h^*-vectors for matroids

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(3) Fundamental result

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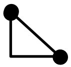
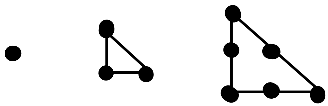
E.g. $P =$  , then  $\#|\ell P \cap \mathbb{Z}^2|$, $\mathrm{Ehr}_P(t) = \dfrac{1}{(1-t)^3}$

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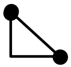
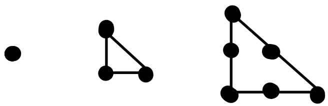
E.g. $P =$ , then  $\#|\ell P \cap \mathbb{Z}^2|$, $\mathrm{Ehr}_P(t) = \frac{1}{(1-t)^3}$

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Braun, Ferroni–Higashitani: surveys on h^*-vectors.

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$\Longrightarrow$ Stanley since Hilbert series $\sum_{\ell \geq 0} (\dim_{\Bbbk} S_\ell(Y_P))\, t^\ell$ is equal to $\mathrm{Ehr}_P(t)$.

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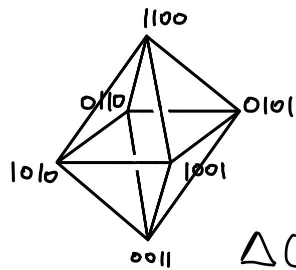
<u>E.g.</u> hypersimplices:

$$\Delta(k,n) := \operatorname{conv}\left\{ \sum_{i \in S} e_i \;\middle|\; S \in \binom{\{1,\dots,n\}}{k} \right\}.$$



$\Delta(2,4)$

<u>Defn</u> A matroid M is a collection $\mathcal{B}$ of subsets of $\{1, \cdots, n\}$ such that $P(M) := \mathrm{conv}\{\sum_{i \in B} e_i \mid B \in \mathcal{B}\}$ is a GP.

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matroid of L encodes the comb. of $L \cap (k^*)^n$

on which f_P is defined.

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(2') One can generalize to define $h^*(M,P)$ for any matroid M & GP P
and $h^*(M,P)$ is always a Macaulay vector (nonnegative).

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Thm [E–Fink–Larson] Kindred subschemes are reduced & CM.

This is only the beginning!

TWENTY PROBLEMS ON h^*-VECTORS FOR MATROIDS

CHRISTOPHER EUR

ABSTRACT. It is said that AI systems may excel at "taking convex hulls of existing branches of knowledge." We consider two branches: (1) Ehrhart theory of lattice polytopes (a well-established and still-active field) and (2) K-theoretic positivity for hyperplane arrangements and matroids (a new and emerging development [EL26, EFL, BF, Liu]). More specifically, we collect twenty problems about h^*-vectors for matroids, as introduced in [EFL26], where many of the problems are motivated by analogous results/questions about h^*-vectors of lattice polytopes surveyed in [FH24].

NOTATION AND CONVENTIONS

We follow the notations and conventions as given in [EFL26] and [FH24]. Explicitly, a *pair* is (M, P) with M a loopless matroid on $[n]$ and $P \subseteq \mathbb{R}^{[n]}$ a generalized permutohedron (GP). Write

$$d = \deg(k \mapsto \chi(\mathrm{M}, \mathcal{L}_P^k)), \qquad h^*(\mathrm{M}, P) = (h_0, \dots, h_d), \qquad s = \deg h^*(\mathrm{M}, P),$$

where $\chi(\mathrm{M}, -)$ is the sheaf Euler characteristic on the matroid K-ring and $\mathcal{L}_P$ is the (nef) class attached to P.

PROBLEMS

Problem 1. Show that, for a fixed matroid M and an integer $\ell > 0$, there are finitely many GP (up to defining the same element in $A^1(\mathrm{M})$) satisfying $h_d^*(\mathrm{M}, P) = \ell$. Further, give bounds for the number of such in terms of invariants like "volume."

Motivated by: The analogous result by Lagarias and Ziegler for lattice polytopes bounds the number of lattice polytopes in $\mathbb{R}^n$ (up to unimodular equivalence) with a given number of interior lattice