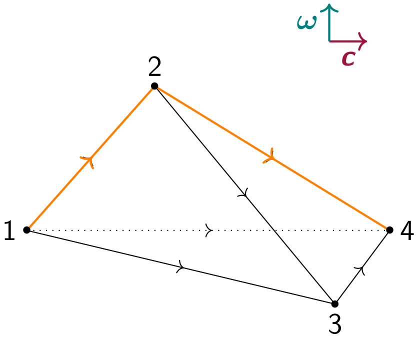

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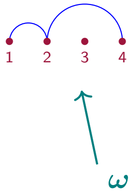

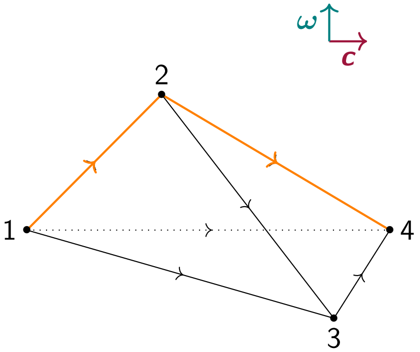

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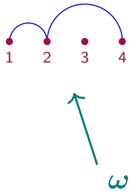

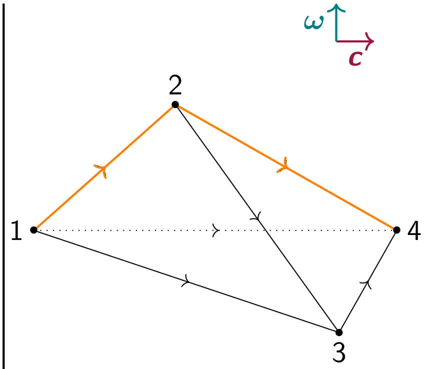

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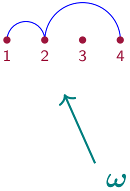

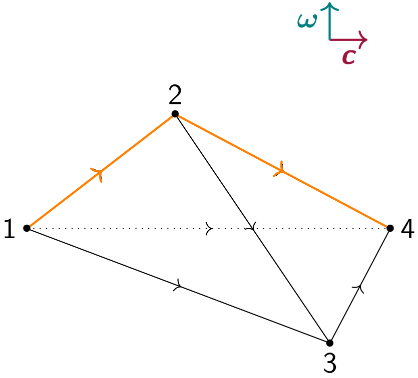

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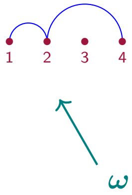

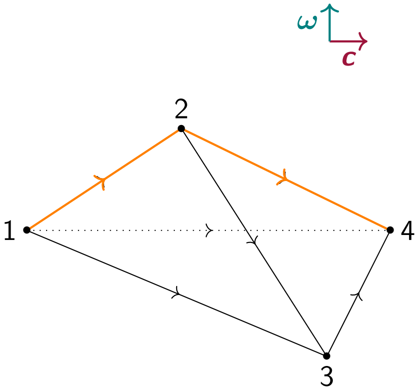

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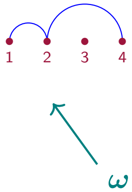

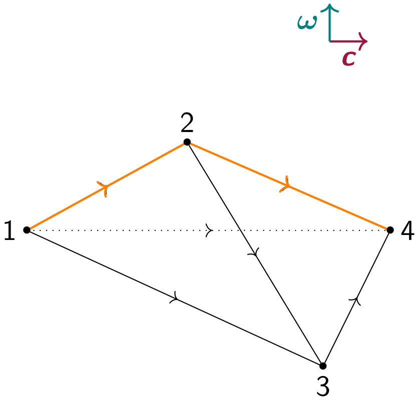

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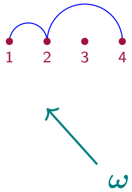

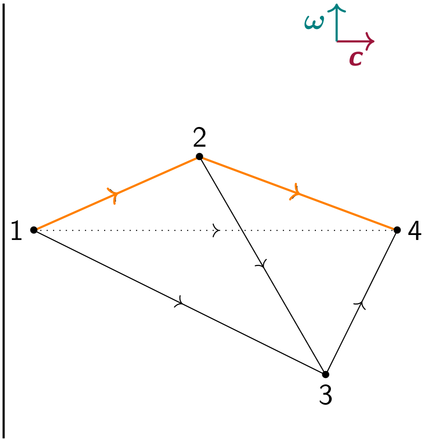

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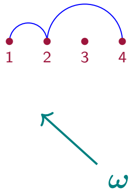

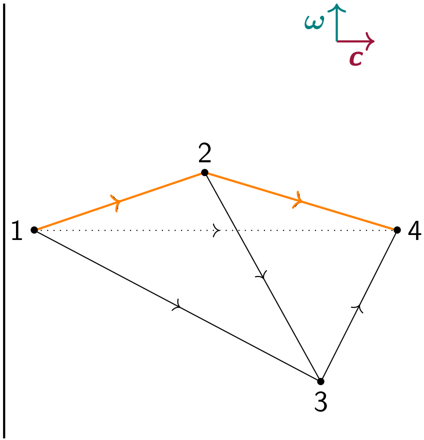

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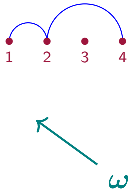

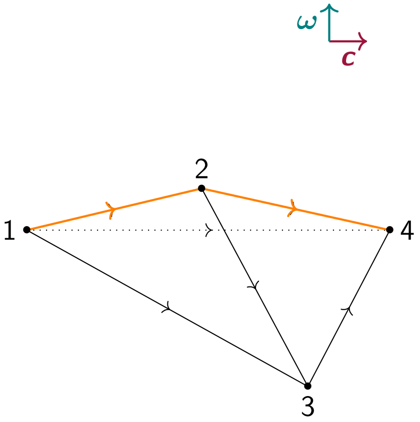

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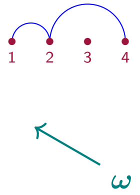

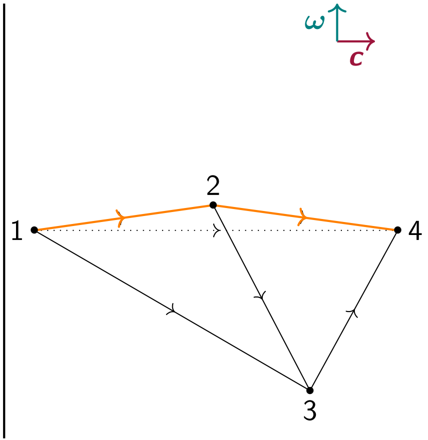

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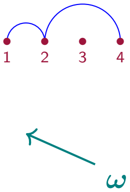

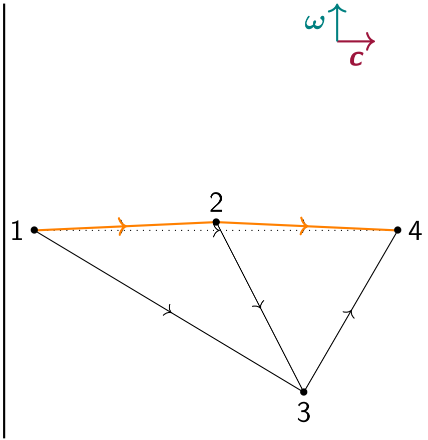

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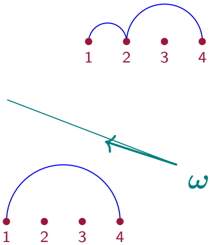

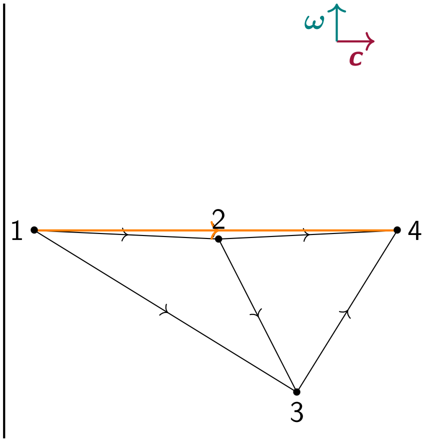

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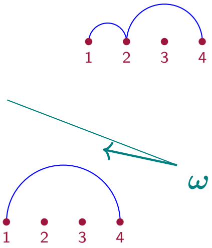

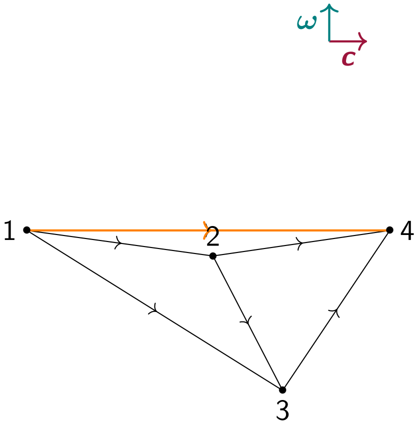

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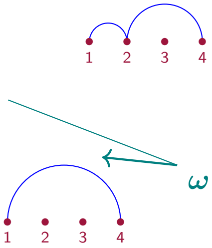

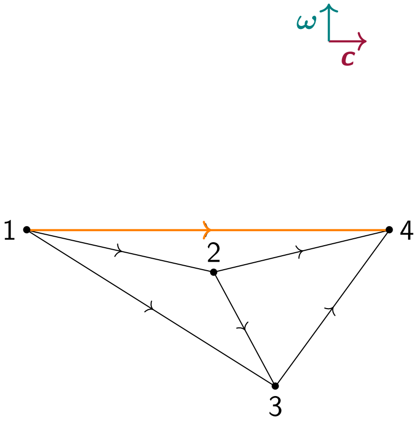

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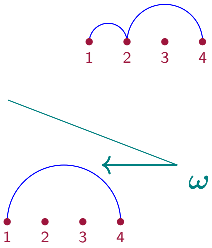

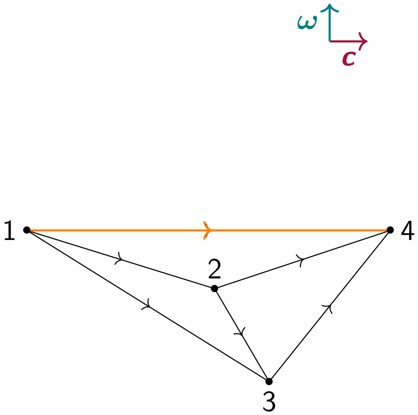

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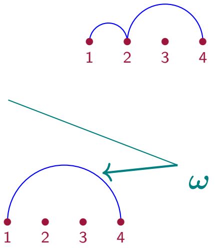

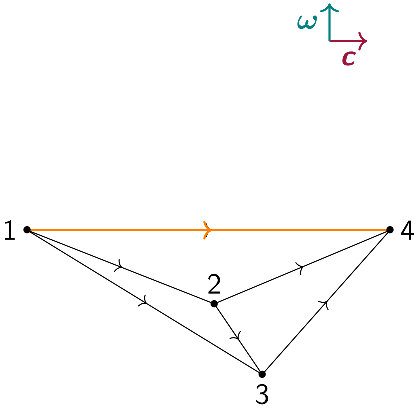

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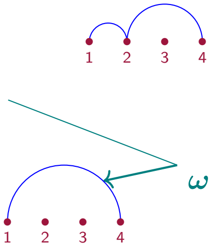

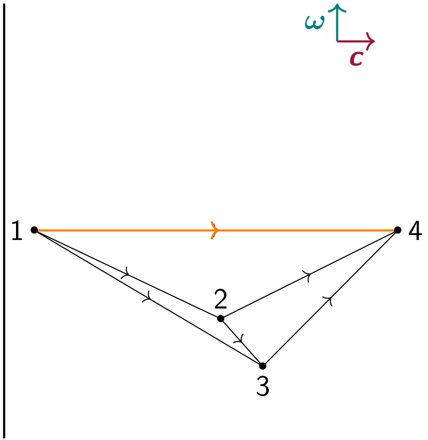

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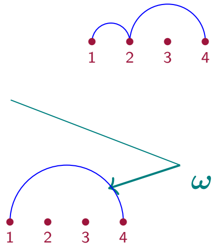

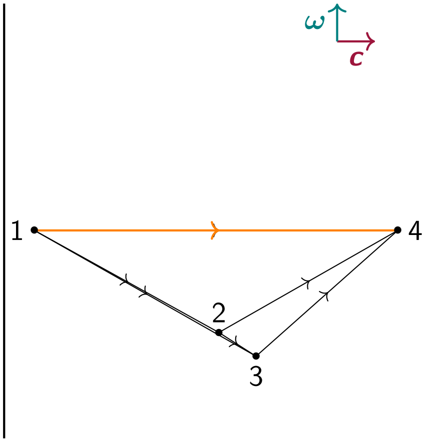

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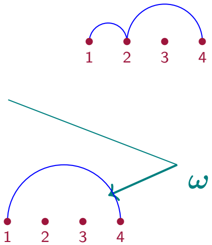

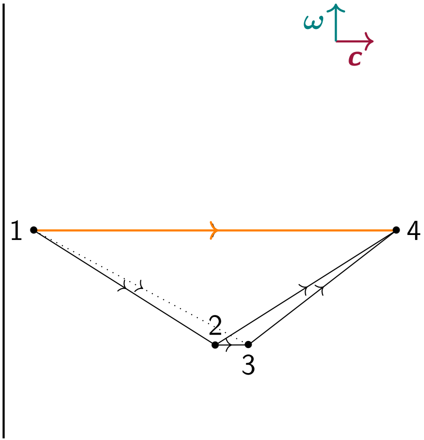

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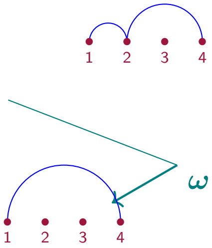

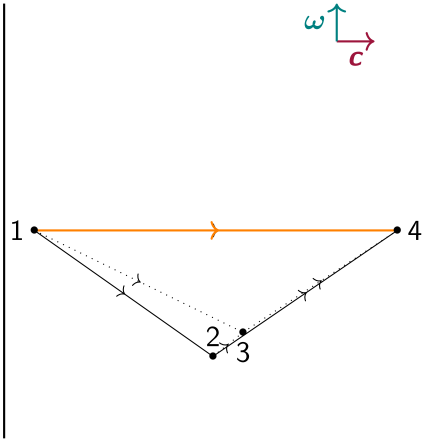

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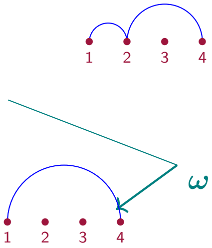

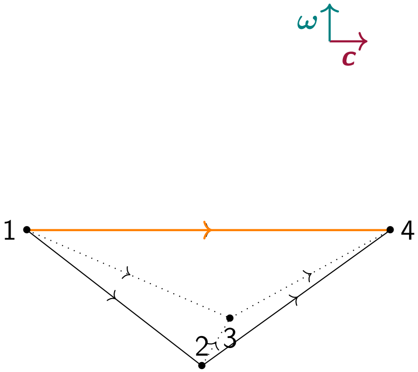

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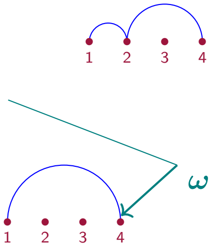

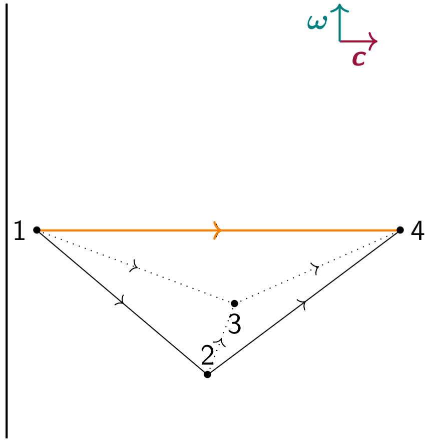

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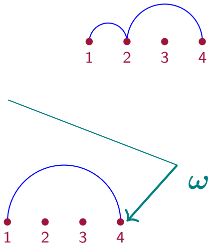

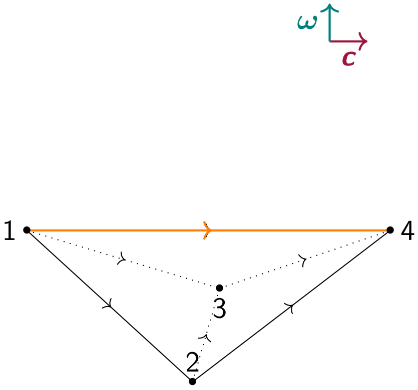

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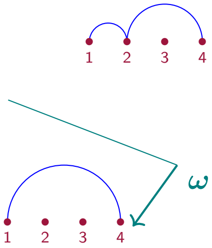

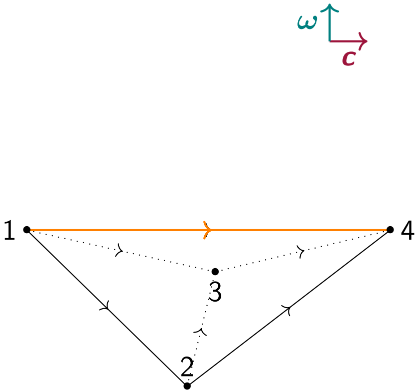

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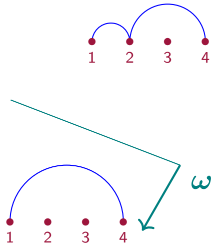

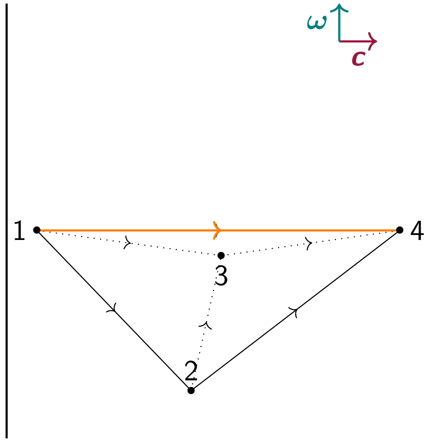

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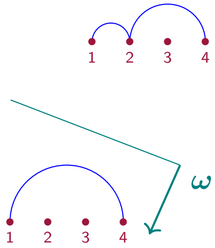

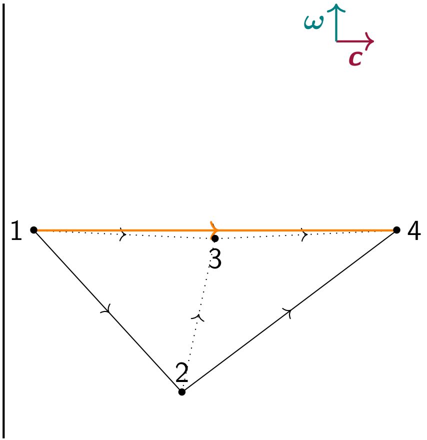

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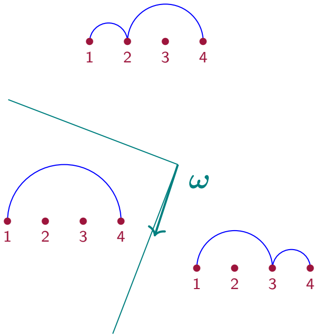

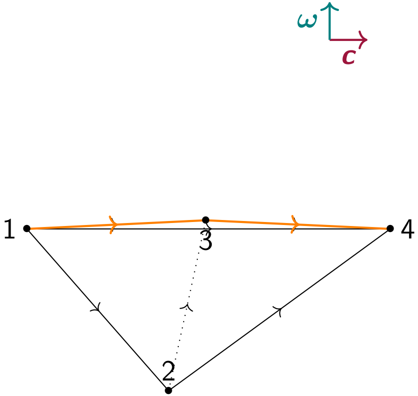

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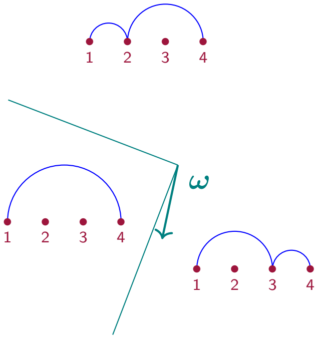

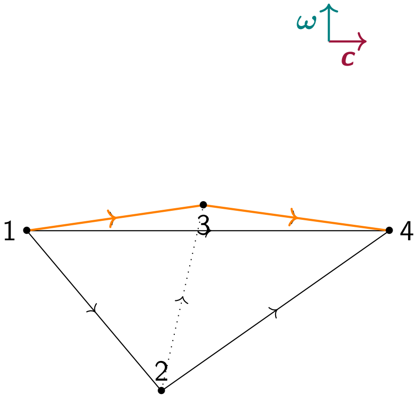

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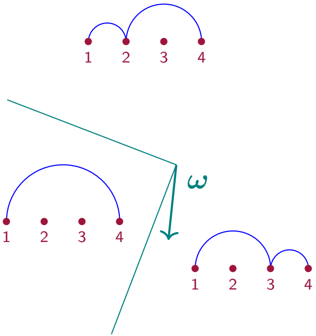

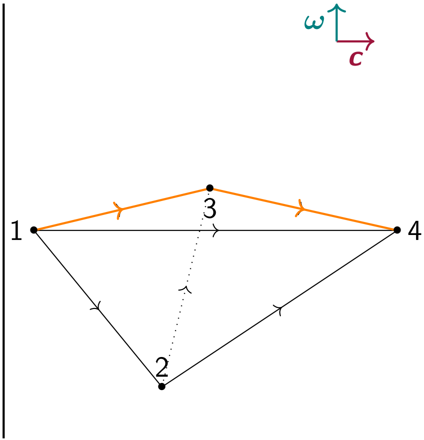

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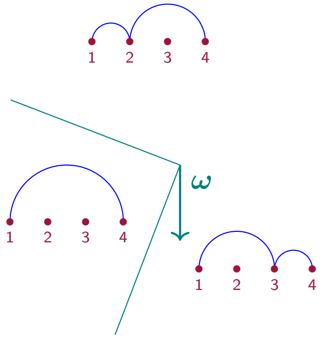

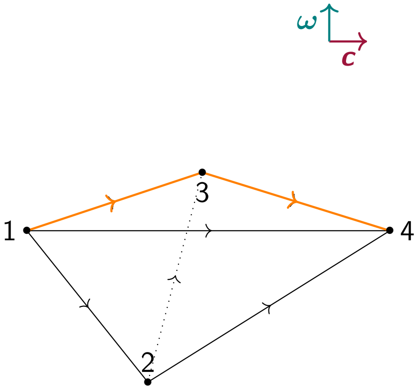

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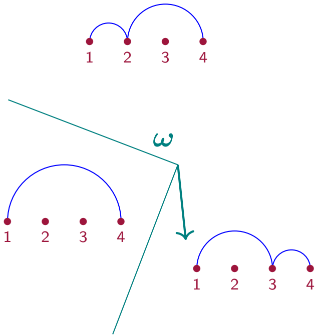

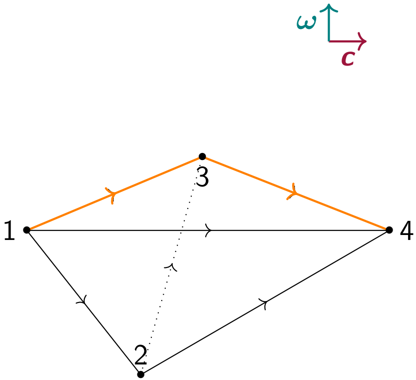

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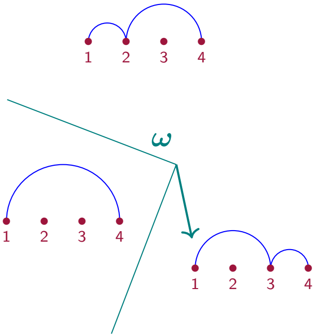

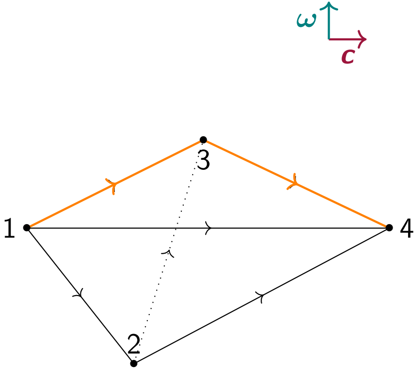

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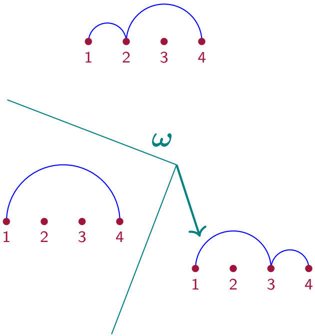

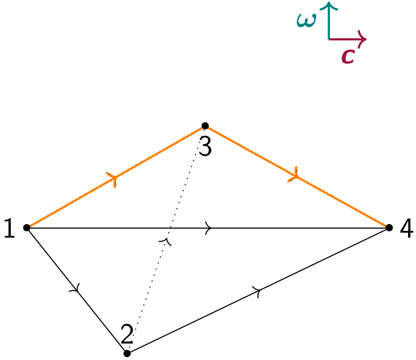

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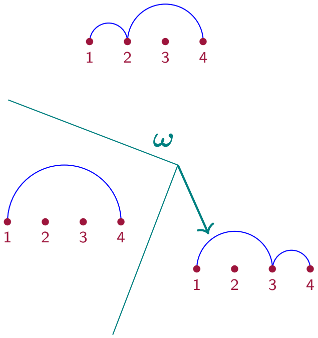

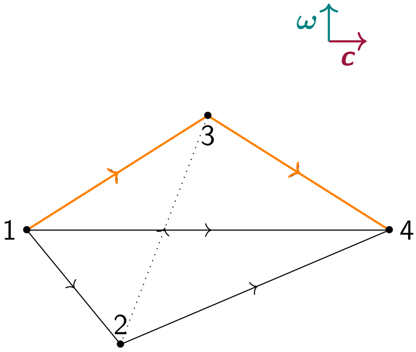

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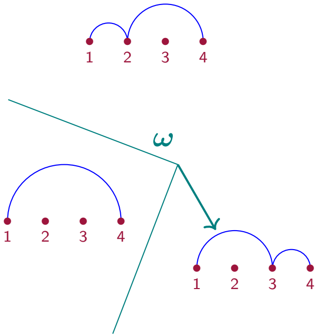

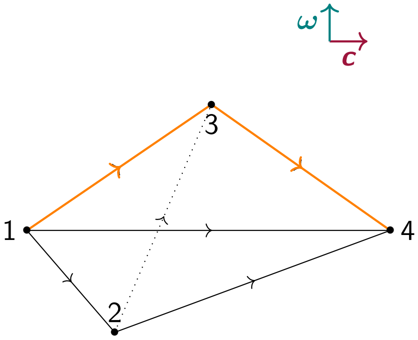

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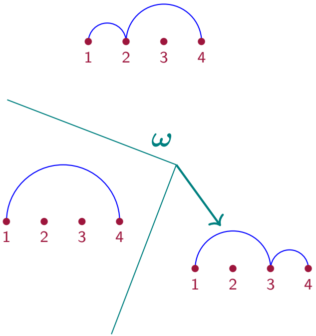

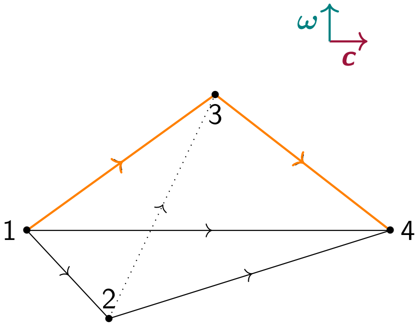

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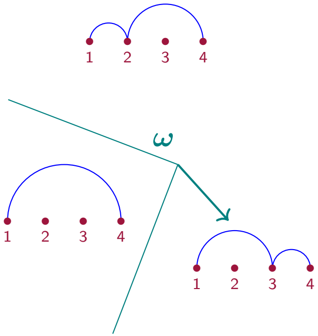

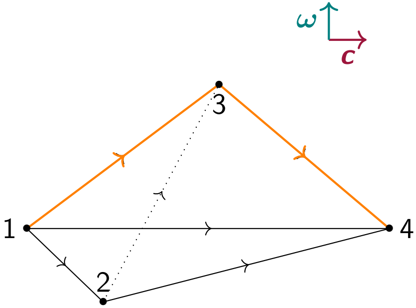

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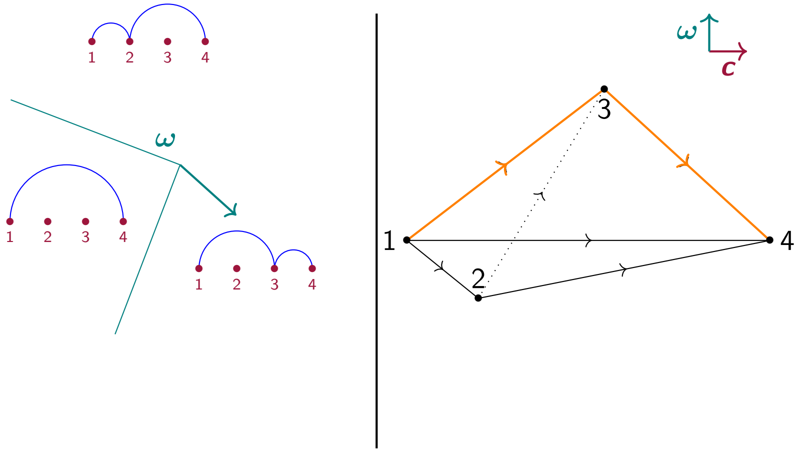

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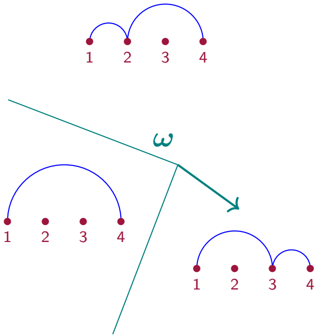

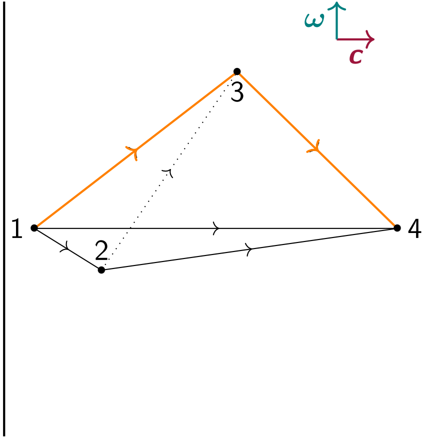

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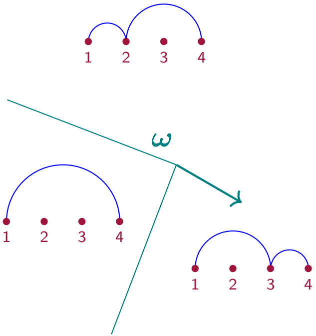

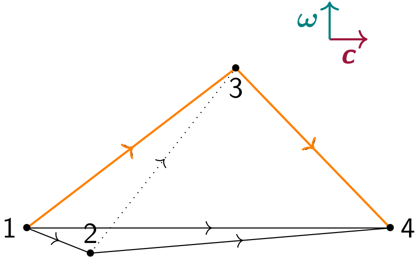

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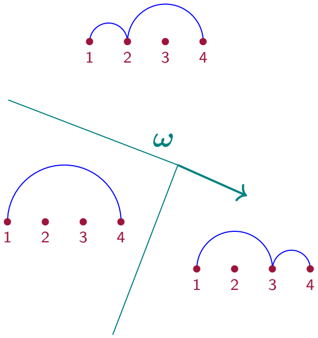

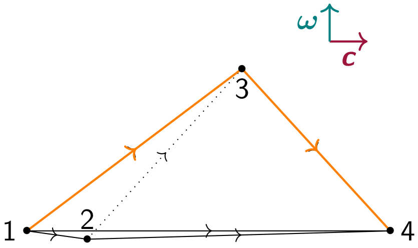

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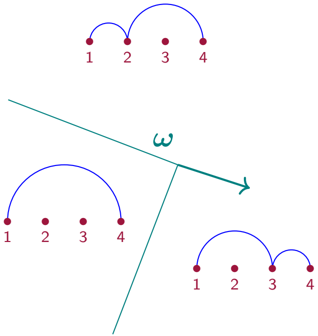

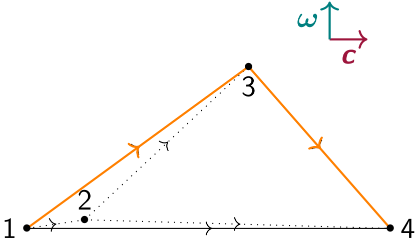

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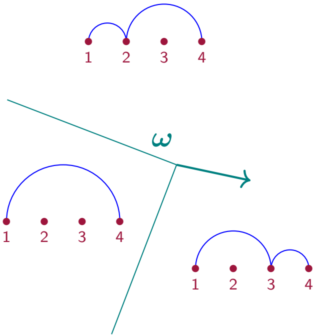

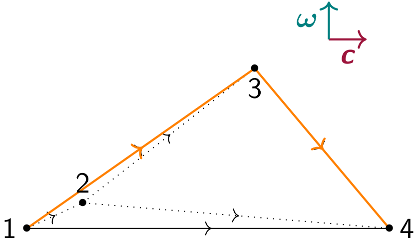

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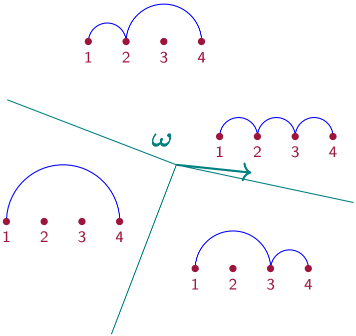

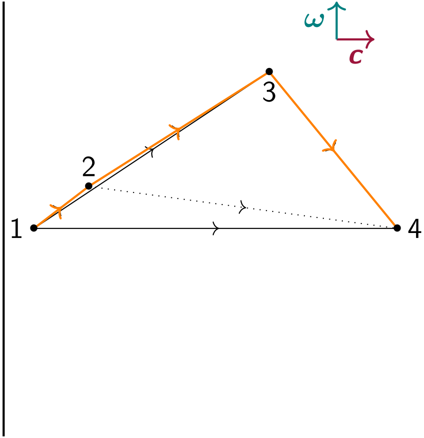

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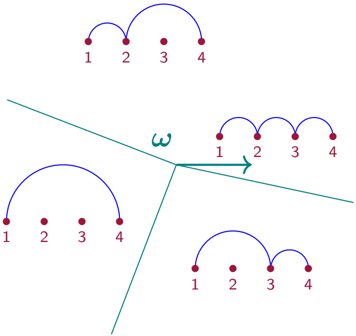

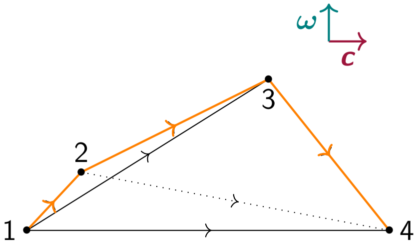

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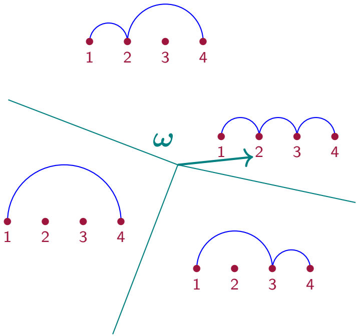

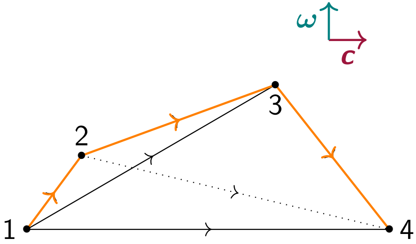

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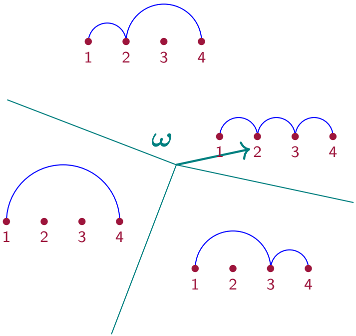

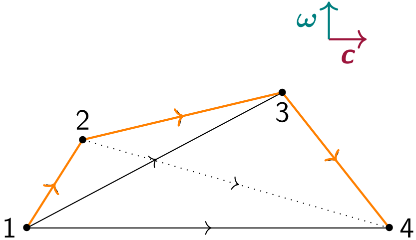

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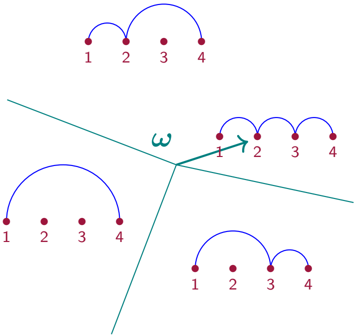

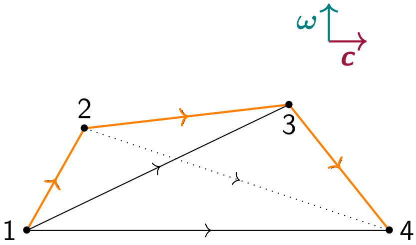

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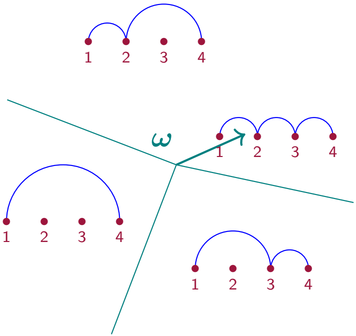

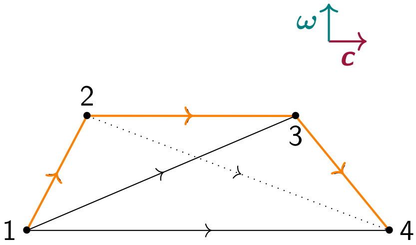

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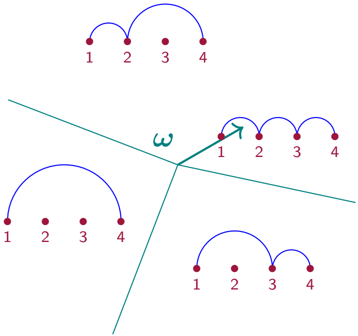

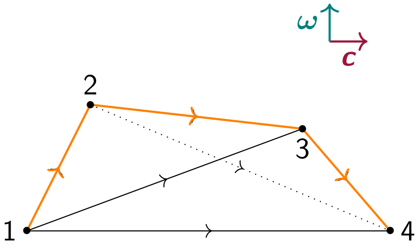

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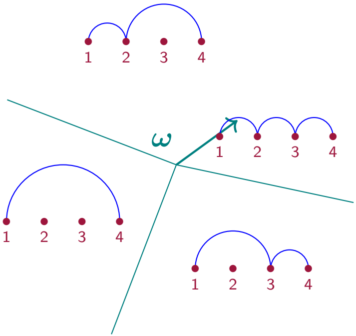

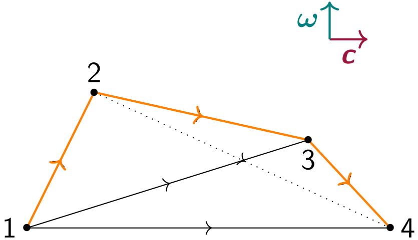

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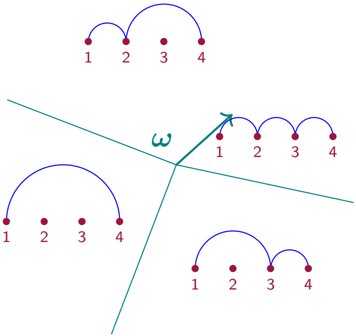

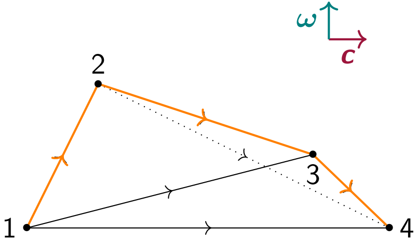

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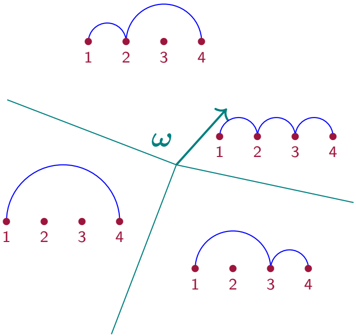

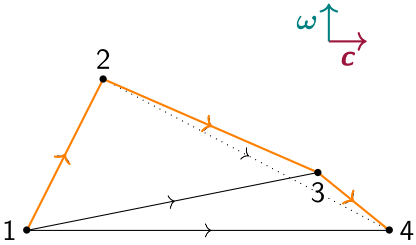

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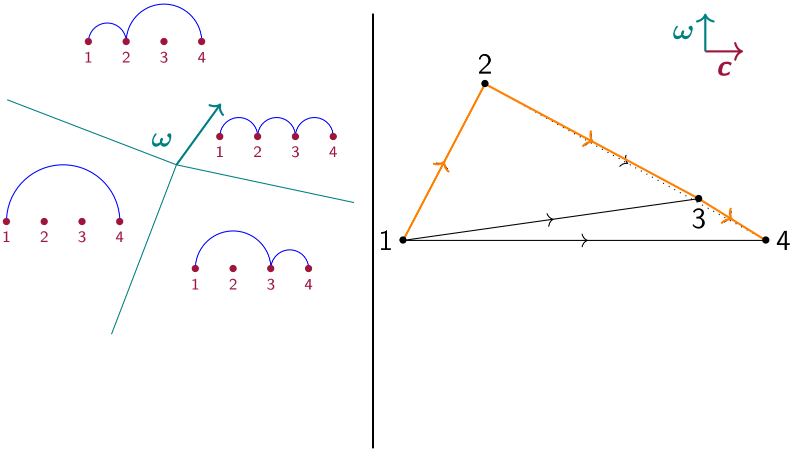

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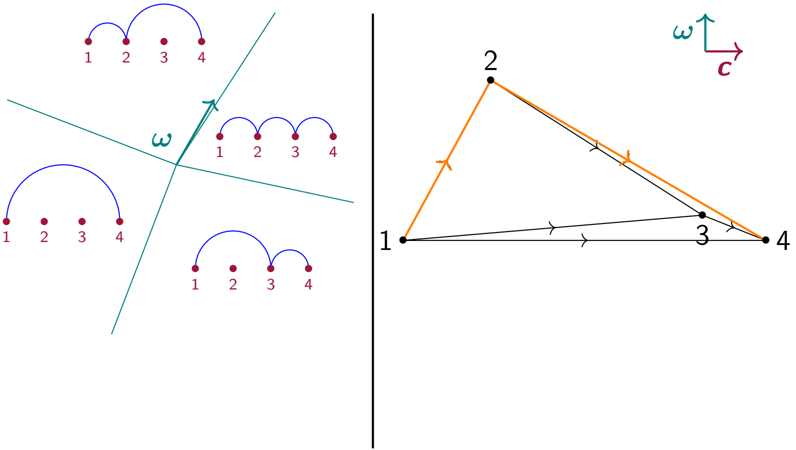

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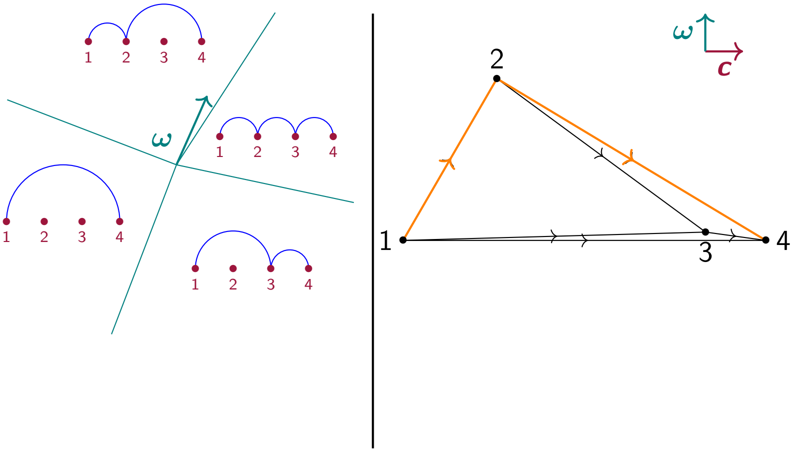

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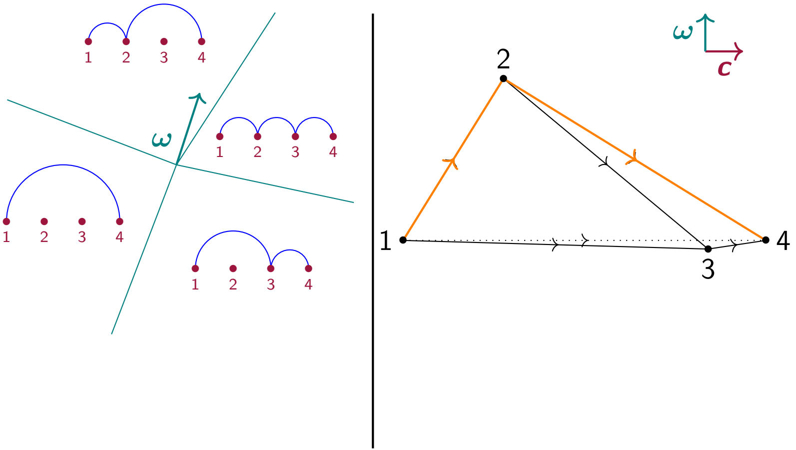

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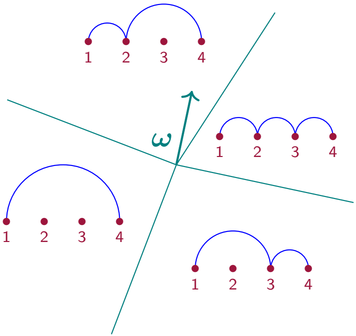

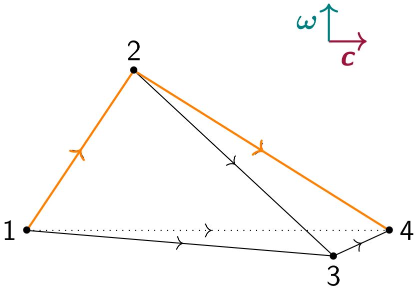

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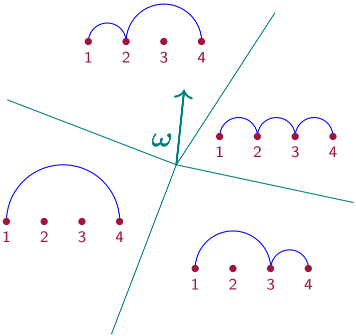

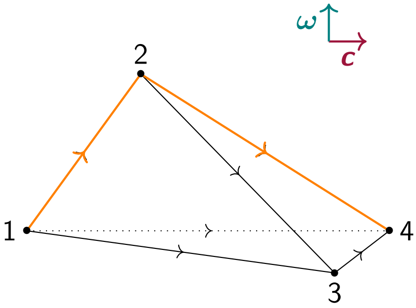

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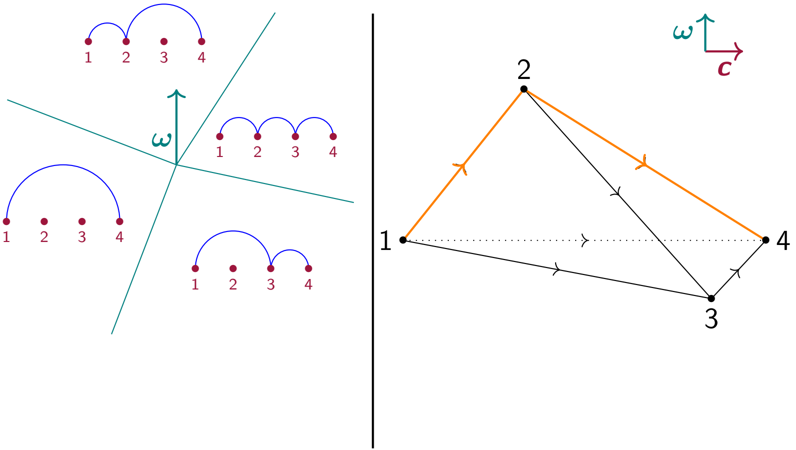

Our starting point

Question (De Loera; 2022)

Let $\mathrm{P} \subseteq \mathbb{R}^d$ be a polytope and $c \in \mathbb{R}^d$.

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Are the sequences $(N_\ell)_{\ell \geq 1}$ and $(N_\ell^{\mathrm{coh}})_{\ell \geq 1}$ unimodal?

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Yes, for simplices and generic c.

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Are the sequences $(N_\ell)_{\ell \geq 1}$ and $(N_\ell^{\mathsf{coh}})_{\ell \geq 1}$ unimodal?

Yes, for simplices and generic c.

There are more positive cases.

Positive answers

The standard cube

Let $P = [0,1]^d$ and $c = (1,\dots,1)$.

- $v_{\text{min}} = (0,\dots,0)$ and $v_{\text{max}} = (1,\dots,1)$

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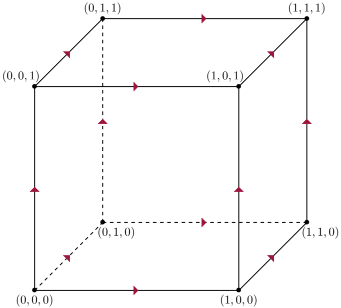
$\Rightarrow$ All c-monotone paths have length d and $N_d = d!$.

- All c-monotone paths are coherent (Billera, Sturmfels; 1992).



More examples

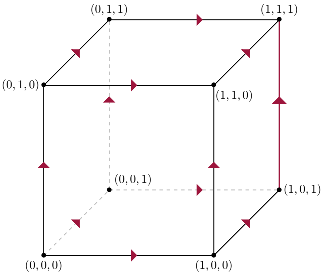
P	$\boldsymbol{c}$	N_ℓ	N_ℓ^{coh}
$\Diamond_d$	generic	$2\sum_{k=0}^{d-2}\binom{2k}{\ell-2}$	$\binom{d-1}{\ell-1}2^{\ell-1}$
$\mathrm{Cyc}_d(n),\ d \geq 4$	$(1,0,\dots,0)$	$\binom{n-2}{\ell-1}$	complicated
$\Delta_d(S),\ \lvert S\rvert = r \geq 2$	$(1,1,\dots,1)$	$\binom{d}{s_1, s_2-s_1,\dots,s_r-s_{r-1}}$ iff $\ell = r$	$\binom{d}{s_1, s_2-s_1,\dots,s_r-s_{r-1}}$ iff $\ell = r$

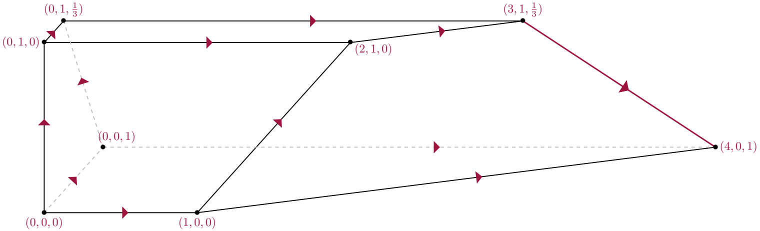
- $\Diamond_d = \mathrm{conv}(\pm \mathbf{e}_i \ :\ 1 \leq i \leq d)$
- $\mathrm{Cyc}_d(n) = \mathrm{conv}\big((t_i, t_i^2, \dots, t_i^d)\ :\ 1 \leq i \leq n\big)$ with $t_1 < t_2 < \cdots < t_n$
- S-hypersimplex $\Delta_d(S) = \mathrm{conv}(\boldsymbol{x} \in \{0,1\}^d\ :\ \sum_i x_i \in S)$, where $S = \{s_1, \dots, s_r\} \subseteq [d]$

Negative answers

Lopsided cubes

$c = (1, 1, 1)$





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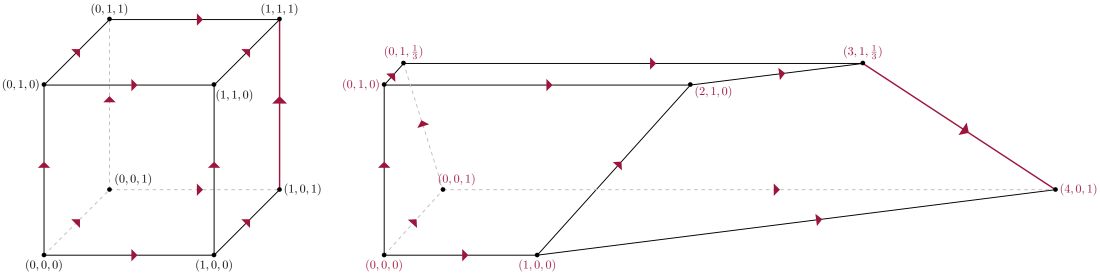

ℓ	2	3	4
$N_\ell = N_\ell^{\mathrm{coh}}$	2	0	4

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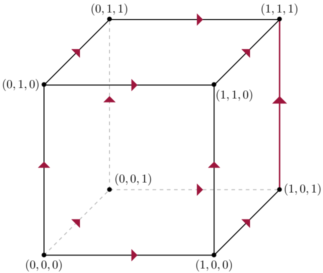
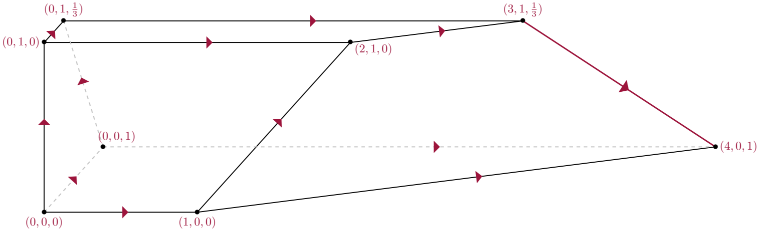


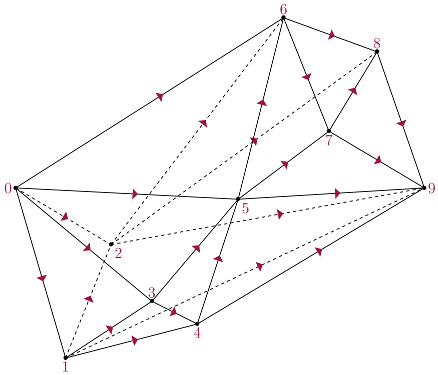


- The example generalizes to arbitrary d with $c = (1, \ldots, 1)$:

ℓ	$d-1$	d	$d+1$
$N_\ell = N_\ell^{\mathrm{coh}}$	$(d-1)!$	0	$\frac{1}{6}(d+1)!$

A simplicial counterexample for monotone paths

$c = (1, 0, 0)$



ℓ	2	3	4	5	6	7	8
N_ℓ	3	8	12	11	12	6	1

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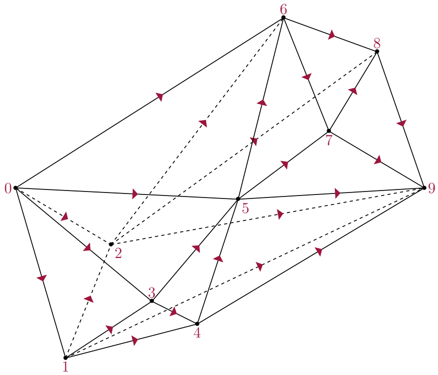

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We do have a simplicial counterexample such that $(N_\ell^{\mathrm{coh}})_\ell$ is not log-concave but unimodal.

Loday's associahedron

$$\mathrm{Asso}_n = \left\{ \boldsymbol{x} \in \mathbb{R}^n \ : \sum_{i=1}^{n} x_i = \binom{n+1}{2}, \ \sum_{i \in I} x_i \geq \binom{|I|+1}{2} \text{ for } \emptyset \neq I = [a,b] \subsetneq [n] \right\}$$

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$n = 6$	ℓ	5	6	7	8	9	10	11	12	13	14	15
(Nelson; 2017), A282698	N_ℓ	1	20	112	232	382	348	456	390	420	334	286
Our computation	N_ℓ^{coh}	1	20	105	206	332	274	332	270	206	122	142

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Problem: Find a characterization of N_ℓ^{coh}.

- We also have a counterexample in dimension 5 that is a $0/1$-polytope.

An almost positive answer

The probabilistic model

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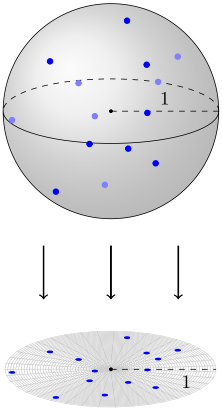
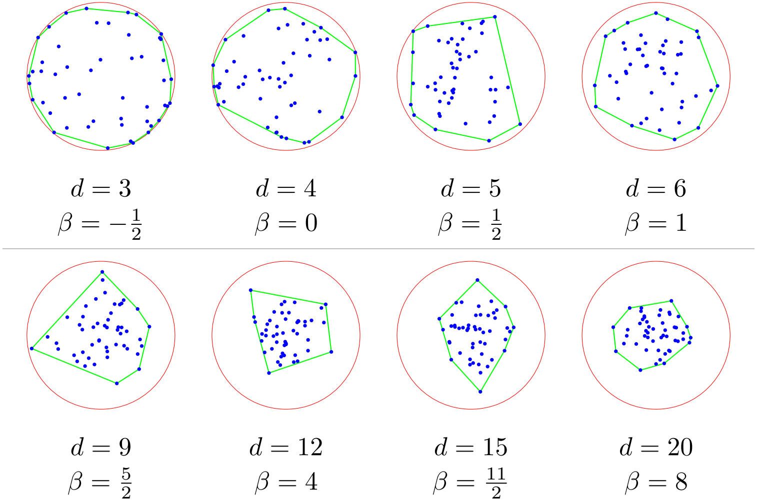
In the following: Focus on coherent paths! $\rightsquigarrow$ Project to a 2-dimensional plane.

Examples of projections





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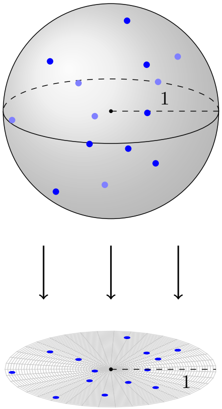
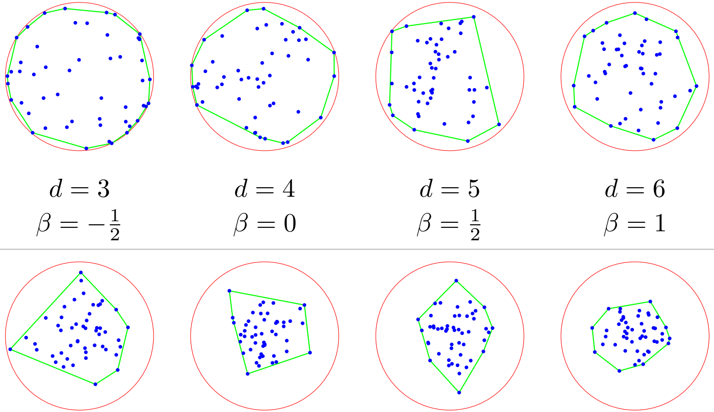




$d = 3$	$d = 4$	$d = 5$	$d = 6$
$\beta = -\frac{1}{2}$	$\beta = 0$	$\beta = \frac{1}{2}$	$\beta = 1$
$d = 9$	$d = 12$	$d = 15$	$d = 20$
$\beta = \frac{5}{2}$	$\beta = 4$	$\beta = \frac{11}{2}$	$\beta = 8$

- Need to understand the number of vertices/edges of the polygons.

β-distributions

X uniformly distributed point on $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$

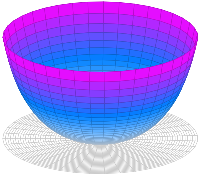
$E \subseteq \mathbb{R}^d$ 2-dimensional plane, $\pi_E : \mathbb{R}^d \to E$ orthogonal projection, $Z = \pi_E(X)$

Theorem (Kabluchko, Temesvári, Thäle; 2019)

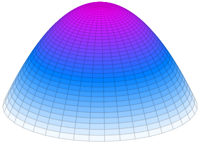
Z is distributed with probability density

$$f_{2,\beta_d}(\boldsymbol{x}) = C_d \, \left(1 - \|\boldsymbol{x}\|^2\right)^{\beta_d} \qquad \textit{for } \boldsymbol{x} \in \mathbb{B}^2,$$

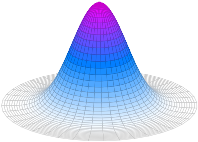
where $\beta_d = \frac{d}{2} - 2$ and C_d is a constant.



$\beta_3 = -\frac{1}{2}$



$\beta_6 = 1$



$\beta_{24} = 10$

Set-up and goal

Let $X_1, \ldots, X_n$ be uniformly independently distributed points on $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$.

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L_n equals the number of upper edges in Q_n. $\rightsquigarrow$

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Instead of counting paths, we consider the length L_n of a random coherent path on P_n.

L_n equals the number of upper edges in Q_n. $\rightsquigarrow$ Enough to understand $f_1(\mathsf{Q}_n)$.

Set-up and goal

Let $X_1, \ldots, X_n$ be uniformly independently distributed points on $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$.

$$\mathsf{P}_n = \mathrm{conv}(X_1, \ldots, X_n)$$

$E \subseteq \mathbb{R}^d$ 2-dimensional plane, $\pi_E : \mathbb{R}^d \to E$ orthogonal projection, $Z_i = \pi_E(X_i)$

$$\mathsf{Q}_n = \mathrm{conv}(Z_1, \ldots, Z_n)$$

Instead of counting paths, we consider the length L_n of a random coherent path on P_n.

L_n equals the number of upper edges in Q_n. $\rightsquigarrow$ Enough to understand $f_1(\mathsf{Q}_n)$.

Goal
Show that $f_1(\mathsf{Q}_n)$ satisfies a central limit theorem.

A central limit theorem

Theorem (J., Poullot; 2025)

Let $d \geq 4$. $Z_1, \dots, Z_n$ *independent* β_d*-distributed points.*
Let

$$\mathrm{Q}_n = \operatorname{conv}(Z_1, \dots, Z_n)$$

and $f_1(\mathrm{Q}_n)$ *be the number of edges of* Q_n. *Then, for* $U \sim \mathcal{N}(0,1)$,

$$d_K\left(\frac{f_1(\mathrm{Q}_n) - \mathbb{E} f_1(\mathrm{Q}_n)}{\sqrt{\operatorname{Var} f_1(\mathrm{Q}_n)}}, U\right) \leq c(\log(n))^{\frac{7}{2} - \frac{1}{2(d-1)}} \left(\frac{1}{n}\right)^{\frac{1}{2(d-1)}} \to 0 \quad \textit{as} \quad n \to \infty.$$

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- $d_K(X, Y) = \sup_{x \in \mathbb{R}} |\mathbb{P}(X \leq x) - \mathbb{P}(Y \leq x)|$

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- $d_K(X, Y) = \sup_{x \in \mathbb{R}} |\mathbb{P}(X \leq x) - \mathbb{P}(Y \leq x)|$
- $\mathbb{E}f_1(\mathrm{Q}_n) \sim n^{\frac{1}{d-1}}$ (Kabluchko, Thäle, Zaporozhets; 2020)

The variance

Theorem (J., Poullot; 2025)

$$c_0 n^{\frac{1}{d-1}-a} \leq \mathrm{Var} f_1(Q_n) \leq c_1 (\log(n))^{3-\frac{1}{d-1}} n^{\frac{1}{d-1}} \quad \textit{as} \quad n \to \infty$$

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Proof ingredients:

- lower bound: floating bodies, some kind of Sylvester's 4-point problem
- upper bound: Efron-Stein jackknife inequality

How to upper bound the variance

Goal: $\mathrm{Var} f_1(\mathsf{Q}_n) \leq c_1(\log(n))^{3-\frac{1}{d-1}} n^{\frac{1}{d-1}}$

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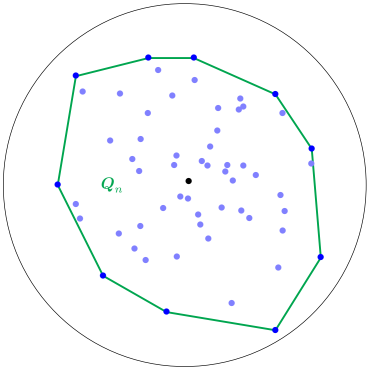
$$\mathrm{Var} f_1(Q_n) \leq (n+1)\,\mathbb{E}\big(\underbrace{f_1(Q_{n+1}) - f_1(Q_n)}_{=Df_1(Q_{n+1})}\big)^2$$



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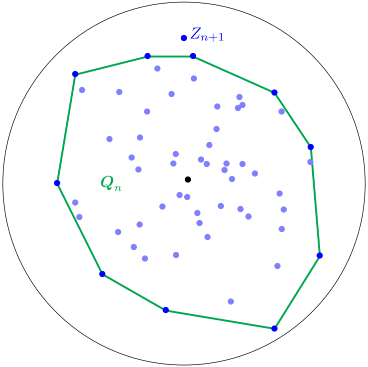

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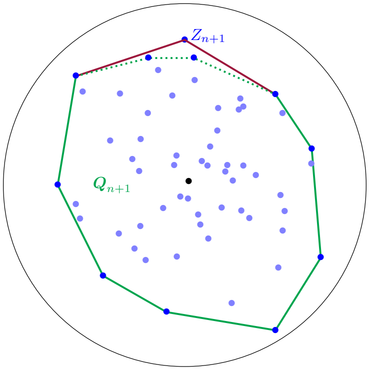


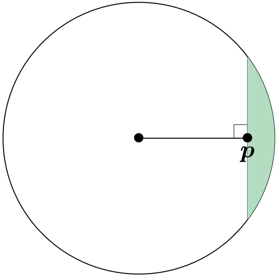
The upper bound: floating bodies

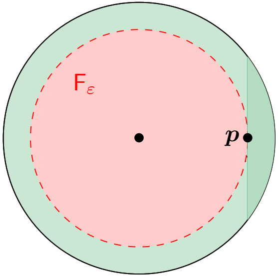
Let $\mathbf{p} \in \mathbb{B}^2$.

$$C_{\mathbf{p}} = \{x \in \mathbb{B}^2 \ : \ \langle x - \mathbf{p}, \mathbf{p} \rangle \geq 0\}$$

is called ε-cap if $\mu(C_{\mathbf{p}}) = \varepsilon$.

The ε-floating body is the complement of the union of all ε-caps.





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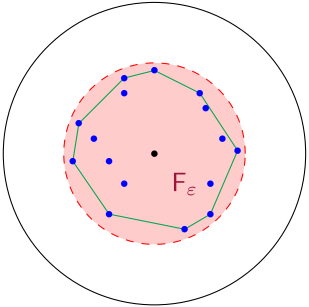
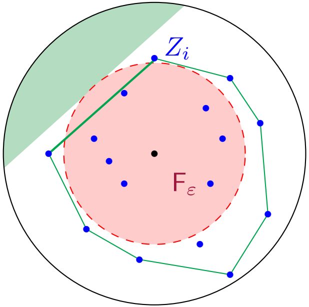
Step 1: Show that the ϵ-floating body is contained in Q_n with probability $\geq 1 - n^{-s}$ (for any $s > 0$ and $\varepsilon = (\frac{1}{d-1} + s)\frac{\log(n)}{n}$).





The upper bound: Conditioning

Goal: $\mathbb{E}Df_1(\mathrm{Q}_{n+1})^2 \le c_1(\log(n))^{3-\frac{1}{d-1}}n^{-1+\frac{1}{d-1}}$.

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Step 2:

$$\begin{aligned}\mathbb{E} Df_1(\mathsf{Q}_{n+1})^2 =& \mathbb{E}\big(Df_1(\mathsf{Q}_{n+1})^2 \,\big|\, \mathsf{F}_\varepsilon \not\subseteq \mathsf{Q}_{n+1}\big)\mathbb{P}(\mathsf{F}_\varepsilon \not\subseteq \mathsf{Q}_{n+1}) \\ &+ \mathbb{E}\big(Df_1(\mathsf{Q}_{n+1})^2 \,\big|\, \mathsf{F}_\varepsilon \subseteq \mathsf{Q}_{n+1}\big)\mathbb{P}(\mathsf{F}_\varepsilon \subseteq \mathsf{Q}_{n+1})\end{aligned}$$

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- For $\mathsf{F}_\varepsilon \not\subseteq \mathsf{Q}_n$, use the trivial estimate $|Df_1(\mathsf{Q}_{n+1})| \leq n$.

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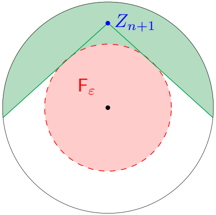
- For $\mathsf{F}_\varepsilon \not\subseteq \mathsf{Q}_n$, use the trivial estimate $|Df_1(\mathsf{Q}_{n+1})| \le n$.
- For $\mathsf{F}_\varepsilon \subseteq \mathsf{Q}_n$, bound $|Df_1(\mathsf{Q}_{n+1})|$ by the number of points in the visibility region of Z_{n+1}.



How to show a CLT: The main tool

Theorem (Shao, Zhang, Zhang; 2025; Lachièze-Rey, Peccati; 2017)

$$d_K\left(\frac{f_1(\mathrm{Q}_n)-\mathbb{E}(f_1(\mathrm{Q}_n))}{\sqrt{\mathrm{Var} f_1(\mathrm{Q}_n)}}, U\right) \le c\,\frac{1}{\mathrm{Var} f_1(\mathrm{Q}_n)}\Big(\sqrt{n}\gamma_1(\mathrm{Q}_n)+n\sqrt{\gamma_2(\mathrm{Q}_n)}+n\sqrt{n}\sqrt{\gamma_3(\mathrm{Q}_n)}\Big),$$

where γ_1, γ_2, γ_3 *are higher moments of first and second order difference operators.*

- Second order difference operators measure the effect of the removal of two points and their interaction.

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where γ_1, γ_2, γ_3 *are higher moments of first and second order difference operators.*

- Second order difference operators measure the effect of the removal of two points and their interaction.
- The theorem is true in a far more general setting.

Open problems and ongoing work

- special classes of polytopes
- coherent paths on Loday's associahedron
- other probabilistic models (random edge polytopes, simple polytopes)
- existence of good directions

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One last remark

I think we should use probabilistic methods more often in order to validate conjectures.

Thank you!

The γ's

$$d_K\left(\frac{f_1(Q_n)-\mathbb{E}(f_1(Q_n))}{\sqrt{\mathrm{Var} f_1(Q_n)}}, U\right) \le c\,\frac{1}{\mathrm{Var} f_1(Q_n)}\Big(\sqrt{n}\gamma_1(Q_n) + n\sqrt{\gamma_2(Q_n)} + n\sqrt{n}\sqrt{\gamma_3(Q_n)}\Big),$$

where

$$\begin{aligned}
D_{ij} f_1(\boldsymbol{X}) &= f_1(\boldsymbol{X}) - f_1(\boldsymbol{X}^i) - f_1(\boldsymbol{X}^j) + f_1(\boldsymbol{X}^{ij}) \\
\gamma_1 &= \mathbb{E}\Big(|Df_1(\boldsymbol{X})|^4\Big), \\
\gamma_2 &= \sup_{(\boldsymbol{Y},\boldsymbol{Z})} \mathbb{E}\Big(\mathbf{1}\big(D_{12} f_1(\boldsymbol{Y}) \neq 0\big)\, D_1 f_1(\boldsymbol{Z})^4\Big), \\
\gamma_3 &= \sup_{(\boldsymbol{Y},\boldsymbol{Y}',\boldsymbol{Z})} \mathbb{E}\Big(\mathbf{1}\big(D_{12} f_1(\boldsymbol{Y}) \neq 0\big)\,\mathbf{1}\big(D_{13} f_1(\boldsymbol{Y}') \neq 0\big)\, D_2 f_1(\boldsymbol{Z})^4\Big),
\end{aligned}$$

where the suprema run over all recombinations $(\boldsymbol{Y}, \boldsymbol{Z})$ resp. $(\boldsymbol{Y}, \boldsymbol{Y}', \boldsymbol{Z})$ of $\{\boldsymbol{X}, \boldsymbol{X}', \widetilde{\boldsymbol{X}}\}$.