A Macdonald expansion of q-chromatic symmetric functions and the Stanley–Stembridge Conjecture

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Sean Griffin

University of North Texas

Joint work with Anton Mellit, Marino Romero, Kevin Weigl, and Joshua Jeishing Wen (Vienna gang)

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European Research Council
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Theorem (Hikita)

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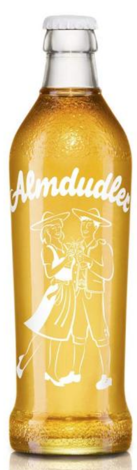

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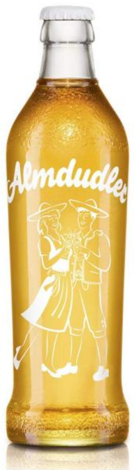

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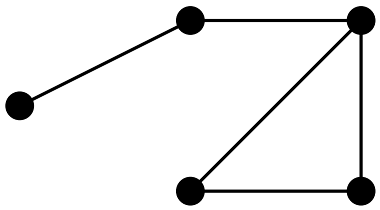
- For each proper coloring $\kappa : V \to \mathbb{N}$,

$G =$ 

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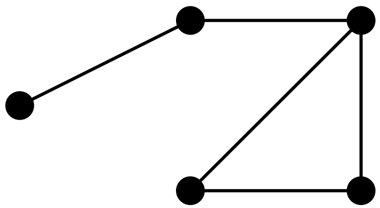
$G =$ 

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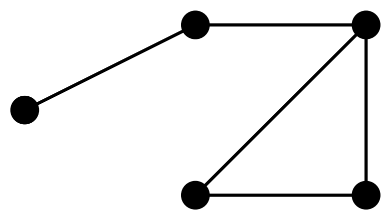
$G =$ 

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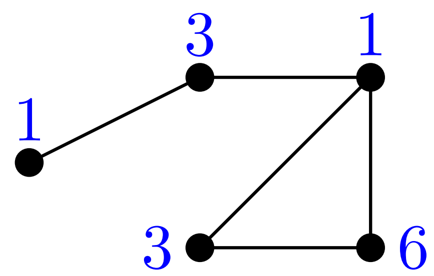
$G =$ 

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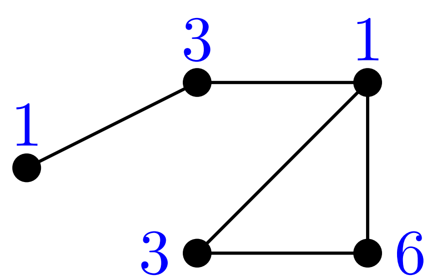
$G =$ 

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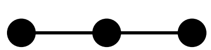
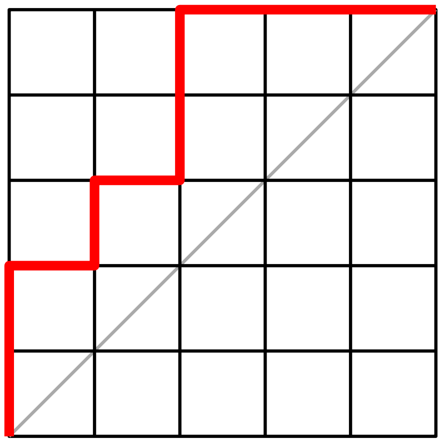
Ex: $G =$ 

$$\chi_G(x) = 4s_{(1,1,1)} + s_{(2,1)}$$

$$= e_{(2,1)} + 3e_{(3)}$$

Unit interval graphs

Every Dyck path determines a **unit interval graph**:

$D =$ 

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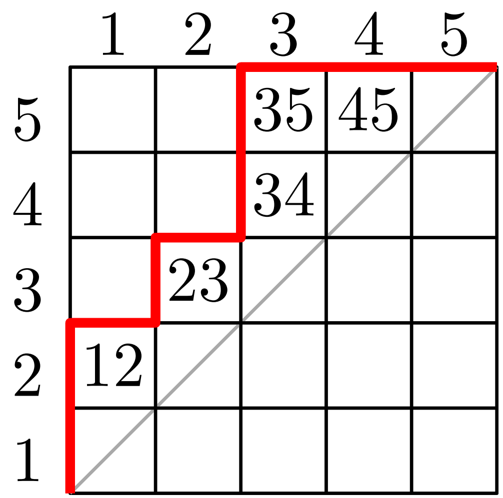
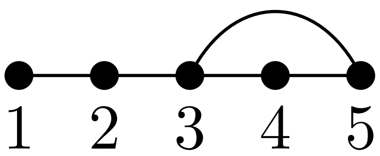
$D =$ 

$G =$ 

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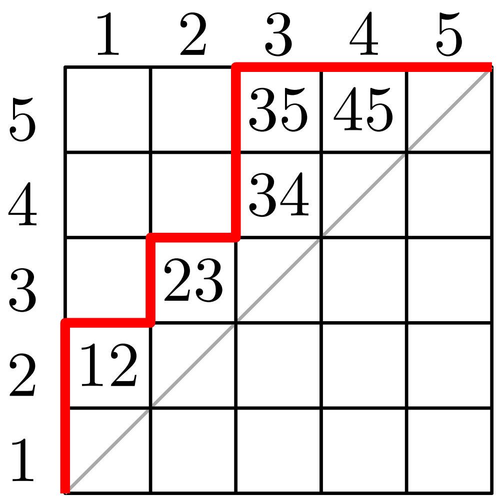

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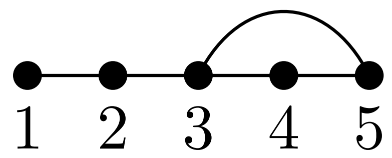

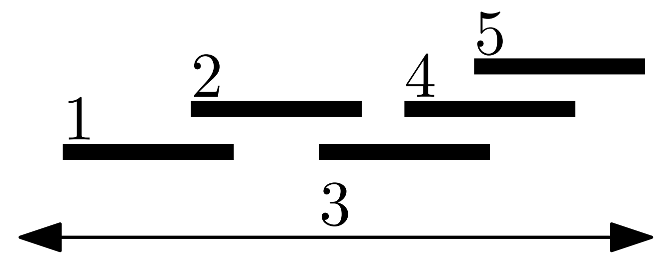

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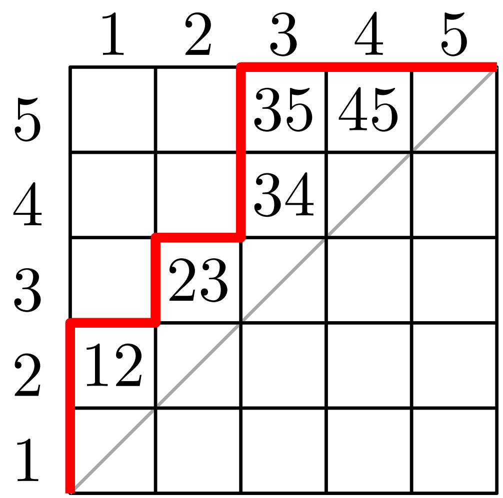
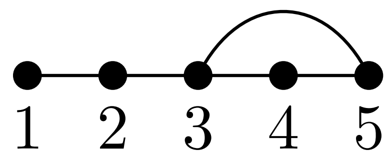
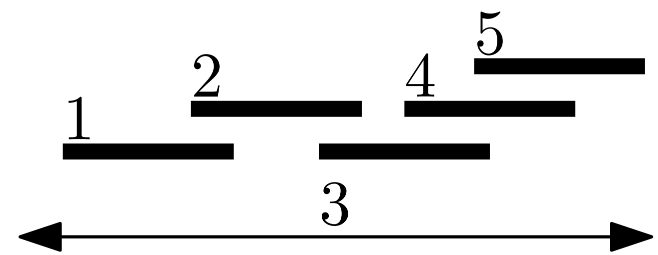
$D =$ 

$G =$ 



Stanley–Stembridge Conjecture, 1995

If G is a unit interval graph, $\chi_G(x)$ is e-**positive**, meaning

$$\chi_G(x) = \sum_{\lambda} a_\lambda e_\lambda$$

such that $a_\lambda \in \mathbb{Z}_{\geqslant 0}$ for all λ.

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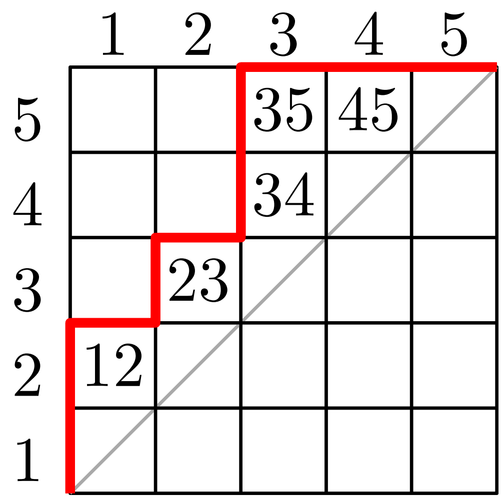
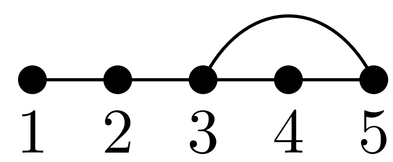
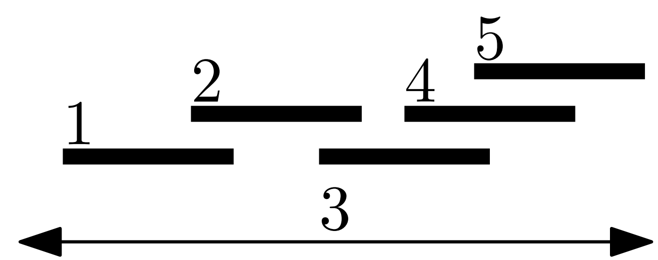
$D =$ 

$G =$ 



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- Originally stated for incomparability graphs of $(3+1)$-free posets. Conjecture reduced to unit interval graphs by Guay-Paquet.

Shareshian–Wachs Conjecture

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$$\chi_G(x; q) := \sum_{\kappa \text{ proper}} q^{\mathrm{asc}(\kappa)} x^\kappa.$$

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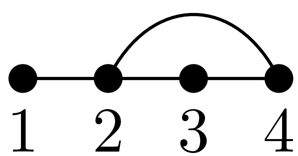
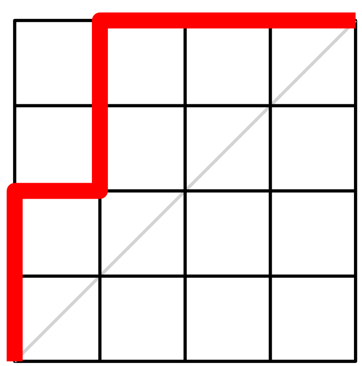
Ex: $G =$ 

$$\chi_G(x; q) = q[2]_q^2 e_{(3,1)} + (1 + q^2)[2]_q^2 e_{(4)}$$



q-chromatic symmetric functions

$\chi_G(x;q)$ are not symmetric in general for other graphs (see Gillespie–Pappe–Salois)

Theorem (Shareshian–Wachs, 2016)

If G is a unit interval graph, then $\chi_G(x;q)$ is symmetric and s-**positive**.

Conjecture (Shareshian–Wachs, 2016)

If G a unit interval graph, then $\chi_G(x;q)$ is e-**positive**.

- Related to geometric rep theory of rss Hessenberg varieties (Shareshian–Wachs, Brosnan–Chow) and affine Grassmannians (Kato).

Hikita's formula: setup

Let $\chi_G(x;q) = \sum_\lambda c_\lambda(q) e_\lambda$.

Hikita (2024) found an *explicit* formula for the coefficient $c_\lambda(q)$ and proved $c_\lambda(1) \geqslant 0$.

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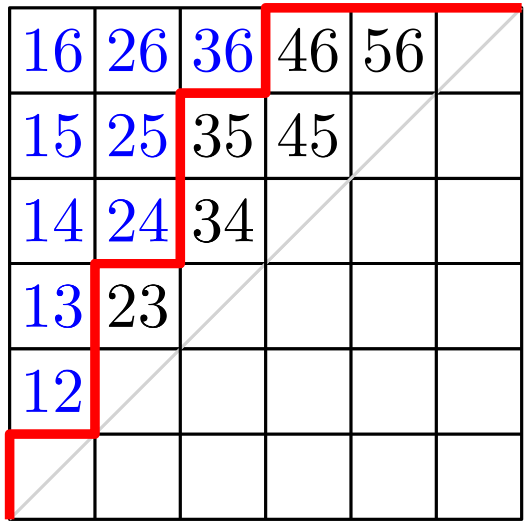
- $i < j$ is usual order on $[n]$
- $i \ll j$ if $i < j$ and above D.
- $i \prec j$ if $i < j$ and below D.

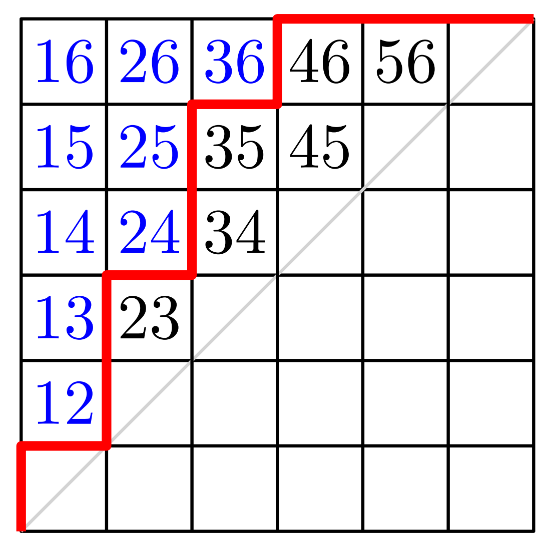
(unit interval graph)



Hikita's formula: setup

- Let SYT^D_λ = standard tableaux T of shape λ whose columns increase with respect to $\ll$.

$$\begin{array}{|c|c|c|} \cline{1-1} 5 \\ \cline{1-2} 2 & 6 \\ \hline 1 & 3 & 4 \\ \hline \end{array} \in SYT^D_\lambda$$



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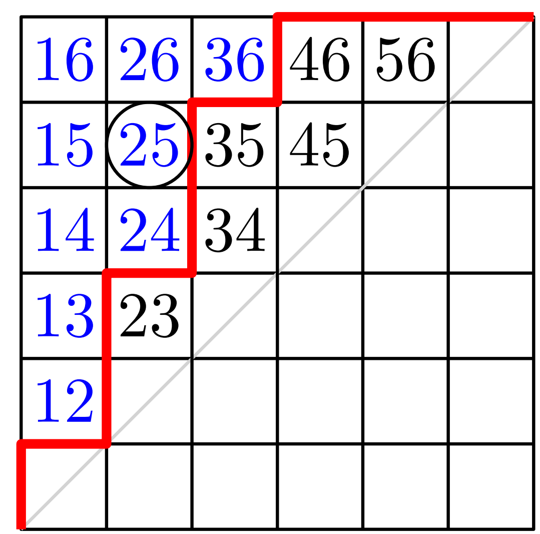

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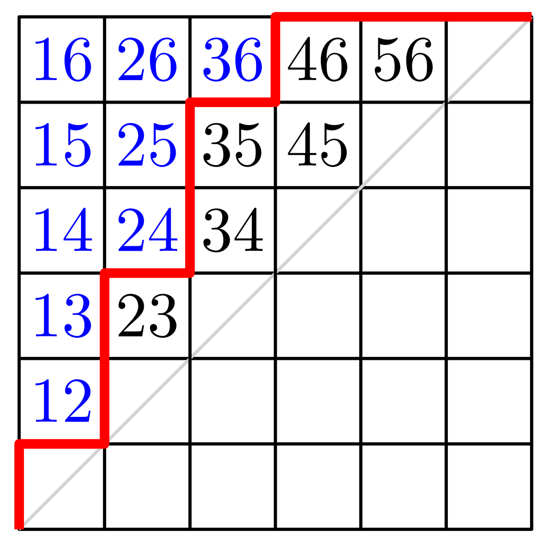
$$\begin{array}{|c|cccc}\hline 4 & & & & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{6} \\ \hline\end{array} \notin \overline{SYT}_\lambda^D$$



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$\in \overline{SYT}_\lambda^D$

4				
1	2	3	5	6

$\notin \overline{SYT}_\lambda^D$

16	26	36	46	56	
15	25	35	45		
14	24	34			
13	23				
12					

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Theorem (Hikita, 2024)

Let G be the graph corresponding to Dyck path D, then

$$\chi_G(x;q) = q^{\binom{n-1}{2}-1-|D|} \sum_{\lambda} \left(q^{\ell(\lambda)} \prod_i [\lambda_i]_q \sum_{T \in \overline{SYT}_\lambda^D} \prod_{\substack{i<j \\ c_j(T) \neq c_i(T)+1}} \frac{q^{c_j(T)} - q^{c_i(T)}}{q^{c_j(T)} - q^{c_i(T)+1}} \right) e_\lambda.$$

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Corollary (Hikita, 2024)

At $q=1$, the chromatic symmetric function is

$$\chi_G(x) = \chi_G(x;1) = \sum_{\lambda}\left(\prod_i \lambda_i \sum_{T\in\overline{SYT}_\lambda^D}\prod_{\substack{i<j\\ c_j(T)\neq c_i(T)+1}}\frac{|c_j(T)-c_i(T)|}{|c_j(T)-c_i(T)-1|}\right)e_\lambda,$$

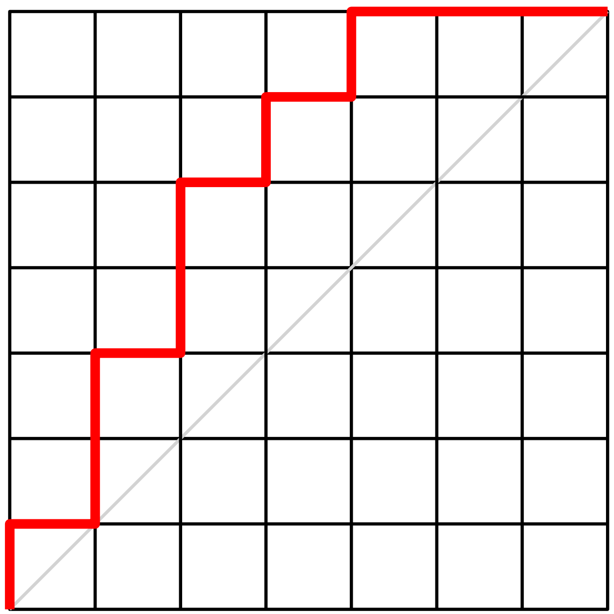
thus the Stanley–Stembridge Conjecture is true.

Example of e expansion

Hikita's formula gives the following coefficient formulas:

$$[e_{331}]\chi_D(x;q) = q^2[4]_q[2]_q + q^3\frac{[4]_q[2]_q}{[3]_q} + q^3\frac{[2]_q^4}{[3]_q}$$

$$[e_{331}]\chi_D(x;1) = 4\cdot 2 + \frac{4\cdot 2}{3} + \frac{2^4}{3}$$

$D =$ 

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We give a new proof of Hikita's formula.

- Gives a direct way of obtaining the formula
- Expansion into Macdonald polynomials
- Also gives e expansion, s expansion, and Hall–Littlewood expansion.
- Related to geometry of Hilbert schemes

Macdonald polynomials and $\mathbb{A}_{q,t}$ algebra

Macdonald polynomials

Our main result is an expansion of $\chi_G(x;q)$ in terms of **Macdonald symmetric functions.**

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- At $t=0$, $\widetilde{H}_\mu(x;q,0) =$ (modified) Hall–Littlewood polynomial
- At $t=1$, $\dfrac{\widetilde{H}_\lambda[(q-1)x;q,1]}{(q-1)^n} = (-1)^n \prod_i [\lambda_i]_q! e_\lambda$.

$\mathbb{A}_q$ algebra

$\mathbb{A}_q$ is a $\mathbb{Q}(q)$-algebra with idempotents labeled by $\mathbb{Z}_{\geqslant 0}$ with elements

$$\underset{0}{\bullet} \qquad \underset{1}{\bullet} \qquad \underset{2}{\bullet} \qquad \underset{3}{\bullet} \quad \cdots$$

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- $d_\pm : k \to k \pm 1$
- $T_i : k \to k \ (1 \leqslant i < k)$ mod some relations
- $y_i : k \to k \ (1 \leqslant i \leqslant k)$

$$\begin{tikzcd}
0 \arrow[r, bend left, "d_+"] & 1 \arrow[l, bend left, "d_-"] \arrow[r, bend left, "d_+"] & 2 \arrow[l, bend left, "d_-"] \arrow[r, bend left, "d_+"] & 3 \arrow[l, bend left, "d_-"] & \cdots
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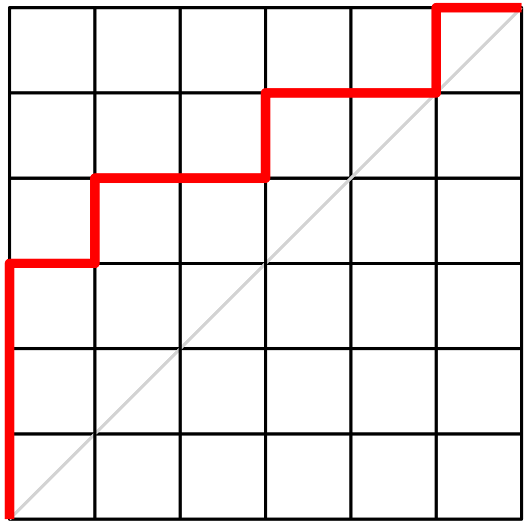
$\mathbb{A}_{q,t}$ is a "doubling" of $\mathbb{A}_q$ defined by Carlsson–Mellit

Has a representation $V = \bigoplus_{k \geqslant 0} V_k$

$$V_k = \Lambda \otimes \mathbb{Q}(q,t)[y_1, \ldots, y_k]$$

$$V_0 = \Lambda$$

Chromatics via $\mathbb{A}_{q,t}$

$D =$ 

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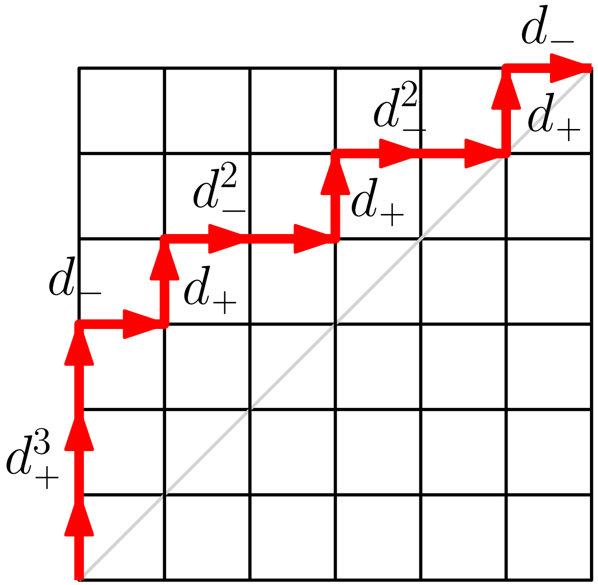
$D =$ 

$$F_D(x;q) := d_-^{b_\ell} d_+^{a_\ell} \cdots d_-^{b_1} d_+^{a_1}(1).$$

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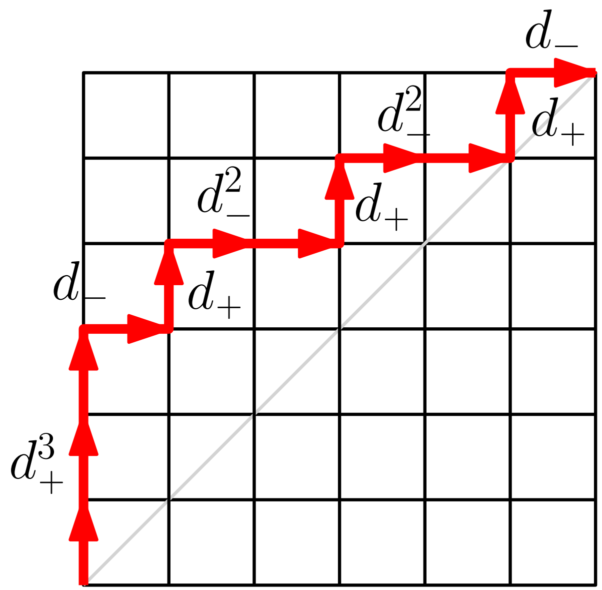
$D =$ 

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Proposition (Carlsson–Mellit, 2015)

$$\chi_G(x;q) = (q-1)^{-n} F_D[(q-1)x].$$

(See e.g. Precup–Sommers and Masuda–Sato for a geometric meaning of the identity)

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There are elements $\widetilde{H}_{\lambda,w}$ of V, proven by Bechtloff-Weising–Orr to correspond to **partially symmetric Macdonald functions**.

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- $\widetilde{H}_{\lambda,w}(x; y_1, \dots, y_k) \in V_k$ where
  $w = (w_1, \dots, w_k)$ a row-decreasing sequence of cells in λ that form a horizontal strip.

$$\lambda = \begin{array}{|c|c|c|c|c|}
\cline{1-1}
w_2 \\
\cline{1-3}
 & w_3 & w_1 \\
\hline
 & & & & w_4 \\
\hline
\end{array}$$

$\mathbb{A}_{q,t}$ action

There are elements $\widetilde{H}_{\lambda,w}$ of V, proven by Bechtloff-Weising–Orr to correspond to **partially symmetric Macdonald functions**.

- $\widetilde{H}_{\lambda,w}(x; y_1, \ldots, y_k) \in V_k$ where
  $w = (w_1, \ldots, w_k)$ a row-decreasing sequence of cells in λ that form a horizontal strip.

$$\lambda = \begin{ytableau} w_2 \\ {} & w_3 & w_1 \\ {} & {} & {} & {} & w_4 \end{ytableau}$$

- When $w = \varnothing$, you recover Macdonald polynomials:

$$\widetilde{H}_{\lambda,\varnothing} = \widetilde{H}_\lambda(x)/\widetilde{H}_\lambda(-1)$$

where $\widetilde{H}_\lambda(-1) = (-1)^{|\lambda|} \prod_{(i,j)\in\lambda} q^i t^j$.

$d_\pm$ action on the basis

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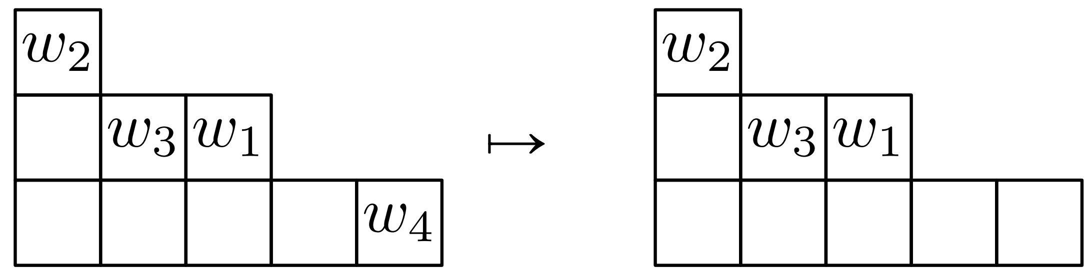

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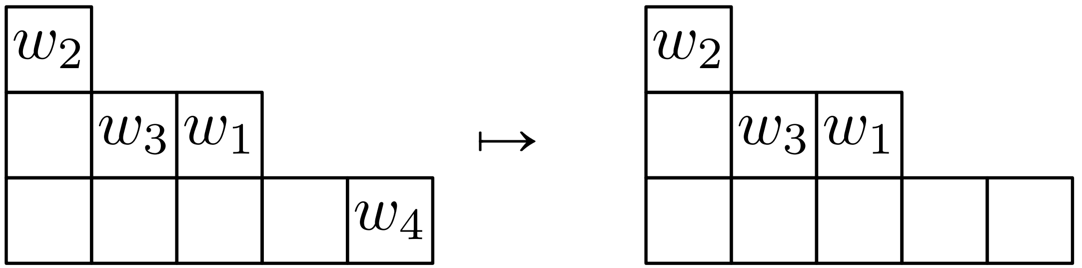

- $d_+\widetilde{H}_{\lambda,w} = -q^k \sum_c c\, d_{\lambda+c,\lambda} \prod_i \frac{c - tw_i}{c - qtw_i} \widetilde{H}_{\lambda+c,cw}.$

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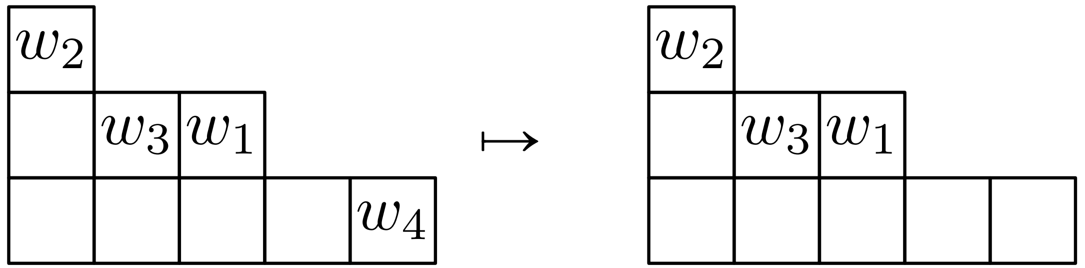

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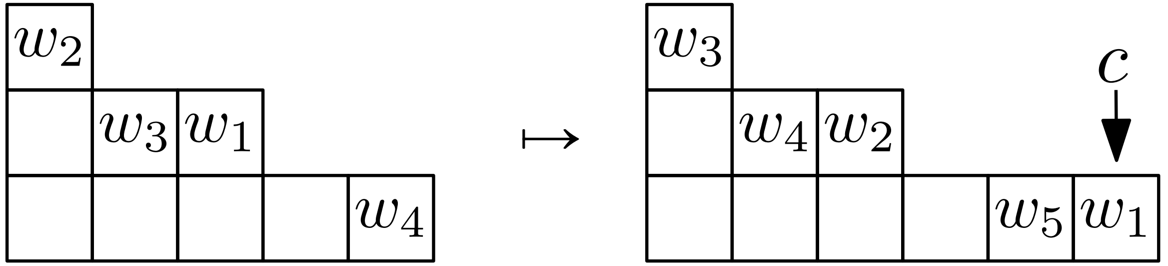

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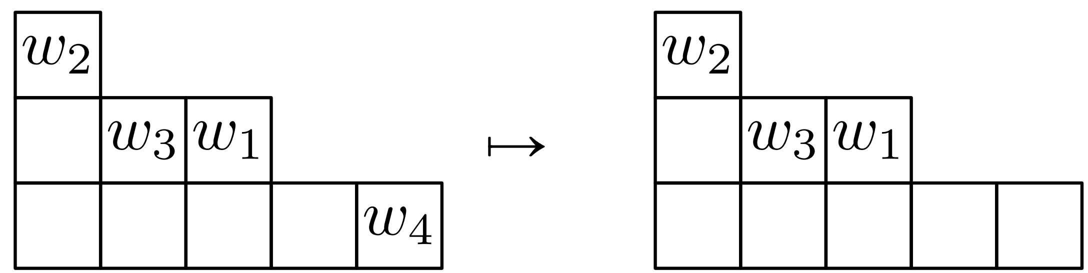

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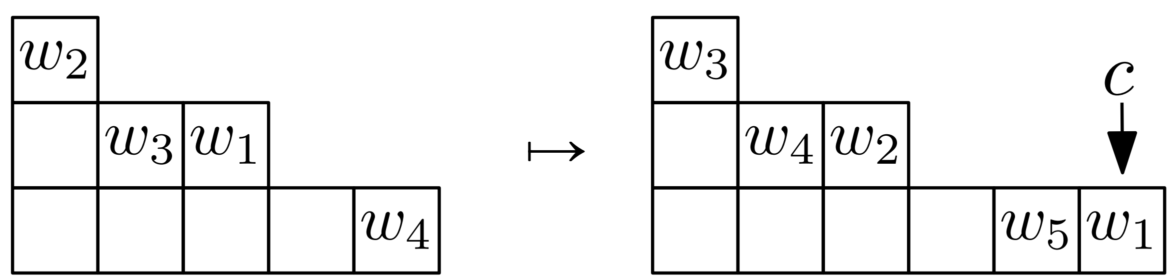

where $d_{\lambda+c,\lambda} = \prod_{x\in \mathrm{Row}(c)} \frac{q^{a(x)} - t^{\ell(x)+1}}{q^{a(x)+1} - t^{\ell(x)+1}} \prod_{x\in \mathrm{Col}(c)} \frac{q^{a(x)+1} - t^{\ell(x)}}{q^{a(x)+1} - t^{\ell(x)+1}}.$

Macdonald expansion

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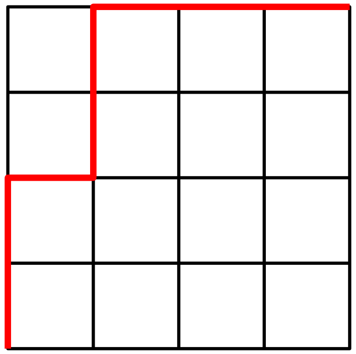

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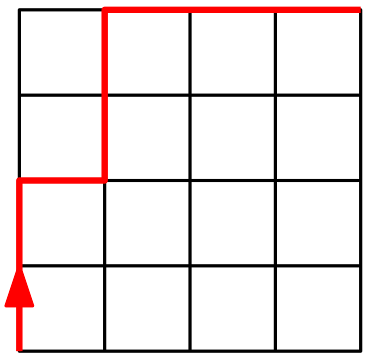

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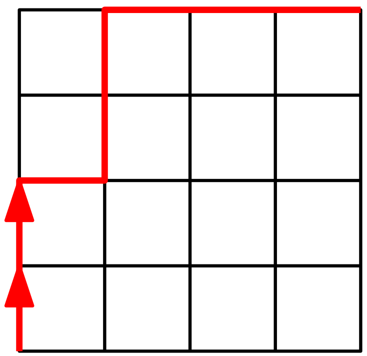

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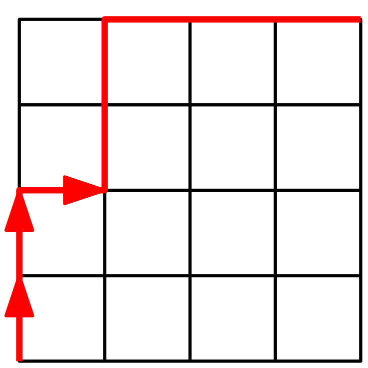

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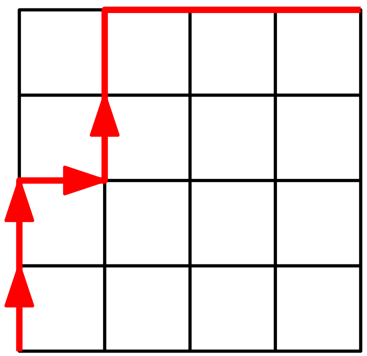

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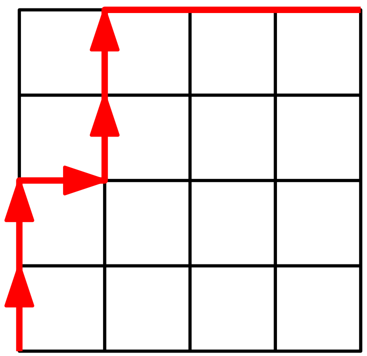

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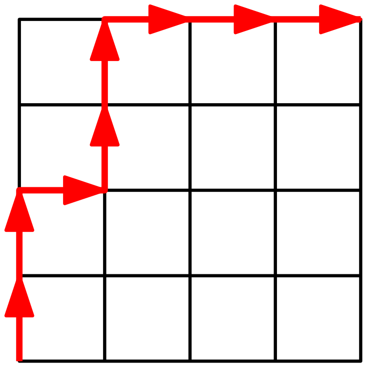

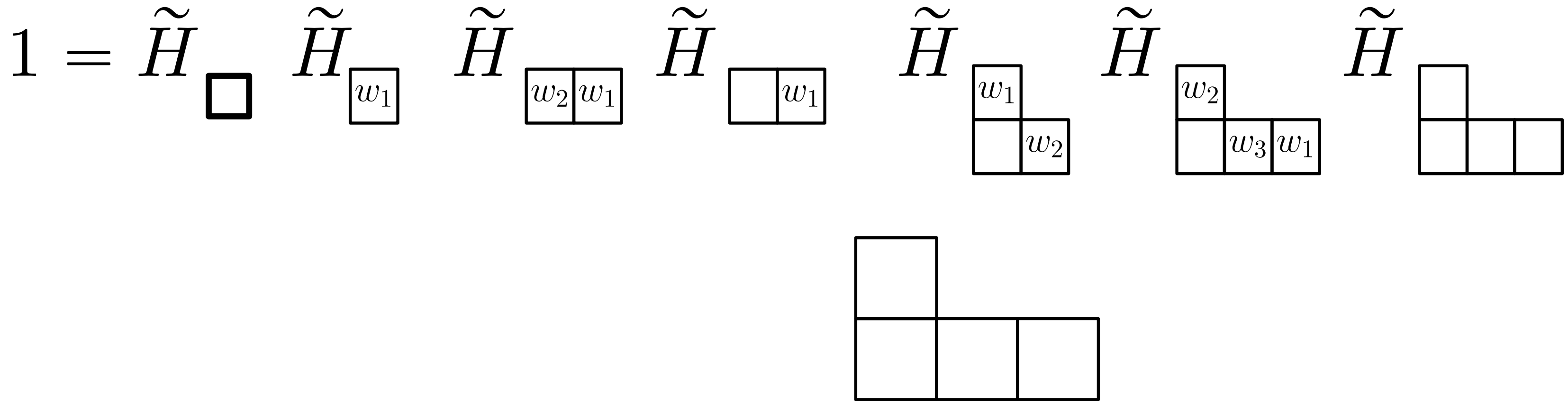

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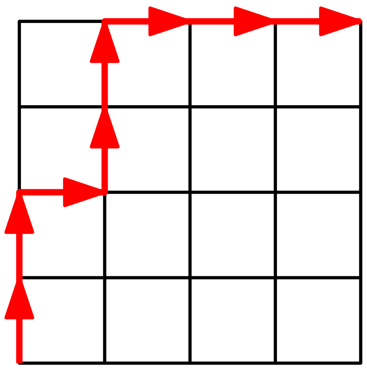

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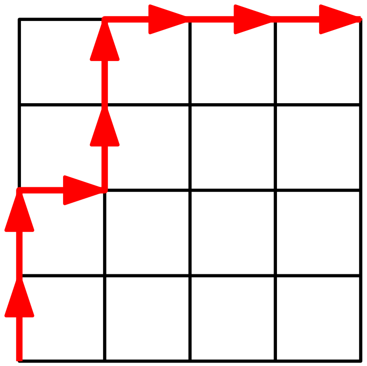

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Theorem (G–Mellit–Romero–Weigl–Wen, 2025)

$$F_D(x;q) = q^{\binom{n}{2}-|D|} \sum_{\lambda} \left(\sum_{T \in \mathrm{SYT}^D_\lambda} A_T(q,t) \right) \widetilde{H}_\lambda(x;q,t).$$

where $A_T(q,t)$ are rational functions in q, t (with explicit formulas).

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Further directions

Reprove recent extensions of Stanley-Stembridge using the $\mathbb{A}_{q,t}$ algebra:

- Huh, Hwang, Kim, Kim, and Oh (2025) proved the e-positivity of Abreu–Nigro's g-functions.
- Tom–Vailaya (2026) proved e-positivity of glued sequences of unit interval graphs and cycles.
- Shareshian–Wachs Conjecture (positivity when you keep q) is still open!

Thanks for your attention!

$$d_-^3 d_+ d_-^2 d_+ d_-^4 d_+ d_-^2 d_+^5 d_- d_+^2 d_- d_+^2 d_- d_+^2 (1)$$

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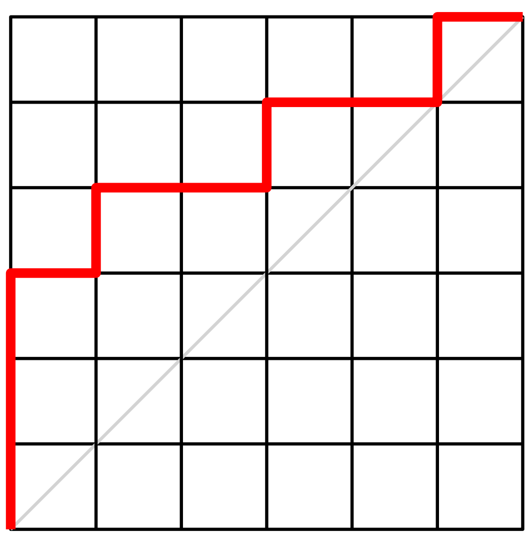
$D =$ 

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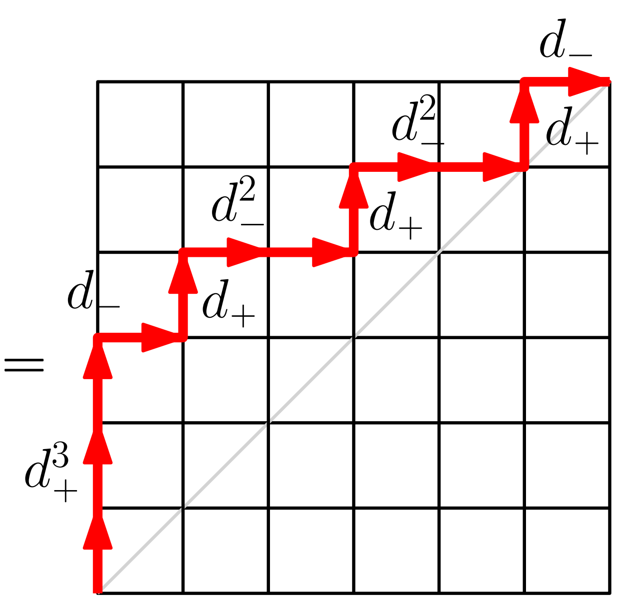
$D =$ 

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$$(I_n \subset I_{n-1} \subset \cdots \subset I_{n-k} \subset \mathbb{C}[x,y])$$

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By the localization theorem, each (localized,equivariant) K-theory ring has a basis (over K-theory of a point) given by the classes of torus-fixed points, $I_{\lambda,w}$.

Aside: Geometric $d_\pm$ operators

The torus fixed points $I_{\lambda,w}$ are flags of monomial ideals.

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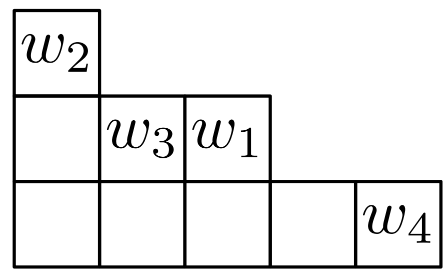
$\lambda =$ 

$\widetilde{H}_{\lambda,w} \in V$ correspond to $[I_{\lambda,w}]$ under the isomorphism above.

$d_- \leftrightarrow$ pushforward along

$$(I_n \subset \cdots \subset I_{n-k}) \mapsto (I_{n-1} \subset \cdots \subset I_{n-k})$$

$q^k(q-1)d_+ \leftrightarrow$ pullback along

$$(I_n \subset \cdots \subset I_{n-k}) \mapsto (I_n \subset \cdots \subset I_{n-k+1})$$

Hall-Littlewood expansion

At $t = 0$, we get a formula for $\chi_G(x;q)$ in terms of Hall–Littlewood symmetric functions.

$$\widetilde{H}_\lambda[(q-1)x;q,0] = q^{|\lambda'|+n(\lambda')} Q_{\lambda'}(x;q^{-1}).$$

Corollary (G–Mellit–Romero–Weigl–Wen, 2025)

$\chi_G(x;q)$ has the following expansion in terms of Q:

$$q^{\binom{n+1}{2}-|\mathbf{r}|} \sum_\lambda \frac{Q_{\lambda'}(x;q^{-1})}{(q-1)^{\lambda_1}}$$
$$\times \left(\sum_{T\in \mathrm{SYT}^D_\lambda} \prod_{i\notin T_1} q^{-L(T,i)-d(T,i)} \prod_{\substack{i\notin T_1,\\ (T_{<i})_s=\varnothing}} [d(T,i)]_q \prod_{i:(T_{<i})_s\neq\varnothing} [L(T,i)-\mathrm{sh}(T_{<i})_{s+1}]_q \right)$$

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040ec09b1e35df139433887a97daa66f/5/1/51FELj6GO0L.jpg