Kostant's Partition Function: Support, Structure, and Surprises

Pamela E. Harris

UNIVERSITY OF WISCONSIN MILWAUKEE

FPSAC Seattle 2026

Formal Power Series and Algebraic Combinatorics

July 16, 2026

Goal & Outline

What is a vector partition function?

- Introduce Kostant's partition function and its q-analog.
- Connection to the representation theory of Lie algebras.
- Connection to polyhedral geometry.

Open problems will be sprinkled throughout.

Vector Partition Functions

Let A be an $m \times d$ integral matrix.

Goal: Compute the vector partition function

$$\phi_A(\mathbf{b}) = \#\{\mathbf{x} \in \mathbb{N}^d \; : \; A\mathbf{x} = \mathbf{b}\}$$

defined for $\mathbf{b}$ in the nonnegative linear span of the columns of A.

Applications in

- Number Theory (partitions)
- Representation theory (weight multiplicities)
- Commutative Algebra (Hilbert Series)
- Algebraic Geometry (toric varieties)
- Discrete Geometry (polyhedra)
- Optimization (integer programming)

Kostant's partition function

Bertram Kostant (1928-2017)

Kostant's partition function is a vector partition function where the columns of the matrix A are the positive roots of a Lie algebra.

Classical Lie algebras

- Type A: $\mathfrak{sl}_{r+1}(\mathbb{C}) = \{X \in M_{r+1}(\mathbb{C}) : Tr(X) = 0\}$.
- Type B: $\mathfrak{so}_{2r+1}(\mathbb{C}) = \{X \in M_{2r+1}(\mathbb{C}) : X^t = -X\}$.
- Type C: $\mathfrak{sp}_{2r}(\mathbb{C}) = \{X \in M_{2r}(\mathbb{C}) : X^t J = -JX\}$, where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ with I the $r \times r$ identity matrix.
- Type D: $\mathfrak{so}_{2r}(\mathbb{C}) = \{X \in M_{2r}(\mathbb{C}) : X^t = -X\}$.

In these cases the Lie bracket is the commutator bracket. That is, if $X, Y \in \mathfrak{g}$, then $[X, Y] = XY - YX$.

Roots of the classical Lie algebra of type A_r

We specialize to the Lie algebra of type A_r:

$$\mathfrak{sl}_{r+1}(\mathbb{C}) = \{X \in M_{r+1}(\mathbb{C}) : Tr(X) = 0\}$$

Let e_i be a standard basis element of $\mathbb{R}^{r+1}$.

If

$$\alpha_i = e_i - e_{i+1}$$

then the positive roots are

$$\phi^+ = \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \cup \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j : 1 \leq i < j \leq r\}.$$

All roots:

$$\phi = \phi^+ \cup (-\phi^+)$$

The vectors in ϕ^+ are the columns of matrix A.

Kostant's partition function in type A_2

The positive roots are

$$\Phi^+_{A_2} = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

The vectors in ϕ^+ are the columns of matrix A. Thus

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}.$$

Given a vector $\mathbf{b} \in Col(A)$, Kostant's partition function would return the number of ways to express $\mathbf{b}$ as a linear combination of the vectors in Φ^+ where the coefficients are nonnegative integers.

Denote this count by $\wp(\mathbf{b})$.

Kostant's partition function in type A_2

Lemma

If $\mathbf{b} = n_1\alpha_1 + n_2\alpha_2 = \begin{bmatrix} n_1 \\ n_2 - n_1 \\ -n_2 \end{bmatrix}$ *with* $n_1, n_2 \in \mathbb{N} = \{0, 1, 2, \ldots\}$,

then

$$\wp(\mathbf{b}) = min(n_1, n_2) + 1.$$

Idea of the proof

The only positive root involving all of the simple roots α_1 and α_2 is

$$\alpha_1 + \alpha_2.$$

So the whole proof is about counting how many copies of $\alpha_1 + \alpha_2$ can be used.

A concrete example

Let

$$\mathbf{b} = 4\alpha_1 + 2\alpha_2.$$

If we use k copies of the mixed root $\alpha_1 + \alpha_2$, then

$$\mathbf{b} = (4 - k)\alpha_1 + (2 - k)\alpha_2 + k(\alpha_1 + \alpha_2).$$

Nonnegativity condition

All coefficients must be nonnegative, so

$$4 - k \geq 0 \qquad \text{and} \qquad 2 - k \geq 0.$$

Hence

$$0 \leq k \leq \min(4, 2) = 2.$$

List the valid choices for k

For $\mathbf{b} = 4\alpha_1 + 2\alpha_2$, the possible values are

$$k = 0, 1, 2.$$

Each value of k gives one partition

$$\begin{aligned}
k = 0: &\qquad \mathbf{b} = 4\alpha_1 + 2\alpha_2 + 0(\alpha_1 + \alpha_2), \\
k = 1: &\qquad \mathbf{b} = 3\alpha_1 + 1\alpha_2 + 1(\alpha_1 + \alpha_2), \\
k = 2: &\qquad \mathbf{b} = 2\alpha_1 + 0\alpha_2 + 2(\alpha_1 + \alpha_2).
\end{aligned}$$

Why there are no other partitions?

A partition of $4\alpha_1 + 2\alpha_2$ must have the form

$$(4-k)\alpha_1 + (2-k)\alpha_2 + k(\alpha_1 + \alpha_2).$$

Allowed

$$0 \leq k \leq 2.$$

So $k = 0, 1, 2$ are valid.

Not allowed

If $k = 3$, then

$$2 - k = 2 - 3 = -1,$$

so the coefficient of α_2 would be negative.

Thus the list on the previous slide is complete.

Conclusion: the Lemma in action

We found exactly three valid choices:

$$k = 0, 1, 2.$$

Therefore

$$\wp(4\alpha_1 + 2\alpha_2) = 3.$$

This agrees with the lemma:

$$\wp(n_1\alpha_1 + n_2\alpha_2) = \min(n_1, n_2) + 1,$$

because here

$$\min(4, 2) + 1 = 2 + 1 = 3.$$

Kostant's partition function in type A_3

The positive roots are

$$\Phi^+ = \{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_1+\alpha_2,\ \alpha_2+\alpha_3,\ \alpha_1+\alpha_2+\alpha_3\}$$
$$= \left\{ \begin{bmatrix} 1\\-1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\-1\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\1\\-1 \end{bmatrix}, \begin{bmatrix} 1\\0\\-1\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\0\\-1 \end{bmatrix}, \begin{bmatrix} 1\\0\\0\\-1 \end{bmatrix} \right\}$$

Thus $A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1\\ -1 & 1 & 0 & 0 & 1 & 0\\ 0 & -1 & 1 & -1 & 0 & 0\\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$.

Audience participation: What is the value of $\wp(5\alpha_1 + 2\alpha_2 + 7\alpha_3)$?

Partitions of $5\alpha_1 + 2\alpha_2 + 7\alpha_3$

Use $2(\alpha_1+\alpha_2+\alpha_3)$:

1. $\mathbf{3}\alpha_1 + \mathbf{0}\alpha_2 + \mathbf{5}\alpha_3 + \mathbf{0}(\alpha_1+\alpha_2) + \mathbf{0}(\alpha_2+\alpha_3) + \mathbf{2}(\alpha_1+\alpha_2+\alpha_3)$.

Use $1(\alpha_1+\alpha_2+\alpha_3)$:

2. $\mathbf{4}\alpha_1 + \mathbf{1}\alpha_2 + \mathbf{5}\alpha_3 + \mathbf{0}(\alpha_1+\alpha_2) + \mathbf{1}(\alpha_2+\alpha_3) + \mathbf{1}(\alpha_1+\alpha_2+\alpha_3)$
3. $\mathbf{3}\alpha_1 + \mathbf{1}\alpha_2 + \mathbf{6}\alpha_3 + \mathbf{1}(\alpha_1+\alpha_2) + \mathbf{0}(\alpha_2+\alpha_3) + \mathbf{1}(\alpha_1+\alpha_2+\alpha_3)$
4. $\mathbf{4}\alpha_1 + \mathbf{1}\alpha_2 + \mathbf{6}\alpha_3 + \mathbf{0}(\alpha_1+\alpha_2) + \mathbf{0}(\alpha_2+\alpha_3) + \mathbf{1}(\alpha_1+\alpha_2+\alpha_3)$.

Use $0(\alpha_1+\alpha_2+\alpha_3)$:

5. $\mathbf{5}\alpha_1 + \mathbf{0}\alpha_2 + \mathbf{5}\alpha_3 + \mathbf{0}(\alpha_1+\alpha_2) + \mathbf{2}(\alpha_2+\alpha_3) + \mathbf{0}(\alpha_1+\alpha_2+\alpha_3)$
6. $\mathbf{4}\alpha_1 + \mathbf{0}\alpha_2 + \mathbf{6}\alpha_3 + \mathbf{1}(\alpha_1+\alpha_2) + \mathbf{1}(\alpha_2+\alpha_3) + \mathbf{0}(\alpha_1+\alpha_2+\alpha_3)$
7. $\mathbf{5}\alpha_1 + \mathbf{1}\alpha_2 + \mathbf{6}\alpha_3 + \mathbf{0}(\alpha_1+\alpha_2) + \mathbf{1}(\alpha_2+\alpha_3) + \mathbf{0}(\alpha_1+\alpha_2+\alpha_3)$
8. $\mathbf{3}\alpha_1 + \mathbf{0}\alpha_2 + \mathbf{7}\alpha_3 + \mathbf{2}(\alpha_1+\alpha_2) + \mathbf{0}(\alpha_2+\alpha_3) + \mathbf{0}(\alpha_1+\alpha_2+\alpha_3)$
9. $\mathbf{4}\alpha_1 + \mathbf{1}\alpha_2 + \mathbf{7}\alpha_3 + \mathbf{1}(\alpha_1+\alpha_2) + \mathbf{0}(\alpha_2+\alpha_3) + \mathbf{0}(\alpha_1+\alpha_2+\alpha_3)$
10. $\mathbf{5}\alpha_1 + \mathbf{2}\alpha_2 + \mathbf{7}\alpha_3 + \mathbf{0}(\alpha_1+\alpha_2) + \mathbf{0}(\alpha_2+\alpha_3) + \mathbf{0}(\alpha_1+\alpha_2+\alpha_3)$.

There has to be a better way...

Students leading the way

Gabriel Ngwe

Cielo Perez

Aesha Siddiqui

All were undergraduate students at Williams College between second to senior year.

Kostant's partition function in type A_3

Theorem (G. Ngwe, C. Perez, A. Siddiqui, 2017)

If $n_1, n_2, n_3 \in \mathbb{N}$, then the value of $\wp(n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3)$ is given by computing

$$1 + min(n_1, n_2) + min(n_2, n_3) + min(n_1, n_2, n_3) + \sum_{i=1}^{min(n_1,n_2)} min(n_2 - i, n_3)$$

$$+ \sum_{i=1}^{min(n_1,n_2,n_3)} min(n_1 - i, n_2 - i) + \sum_{i=1}^{min(n_1,n_2,n_3)} min(n_2 - i, n_3 - i)$$

$$+ \sum_{i=1}^{min(n_1,n_2,n_3)} \sum_{j=1}^{min(n_1-i,n_2-i)} min(n_2 - i - j, n_3 - i).$$

Computational formula in type A_3

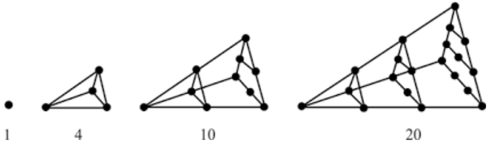
Theorem (G. Ngwe, C. Perez, A. Siddiqui, 2017)

If $\mathbf{b} = n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3$, *where* $n_1, n_2, n_3 \in \mathbb{N}$, *then*

$$\wp(\mathbf{b}) = \begin{cases} \frac{1}{6}(n_2+1)(n_2+2)(n_2+3) & \text{if } n_1, n_3 \geq n_2 \\ 1 + 2n_3 + n_2 + 3\binom{n_3}{2} + \binom{n_3}{3} + 2n_3(n_2 - n_3) & \text{if } n_1 \geq n_2 \geq n_3 \\ 1 + n_1 + 2n_3 + 2n_1n_3 + (-2 - n_1 - n_3 + \frac{2n_3}{3} + \frac{1}{3})(\frac{n_3(n_3+1)}{2}) & \\ +n_3^2(n_1+1) & \text{if } n_2 \geq n_1 \geq n_3 \text{ and } n_2 - n_3 \geq n_1 \\ \frac{1}{6}(6 - n_1^3 + n_2^3 - 2n_3^3 + (3n_3^2 + 3n_1^2)(-1 + n_2) + n_2(2 - 3n_2) & \\ +n_1(4 + 6n_2 - 3n_2^2) + n_3(5 - 3n_1^2 + 6n_2 - 3n_2^2 + n_1(3 + 6n_2)) & \text{if } n_2 \geq n_1 \geq n_3 \text{ and } n_2 - n_3 \leq n_1 \end{cases}$$

Corollary (Tetrahedral numbers)

If $\mathbf{b} = n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3$, *with* $n_1, n_3 \geq n_2$ *and* $n_1, n_2, n_3 \in \mathbb{N}$, *then* $\wp(n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3) = \frac{(n_2+1)(n_2+2)(n_2+3)}{6}$.



Partitions as lattice points in a tetrahedron

For $\mathbf{b} = 5\alpha_1 + 2\alpha_2 + 7\alpha_3$, choose the non-simple root coefficients

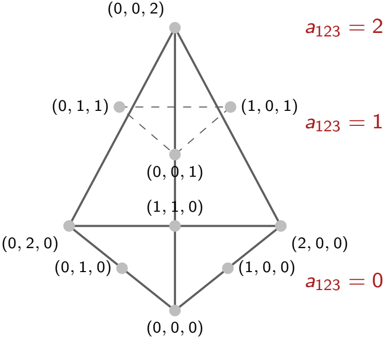
$$(a_{12}, a_{23}, a_{123}) = (\#(\alpha_1+\alpha_2), \#(\alpha_2+\alpha_3), \#(\alpha_1+\alpha_2+\alpha_3)).$$

Because the coefficient of α_2 is 2, we must have

$$a_{123} + a_{23} + a_{12} \leq 2.$$

Once these three coefficients are chosen, the simple-root coefficients are forced:

$$\begin{aligned} \alpha_1 &: 5 - a_{123} - a_{12}, \\ \alpha_2 &: 2 - a_{123} - a_{23} - a_{12}, \\ \alpha_3 &: 7 - a_{123} - a_{23}. \end{aligned}$$



$\boxed{\text{Partitions} \longleftrightarrow \text{lattice points}}$

Top layer: use two three-term roots

Layer $a_{123} = 2$**.** Here the only possible choice is

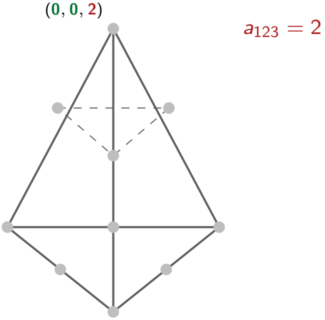
$$(a_{12}, a_{23}, a_{123}) = (\mathbf{0}, \mathbf{0}, \mathbf{2})$$

corresponding to

$$\mathbf{3}\alpha_1 + \mathbf{0}\alpha_2 + \mathbf{5}\alpha_3 + \mathbf{0}(\alpha_1 + \alpha_2) + \mathbf{0}(\alpha_2 + \alpha_3) + \mathbf{2}(\alpha_1 + \alpha_2 + \alpha_3).$$

So

$$a_{123} = 2 \quad \Rightarrow \quad \text{one point at the top.}$$



Middle layer: use one three-term root

Layer $a_{123} = 1$. Now choose a_{23} and a_{12} with $a_{23} + a_{12} \leq 1$, corresponding to

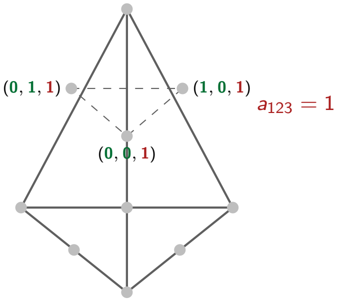
$$\mathbf{4}\alpha_1 + \mathbf{1}\alpha_2 + \mathbf{5}\alpha_3 + \mathbf{0}(\alpha_1 + \alpha_2) + \mathbf{1}(\alpha_2 + \alpha_3) + \mathbf{1}(\alpha_1 + \alpha_2 + \alpha_3)$$

$$\mathbf{3}\alpha_1 + \mathbf{1}\alpha_2 + \mathbf{6}\alpha_3 + \mathbf{1}(\alpha_1 + \alpha_2) + \mathbf{0}(\alpha_2 + \alpha_3) + \mathbf{1}(\alpha_1 + \alpha_2 + \alpha_3)$$

$$\mathbf{4}\alpha_1 + \mathbf{1}\alpha_2 + \mathbf{6}\alpha_3 + \mathbf{0}(\alpha_1 + \alpha_2) + \mathbf{0}(\alpha_2 + \alpha_3) + \mathbf{1}(\alpha_1 + \alpha_2 + \alpha_3).$$

So

$$a_{123} = 1 \quad \Rightarrow \quad \text{three lattice points.}$$



Bottom layer: use no three-term roots

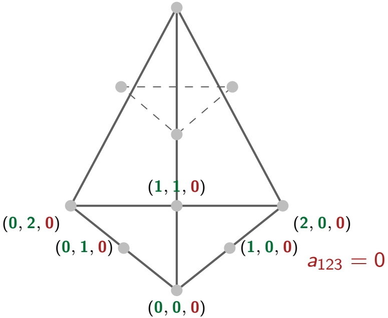
Layer $a_{123} = 0$**.** Now choose a_{23} and a_{12} with $a_{23} + a_{12} \leq 2$, corresponding to:

$$5\alpha_1 + 0\alpha_2 + 5\alpha_3 + 0(\alpha_1+\alpha_2) + 2(\alpha_2+\alpha_3) + 0(\alpha_1+\alpha_2+\alpha_3)$$
$$4\alpha_1 + 0\alpha_2 + 6\alpha_3 + 1(\alpha_1+\alpha_2) + 1(\alpha_2+\alpha_3) + 0(\alpha_1+\alpha_2+\alpha_3)$$
$$5\alpha_1 + 1\alpha_2 + 6\alpha_3 + 0(\alpha_1+\alpha_2) + 1(\alpha_2+\alpha_3) + 0(\alpha_1+\alpha_2+\alpha_3)$$
$$3\alpha_1 + 0\alpha_2 + 7\alpha_3 + 2(\alpha_1+\alpha_2) + 0(\alpha_2+\alpha_3) + 0(\alpha_1+\alpha_2+\alpha_3)$$
$$4\alpha_1 + 1\alpha_2 + 7\alpha_3 + 1(\alpha_1+\alpha_2) + 0(\alpha_2+\alpha_3) + 0(\alpha_1+\alpha_2+\alpha_3)$$
$$5\alpha_1 + 2\alpha_2 + 7\alpha_3 + 0(\alpha_1+\alpha_2) + 0(\alpha_2+\alpha_3) + 0(\alpha_1+\alpha_2+\alpha_3).$$

So

$$a_{123} = 0 \quad \Rightarrow$$

six lattice points.



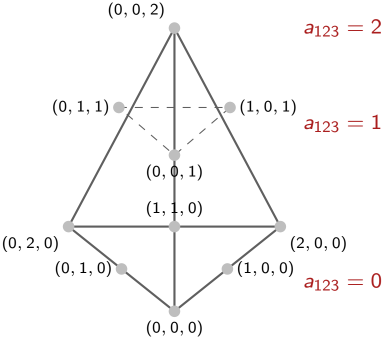
The tetrahedral count

The three layers contain

$$1 + 3 + 6 = 10$$

lattice points, matching the ten partitions of

$$5\alpha_1 + 2\alpha_2 + 7\alpha_3.$$



More generally, when $n_1, n_3 \geq n_2$, the same reasoning gives:

Tetrahedral numbers

$$\wp(n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3) = \binom{n_2+3}{3} = \frac{(n_2+1)(n_2+2)(n_2+3)}{6}.$$

Next let's try A_4...

Open questions

There are no general known closed formulas for Kostant's partition function.

It was shown by Blakley (1964) that there exists a finite decomposition of $\mathbb{N}^d$ such the partition function ϕ_A associated to the matrix A is a polynomial of degree $n - d$ on each piece.

Finding these polynomials and the decomposition of $\mathbb{N}^d$ is still a very active area of research.

Generalizing Kostant's partition function

The q-analog of Kostant's partition function $\wp_q$ is a polynomial-valued function in q whose evaluation at $q = 1$ recovers Kostant's partition function.

Namely,

$$\wp_q(\mathbf{b}) = a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \cdots + a_k q^k$$

where a_i is the number of ways to write the vector $\mathbf{b}$ as a sum of exactly i positive roots.

q-analog of KPF result

Every Lie algebra has a highest root, it is the positive root with largest coefficient sum. In type A_r it is well-known that

$$\wp_q(\tilde{\alpha}) = q(1+q)^{r-1}.$$

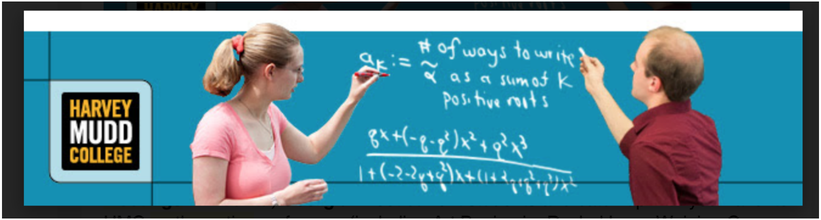
Question: What about in other Lie types?

Erik Insko

Mohamed Omar

We q-counted the number of ways to express the highest root of a classical Lie algebra as a nonnegative integral sum of the positive roots for Lie algebras of type B, C, D.

Photo advertisement



The above photo appeared in the advertisement for Harvey Mudd's alumni dinner at the JMM 2017!

Gravity Diagrams

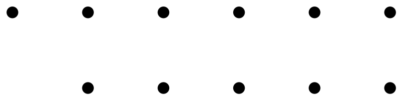
Before stating the main result we introduce a new combinatorial object that represents a collection of positive roots.

A *antigravity diagram* is a diagram with n up-adjusted columns of dots representing multiple copies of the simple roots α_1 through α_n and horizontal and vertical lines (under some conditions) representing positive roots.

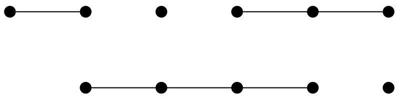
For example the highest root of the Lie algebra of type B_6 is given by

$$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6$$

an antigravity diagram can be



Example



This $(1, 2, 2, 2, 2, 2)$-antigravity diagram corresponds to the partition

$$\{\alpha_1 + \alpha_2, \alpha_3, \alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_6\}$$

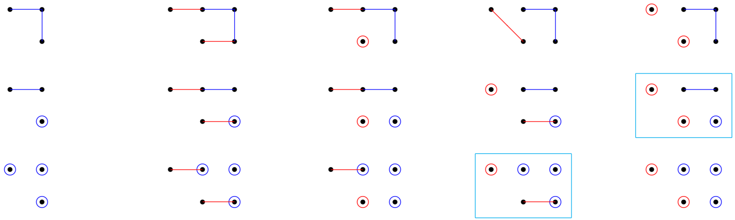
of

$$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6.$$

Extending gravity diagrams

Using gravity diagrams and *extensions* we were able to find the generating formulas for the number of ways to express the highest root as a sum of positive roots.

B_2 B_3



Main Result

Theorem (H., Insko, Omar - 2016)

The closed formulas for the generating functions $\sum_{r\geq 1}\mathcal{P}_{B_r}(q)x^r$, $\sum_{r\geq 1}\mathcal{P}_{C_r}(q)x^r$, *and* $\sum_{r\geq 4}\mathcal{P}_{D_r}(q)x^r$, *are given by*

$$\begin{aligned}
\sum_{r\geq 1}\mathcal{P}_{B_r}(q)x^r &= \frac{qx+(-q-q^2)x^2+q^2x^3}{1-(2+2q+q^2)x+(1+2q+q^2+q^3)x^2},\\
\sum_{r\geq 1}\mathcal{P}_{C_r}(q)x^r &= \frac{qx+(-q-q^2)x^2}{1-(2+2q+q^2)x+(1+2q+q^2+q^3)x^2},\\
\sum_{r\geq 4}\mathcal{P}_{D_r}(q)x^r &= \frac{(q+4q^2+6q^3+3q^4+q^5)x^4-(q+4q^2+6q^3+5q^4+3q^5+q^6)x^5}{1-(2+2q+q^2)x+(1+2q+q^2+q^3)x^2},
\end{aligned}$$

where $\mathcal{P}_{B_r}(q)$, $\mathcal{P}_{C_r}(q)$, *and* $\mathcal{P}_{D_r}(q)$ *denote* $\wp_q(\tilde{\alpha})$ *when* $\tilde{\alpha}$ *is the highest root of the Lie algebras of type* B_r, C_r, *and* D_r, *respectively.*

Explicit Formulas

$$
\begin{aligned}
\text{Type } A_r \ (r \geq 1): \quad \mathcal{P}_{A_r}(q) &= q(1+q)^{r-1}, \\
\text{Type } B_r \ (r \geq 2): \quad \mathcal{P}_{B_r}(q) &= b_1(q) \cdot (f_1(q))^{r-2} + b_2(q) \cdot (f_2(q))^{r-2}, \\
\text{Type } C_r \ (r \geq 3): \quad \mathcal{P}_{C_r}(q) &= c_1(q) \cdot (f_1(q))^{r-1} + c_2(q) \cdot (f_2(q))^{r-1}, \\
\text{Type } D_r \ (r \geq 4): \quad \mathcal{P}_{D_r}(q) &= d_1(q) \cdot (f_1(q))^{r-4} + d_2(q) \cdot (f_2(q))^{r-4},
\end{aligned}
$$

where

$$
f_1(q) = \frac{(q^2+2q+2) + q\sqrt{q^2+4}}{2}, \quad f_2(q) = \frac{(q^2+2q+2) - q\sqrt{q^2+4}}{2}
$$

and

$$
\begin{aligned}
b_1(q) &= \frac{(q^5+q^4+5q^3+4q^2+4q) + (q^4+q^3+3q^2+2q)\sqrt{q^2+4}}{2(q^2+4)}, \\
b_2(q) &= \frac{(q^5+q^4+5q^3+4q^2+4q) - (q^4+q^3+3q^2+2q)\sqrt{q^2+4}}{2(q^2+4)}, \\
c_1(q) &= \frac{\left(q^3+4q\right) + q^2\sqrt{q^2+4}}{2(q^2+4)}, \qquad c_2(q) = \frac{\left(q^3+4q\right) - q^2\sqrt{q^2+4}}{2(q^2+4)}, \\
d_1(q) &= \frac{(q^7+3q^6+10q^5+16q^4+25q^3+16q^2+4q) + (q^6+3q^5+8q^4+12q^3+9q^2+2q)\sqrt{q^2+4}}{2(q^2+4)}, \\
d_2(q) &= \frac{(q^7+3q^6+10q^5+16q^4+25q^3+16q^2+4q) - (q^6+3q^5+8q^4+12q^3+9q^2+2q)\sqrt{q^2+4}}{2(q^2+4)}.
\end{aligned}
$$

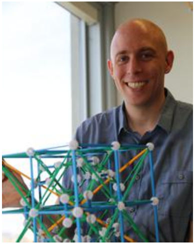
Connection to Juggling!

Setting $q = 1$ in the generating functions for the Lie algebras of type B and C recovers the generating functions used by Butler and Graham to count the number of certain multiplex juggling sequences.

In fact a lot more was true!!!

Math friends

Carolina Benedetti



Christopher Hanusa

Alejandro Morales

Anthony Simpson

A main result

Theorem (Benedetti-Hanusa-H.-Morales-Simpson 2020)

If $n \in \mathbb{N}$ and $\wp$ denotes the type A_r Kostant's partition function, then

$$\wp(n(\alpha_1 + \alpha_2 + \cdots + \alpha_r)) = mjs(\langle n \rangle, \langle n \rangle, n, r).$$

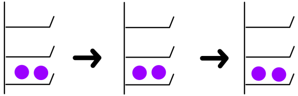
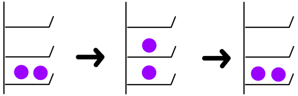
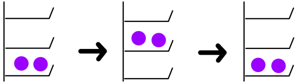
Proof's Key Insight

A throw at time i to height j encodes the vector

$$e_i - e_{i+j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+j-1}.$$

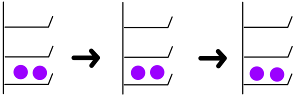
Applying Our Key Insight

$$\mathrm{mjs}(\langle 2\rangle, \langle 2\rangle, 2, 2) = \wp(2\alpha_1 + 2\alpha_2) = 3$$

Juggling sequences	Partitions
	$\{\alpha_1, \alpha_1, \alpha_2, \alpha_2\}$
	$\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$
	$\{\alpha_1 + \alpha_2, \alpha_1 + \alpha_2\}$

Proof of Concept

Multiplex juggling sequence:



- Two throws at time $i = 1$ to height $j = 1$ gives two vectors $e_1 - e_2 = \alpha_1$
- Two throws at time $i = 2$ to height $j = 1$ gives two vectors $e_2 - e_3 = \alpha_2$

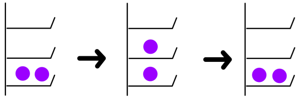
Partition:

Corresponds to the partition of $2\alpha_1 + 2\alpha_2$:

$$\{\alpha_1, \alpha_1, \alpha_2, \alpha_2\}$$

Proof of Concept

Multiplex juggling sequence:



- One throw at time $i = 1$ to height $j = 1$ gives one vector $e_1 - e_2 = \alpha_1$
- One throw at time $i = 1$ to height $j = 2$ gives one vector $e_1 - e_3 = \alpha_1 + \alpha_2$
- One throw at time $i = 2$ to height $j = 1$ gives one vector $e_2 - e_3 = \alpha_2$

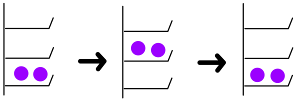
Partition:

Corresponds to the partition of $2\alpha_1 + 2\alpha_2$:

$$\{\alpha_1, \alpha_1 + \alpha_2, \alpha_2\}$$

Proof of Concept

Multiplex juggling sequence:



- Two throws at time $i = 1$ to height $j = 2$ gives two vectors

$$e_1 - e_3 = \alpha_1 + \alpha_2$$

Partition:

Corresponding to the partition of $2\alpha_1 + 2\alpha_2$:

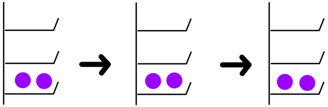
$$\{\alpha_1 + \alpha_2, \alpha_1 + \alpha_2\}$$

Example

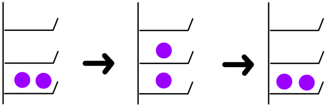
In type A_2: $\tilde{\alpha} = \alpha_1 + \alpha_2$, and $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. The set of partitions of $2\tilde{\alpha}$ are

$$\{2\alpha_1, 2\alpha_2\}, \{2(\alpha_1 + \alpha_2)\}, \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$$

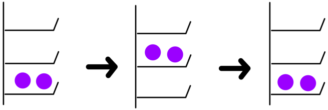
The multiplex juggling sequences with start and end states $\langle 2 \rangle$, hand capacity 2 and length 2 are:



$\{2\alpha_1, 2\alpha_2\}$



$\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$



$\{2(\alpha_1 + \alpha_2)\}$

Expository References

Coauthored with Maria Rodriguez Hertz

Juggling take away

The key insight for the results is the equivalence between the throwing of a ball during a juggling sequence at

$$\text{time } i \text{ to height } j$$

and the positive root

$$e_i - e_{i+j}$$

appearing in a partition of a weight of a Lie algebra of type A_r.

More juggling!

Theorem (Benedetti-Hanusa-H.-Morales-Simpson 2020)

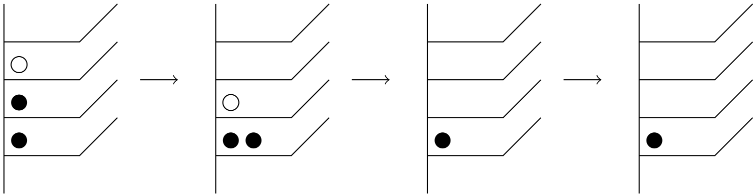
Let μ be a weight of the Lie algebra of type A_r and let $\langle \mu_1, \ldots, \mu_{r+1} \rangle$ be its standard basis vector representation. Then

$$\wp(\mu) = js(\langle \mu_1, \ldots, \mu_r \rangle, \langle \mu_1 + \cdots + \mu_r \rangle, r).$$

Example: If

$$\mu = \alpha_1 + 2\alpha_2 + \alpha_3 = \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4$$

then



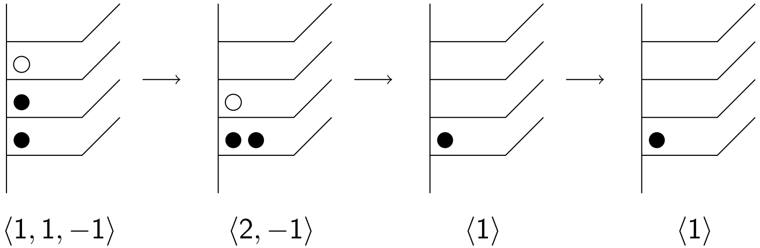
$\langle 1, 1, -1 \rangle$ $\quad$ $\langle 2, -1 \rangle$ $\quad$ $\langle 1 \rangle$ $\quad$ $\langle 1 \rangle$

Example cont.

If

$$\mu = \alpha_1 + 2\alpha_2 + \alpha_3 = \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4$$

then



Positive roots:

$$\varepsilon_1 - \varepsilon_2, \qquad 2(\varepsilon_2 - \varepsilon_3), \qquad \varepsilon_3 - \varepsilon_4$$

Ann. Comb. 24 (2020) 439–473

Published online June 29, 2020
https://doi.org/10.1007/s00026-020-00498-0

Annals of Combinatorics

Kostant's Partition Function and Magic Multiplex Juggling Sequences

Carolina Benedetti, Christopher R. H. Hanusa, Pamela E. Harris, Alejandro H. Morales and Anthony Simpson

Open problems:

- What is the juggling analog of Kostant's partition function for exceptional Lie algebras?
- Study the behavior of the number of throws made in sets $\mathrm{js}(s, e, m, t)$.

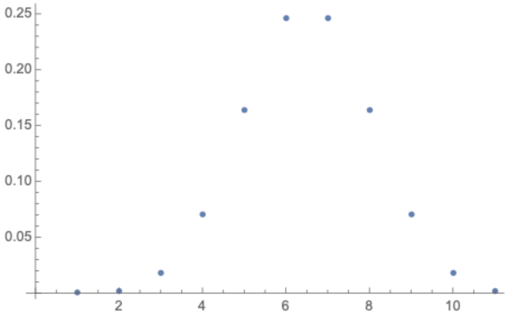
Hot off the mathematical presses!

We now have a juggling analog of Kostant's partition functions for the exceptional Lie algebras!

Kimberly J. Harry, Rebecca Patrias, Pamela E. Harris, Kimberly Hadaway, Lucy Martinez, and Miriam Norris

Open problems

- Find formulas for $\mathrm{js}(s, e, m, t)$ when we restrict the hand capacity or introduce other juggling rules.
- What can we say about the distribution of the number of throws made in $\mathrm{js}(s, e, m, t)$?



Distribution for $\mathrm{js}(\langle 1 \rangle, \langle 1 \rangle, 1, 10)$.

Reps of Lie algebras

The theorem of the highest weight asserts that a finite dimensional complex irreducible representation of $\mathfrak{g}$, a simple Lie algebra, is equivalent to a highest weight representation with dominant integral highest weight λ, denoted $L(\lambda)$.

The adjoint representation is equivalent to $L(\tilde{\alpha})$, where $\tilde{\alpha}$ is the highest root of $\mathfrak{g}$.

Reps of Lie algebras

Finite dimensional representations of semisimple Lie algebras are completely understood, after work of Élie Cartan.

A representation of a semisimple Lie algebra $\mathfrak{g}$ is analyzed by choosing a Cartan subalgebra. Then the representations of $\mathfrak{g}$ can be decomposed into weight spaces. This reduces the analysis of representations to the combinatorics of the possible weights which can occur.

Kostant's weight multiplicity formula

The multiplicity of a weight μ in $L(\lambda)$ is given by

$$m(\lambda, \mu) = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \wp(\sigma(\lambda + \rho) - (\mu + \rho)),$$

where W denotes the Weyl group, $\wp$ denotes Kostant's partition function and $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

Complications

The following complications arise when using Kostant's weight multiplicity formula and its q-analog:

$$m(\lambda, \mu) = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \wp(\sigma(\lambda + \rho) - (\mu + \rho))$$

1. Closed formulas for the value of Kostant's partition function are not known in much generality.
2. The order of the Weyl group grows exponentially as the rank $r \to \infty$.

My work has focused on addressing both of these complications.

q-Analog of KWMF

The same complications arise when using Lusztig's q-analog of Kostant's weight multiplicity formula:

$$m_q(\lambda, \mu) = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \wp_q(\sigma(\lambda + \rho) - \rho - \mu)$$

here $\wp_q$ denotes the q-analog of Kostant's partition function.

Weyl alternation sets

In practice most terms in KWMF are zero. This motivates the following.

Definition

Given a pair of weights (λ, μ), the *Weyl alternation set*, denoted $\mathcal{A}(\lambda, \mu)$, is the set of Weyl group elements which contribute nontrivially to the multiplicity $m(\lambda, \mu)$. Namely,

$$\mathcal{A}(\lambda, \mu) = \{\sigma \in W : \wp(\sigma(\lambda + \rho) - \rho - \mu) > 0\}.$$

The support of KWMF

Theorem (H., 2011)

Let $\tilde{\alpha}$ be the highest root of the Lie algebra of type A_r. Then the number of Weyl group elements contributing to the multiplicity $m(\tilde{\alpha}, 0)$, is given by the rth Fibonacci number.

Theorem (H.-Insko-Williams, 2016)

The number of Weyl group elements contributing nontrivially to the multiplicity of the zero-weight in the adjoint representation of a finite-dimensional classical Lie algebra can be enumerate through linear homogeneous recurrence relations with constant coefficients.

Theorem (Chang-H.-Insko, 2019)

Let λ be the sum of the simple roots of the Lie algebras of type B_r and C_r. Then the number of Weyl group elements contributing to the multiplicity $m(\lambda, 0)$, is given by a multiple of the rth Lucas number.

New results

Kimberly J. Harry

Theorem (Harry, 2024)

For all positive roots μ of the Lie algebra of type A_r, the cardinality of $\mathcal{A}(\tilde{\alpha}, \mu)$ is a product of two Fibonacci numbers.

Theorem (Harry, 2024)

For all positive roots μ of the Lie algebra of type A_r, the q-multiplicity is given by $m_q(\tilde{\alpha}, \mu) = q^{r-height(\mu)}$.

A conjecture of Harry motivated the work at...

MRC 2024

Portia X. Anderson, Esther Banaian, Melanie J. Ferreri, Owen C. Goff, Kimberly P. Hadaway, Pamela E. Harris, Kimberly J. Harry, Nicholas Mayers, Shiyun Wang, and Alexander N. Wilson

Structure of Weyl alternation sets

Writing elements of a Weyl group $\sigma \in W$ as reduced words, i.e., minimal length expressions for σ as a product of generators s_i, and denoting the length of such expressions by $\ell(\sigma)$, the posets on W of interest here are defined as follows.

Definition

The **right weak (Bruhat) order** $(W, \leq_R)$ on the Weyl group W is the transitive closure of the covering relations

$$\sigma \lessdot_R \sigma s_i \text{ whenever } \ell(\sigma s_i) > \ell(\sigma).$$

The **left weak (Bruhat) order** $(W, \leq_L)$ is similarly defined by covering relations

$$\sigma \lessdot_L s_i \sigma \text{ whenever } \ell(s_i \sigma) > \ell(\sigma).$$

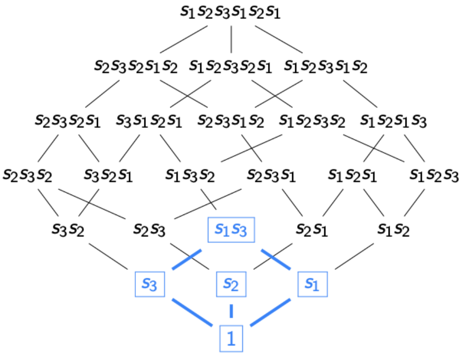
Support is an order ideal

An **order ideal** to be a subposet I of a poset $\mathcal{P}$ with the property that if $y \in I$ and $x \leq y$, then $x \in I$. We have shown:

Theorem

Let λ be an integral dominant weight of a simple Lie algebra $\mathfrak{g}$ with Weyl group W. Then for any weight μ, the Weyl alternation set $\mathcal{A}(\lambda, \mu)$ is a (possibly empty) order ideal in the left and in the right weak Bruhat orders of W.

Example



The left weak order on the type A_3 Weyl group with the set

$$\mathcal{A}(\tilde{\alpha}, -\tilde{\alpha})$$

highlighted where

$$\tilde{\alpha} = \alpha_1 + \alpha_2 + \alpha_3.$$

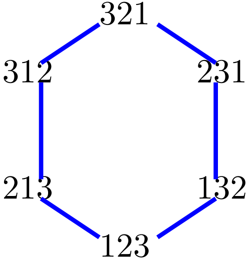
Open Problem: What happens when λ is not dominant? Is the support still an order ideal?

Kostant's partition function and polytopes

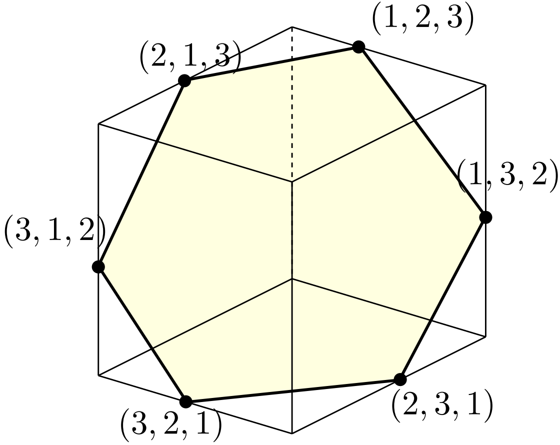
Common theme in combinatorics

Study **combinatorial objects** as part of combinatorial, geometric, algebraic **structures**, ...

Example: Permutations



partial order



vertices of polytope

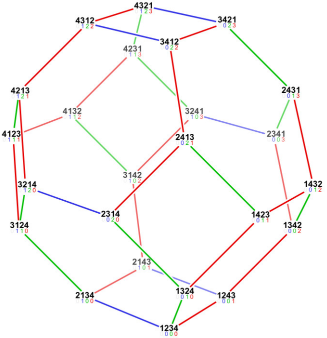
Integral polytopes

P a polytope in $\mathbb{R}^N$ with integral vertices:

P is the **convex hull** of finitely many vertices $\mathbf{v}$ in $\mathbb{Z}^N$

OR

P is the intersection of finitely **half spaces**



(wikipedia)

Permutahedron: convex hull of permutations $(w_1, w_2, \ldots, w_n)$

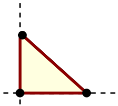
Volume of polytopes

- $\mathrm{vol}(P)$ is normalized volume $= \dim(P)! \cdot$ euclidean volume
- $\# P \cap \mathbb{Z}^N$ number of lattice points (discrete volume)

Example:

standard simplex $\Delta_n = \{(x_1, \ldots, x_n) \mid \sum x_i \leq 1,\ x_i \geq 0\}$

Δ_2



euclidean volume $\frac{1}{2}$

(normalized) volume 1

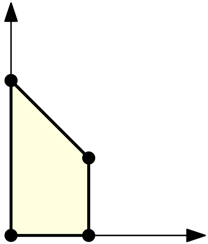
3 lattice points

Pitman-Stanley polytope

$\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$

$$\mathrm{PS}_n(\mathbf{a}) = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} \;\middle|\; (x_1, \ldots, x_n) \preceq (a_1, \ldots, a_n) \right\}$$

Example



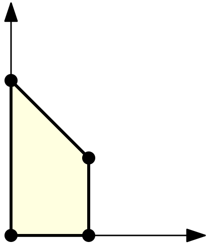
$PS_2(1,1)$

Pitman-Stanley polytope

$\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$

$$\mathrm{PS}_n(\mathbf{a}) = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} \;\middle|\; \begin{array}{c} x_1 \leq a_1 \\ x_1 + x_2 \leq a_1 + a_2 \\ \vdots \\ x_1 + \cdots + x_n \leq a_1 + \cdots + a_n \end{array} \right\}$$

Example



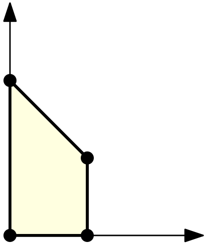
$PS_2(1,1)$

Pitman-Stanley polytope

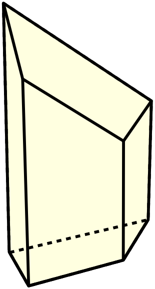
$\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$

$$\mathrm{PS}_n(\mathbf{a}) = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} \;\middle|\; \begin{array}{c} x_1 \leq a_1 \\ x_1 + x_2 \leq a_1 + a_2 \\ \vdots \\ x_1 + \cdots + x_n \leq a_1 + \cdots + a_n \end{array} \right\}$$

Example



$PS_2(1,1)$



$PS_3(1,1,1)$

- 2^n vertices and is n dimensional

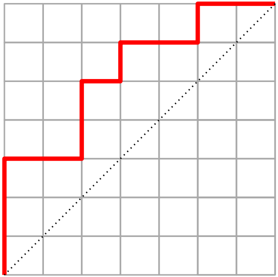
volume Pitman-Stanley polytope

Theorem (Pitman-Stanley 01)

$$\mathrm{vol}\,\mathrm{PS}_n(\mathbf{a}) = \sum_{\mathbf{j} \succeq (1,\ldots,1)} \binom{n}{j_1, \ldots, j_n} a_1^{j_1} \cdots a_n^{j_n}$$

$$= \sum_{\mathsf{f}\ \text{parking function}} a_{\mathsf{f}(1)} \cdots a_{\mathsf{f}(n)}$$

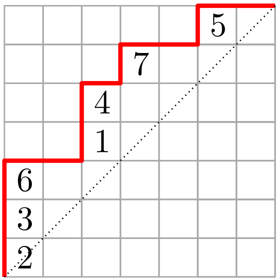
parking functions



A parking function of n is:

- a Dyck path from $(0,0)$ to (n,n)

parking functions

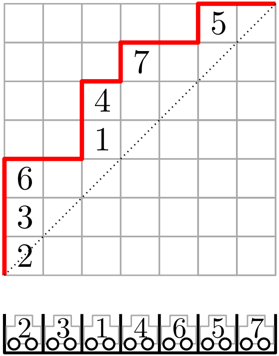


A parking function of n is:

- a Dyck path from $(0,0)$ to (n,n)
- vertical steps labelled $\{1,2,\ldots,n\}$, increasing vertical runs

encodes parking preferences n cars one way street

parking functions



There are $(n+1)^{n-1}$ parking functions of size n.

Flow polytopes

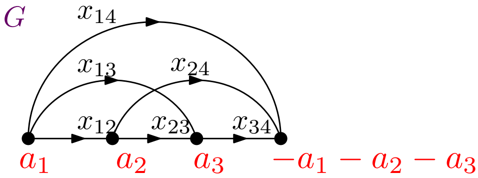
G graph $n+1$ vertices m edges

$\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$

$\mathcal{F}_G(\mathbf{a}) = \{$flows $x(\epsilon) \in \mathbb{R}_{\geq 0}$, $\epsilon \in E(G) \mid$ netflow vertex $i = a_i\}$

Example

$$x_{12} + x_{13} + x_{14} = a_1$$
$$x_{23} + x_{24} - x_{12} = a_2$$
$$x_{34} - x_{13} - x_{23} = a_3$$

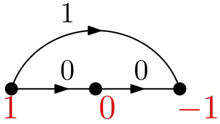


lattice points of $\mathcal{F}_G(\mathbf{a})$ are integral flows on G with netflow $\mathbf{a}$
let $K_G(\mathbf{a}) := \#(\mathcal{F}_G(\mathbf{a}) \cap \mathbb{Z}^{\#E(G)})$

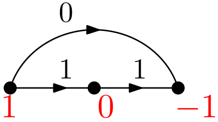
Kostant's vector partition function

When G is complete graph k_{n+1}, $K_{k_{n+1}}(\mathbf{a})$ is called **Kostant's partition function**.

$K_{k_{n+1}}(\mathbf{a}) = \#$ of ways of writing $\mathbf{a}$ as an $\mathbb{N}$-combination of vectors $e_i - e_j$



$(1, 0, -1) = e_1 - e_3$

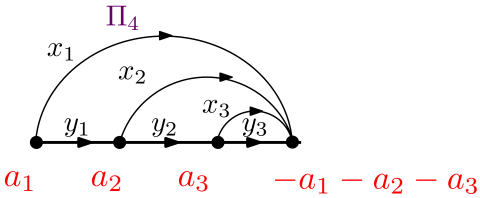


$(1, 0, -1) = (e_1 - e_2) + (e_2 - e_3)$

Examples flow polytopes

$\mathcal{F}_G(\mathbf{a}) = \{$flows $x(\epsilon) \in \mathbb{R}_{\geq 0}$, $\epsilon \in E(G)$ | netflow vertex $i = a_i\}$

Example (Baldoni-Vergne 2008)



$x_1 + y_1 = a_1 \quad \longrightarrow \quad x_1 \leq a_1$

$x_2 + y_2 - y_1 = a_2 \quad \longrightarrow \quad x_1 + x_2 \leq a_1 + a_2$

$x_3 + y_3 - y_2 = a_3 \quad \longrightarrow \quad x_1 + x_2 + x_3 \leq a_1 + a_2 + a_3$

$\mathcal{F}_{\Pi_{n+1}}(\mathbf{a})$ is the Pitman-Stanley polytope!

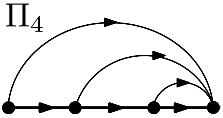
Lidskii volume formula

Theorem (Baldoni-Vergne 08, Postnikov-Stanley 08)
G m edges, $n+1$ vertices, $a_i \geq 0$

$$\mathrm{vol}\mathcal{F}_G(a_1,\ldots,a_n) = \sum_{\mathbf{j}\succeq\mathbf{o}} \binom{m-n}{j_1,\ldots,j_n} a_1^{j_1}\cdots a_n^{j_n} \times K_G(j_1-o_1,\ldots,j_n-o_n,0)$$

where $\mathbf{o}=(o_1,\ldots,o_n)$, $o_v = outdeg(v)-1$

Pitman-Stanley polytope:



$$\mathrm{vol}\mathcal{F}_{\Pi_{n+1}}(\mathbf{a}) = \sum_{\mathbf{j}\succeq(1,\ldots,1)} \binom{n}{j_1,\ldots,j_n} a_1^{j_1}\cdots a_n^{j_n} \cdot 1$$

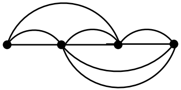
Parking function model Lidskii formula

$$\mathrm{vol}\mathcal{F}_G(1,\ldots,1) = \sum_{\mathbf{j}\succeq\mathbf{o}} \binom{m-n}{j_1,\ldots,j_n} \times K_G(j_1-o_1,\ldots,j_n-o_n,0)$$

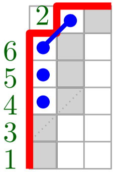
Unified diagram model (B-G D'L-H-H-K-M-Y 18)

- $\sum_{\mathbf{j}\succeq(out_1,\ldots,out_n)}$: lattice paths above diagonal with $(out_1,\ldots,out_n)$ boxes.
- $\binom{m-n}{j_1,\ldots,j_n}$: vertical steps labelled $\{1,2,\ldots,\#E-\#V+1\}$, increasing vertical runs
- $K_G(j_1-o_1,\ldots,j_n-o_n,0)$: dot line diagram under the path

Example



$(out_1, out_2, out_3) = (2,3,1)$



Proof of concept

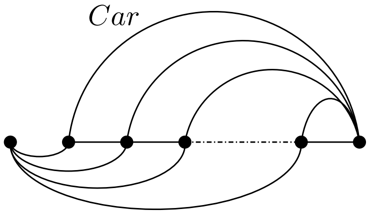
Séminaire Lotharingien de Combinatoire **80B** (2018)
Article #87, 12 pp.

Proceedings of the 30th Conference on Formal Power Series and Algebraic Combinatorics (Hanover)

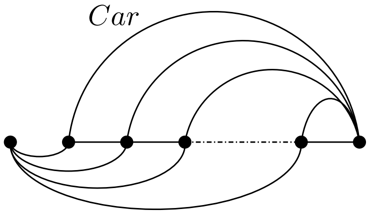
The volume of the caracol polytope

Carolina Benedetti[*1], Rafael S. González D'León[†2], Christopher R. H. Hanusa[‡3], Pamela E. Harris[§4], Apoorva Khare[¶5], Alejandro H. Morales[‖6], and Martha Yip[**7]

Application unified model: Caracol polytope

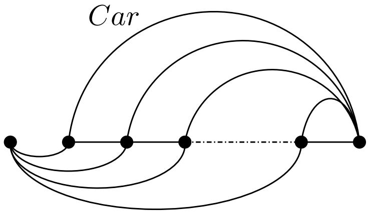


Application unified model: Caracol polytope



$P := \mathcal{F}_{Car}(1, 0, \ldots, 0, -1) \qquad \mathrm{vol}(P) = \mathsf{Cat}_{n-1}$ (Postnikov)

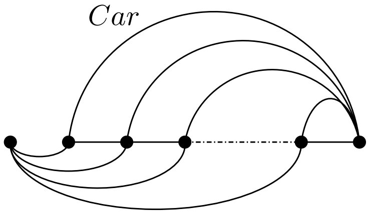
Application unified model: Caracol polytope



$P := \mathcal{F}_{Car}(1, 0, \ldots, 0, -1) \quad \mathrm{vol}(P) = \mathsf{Cat}_{n-1}$ (Postnikov)

$Q := \mathcal{F}_{Car}(0, 1, \ldots, 1, -n) \quad \mathrm{vol}(Q) = n^{n-2}$ (Pitman-Stanley)

Application unified model: Caracol polytope



$P := \mathcal{F}_{Car}(1, 0, \ldots, 0, -1) \quad \mathrm{vol}(P) = \mathsf{Cat}_{n-1}$ (Postnikov)

$Q := \mathcal{F}_{Car}(0, 1, \ldots, 1, -n) \quad \mathrm{vol}(Q) = n^{n-2}$ (Pitman-Stanley)

Theorem (B-G D'L-H-H-K-M-Y 18)

$\mathrm{vol}\mathcal{F}_{Car}(1, 1, \ldots, 1, -n-1) = \mathsf{Cat}_{n-1} \cdot (n+1)^{n-1}$

A new model!

TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 372, Number 5, 1 September 2019, Pages 3369–3404
https://doi.org/10.1090/tran/7743
Article electronically published on May 23, 2019

A COMBINATORIAL MODEL FOR COMPUTING VOLUMES OF FLOW POLYTOPES

CAROLINA BENEDETTI, RAFAEL S. GONZÁLEZ D'LEÓN, CHRISTOPHER R. H. HANUSA, PAMELA E. HARRIS, APOORVA KHARE, ALEJANDRO H. MORALES, AND MARTHA YIP

Dedicated to the memory of Griff L. Bilbro

Open problem

- The Chan-Robbins-Yuen (CRY) polytope

$$\mathcal{F}_{K_{n+2}}(1, 0, 0, \ldots, -1)$$

has volume the product of consecutive Catalan numbers. Zeilberger proved this by evaluating the Morris constant term identity, but no combinatorial proof is known!

Final Remarks

- Vector partition functions have connections to numerous areas of mathematics.
- We focused on the family of vector partition functions defined by Kostant and its connections to weight multiplicities and to flow polytopes.
- There is still so much more to do and maybe you like these types of problems. If you would like to join us in this work, please reach out!

Truly Final Remarks: Support, Structure, Surprises

- I'm deeply grateful for our combinatorics community of **support**. It has made all the difference in my journey.
- Thank you for helping build **structures**, like FPSAC that connect us all and for including me.
- The biggest **surprise** has been the people. So many of you have brought real friendship and joy to my life.

Thank You!

Pamela E. Harris

peharris@uwm.edu

pamelaeharris.com

References

A combinatorial model for computing volumes of flow polytopes, with Carolina Benedetti, Rafael Gonzalez D'Leon, Christopher R. H. Hanusa, Apoorva Khare, Alejandro Morales and Martha Yip. To appear in Transactions of the AMS.

Kostant's Weight Multiplicity Formula and the Fibonacci and Lucas Numbers, with Kevin Chang and Erik Insko. To appear in Journal of Combinatorics.

Pamela E. Harris, Erik Insko, and Anthony Simpson, *Computing weight q-multiplicities for the representations of the exceptional Lie algebras*. Applicable Algebra in Engineering, Communication and Computing, 29(4), August 2018, Volume 29, Issue 4, pp 351-362.

Pamela E. Harris, Haley Lescinsky, Grace Mabie, *Lattice patterns for the support of Kostant's weight multiplicity formula on* $\mathfrak{sl}_3(\mathbb{C})$. Minnesota Journal of Undergraduate Mathematics, [S.l.], v. 4, n. 1, June 2018.

Pamela E. Harris, Erik Insko and Mohamed Omar, *The q-analog of Kostant's partition function and the highest root of the simple Lie algebras*. Australasian Journal of Combinatorics Volume **71(1)** (2018) 68–91.

Pamela E. Harris and Edward L. Lauber, *Weight q-multiplicities for representations of* $\mathfrak{sp}_4(\mathbb{C})$. Journal of Siberian Federal University. Mathematics & Physics 2017, 10(4), 494–502.

Pamela E. Harris, *Computing weight multiplicities*. In: Wootton A., Peterson V., Lee C. (eds) A Primer for Undergraduate Research. Foundations for Undergraduate Research in Mathematics. Birkhäuser, Cham (2017) 193-222.

Pamela E. Harris, Erik Insko and Lauren Kelly Williams, *The adjoint representation of a classical Lie algebra and the support of Kostant's weight multiplicity formula*, Journal of Combinatorics Volume 7 (2016) Number 1 pp. 75-116.

Pamela E. Harris, *Combinatorial problems related to Kostant's weight multiplicity formula*. Ph.D. Dissertation, University of Wisconsin Milwaukee (2012).

Pamela E. Harris, *On the adjoint representation of* $\mathfrak{sl}_n$ *and the Fibonacci numbers*, C. R. Math. Acad. Sci. Paris 349 (2011) pp. 935-937.