On the finiteness of the group associated with weighted walks in multidimensional orthants

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Joint work with Emmanuel Humbert and Kilian Raschel

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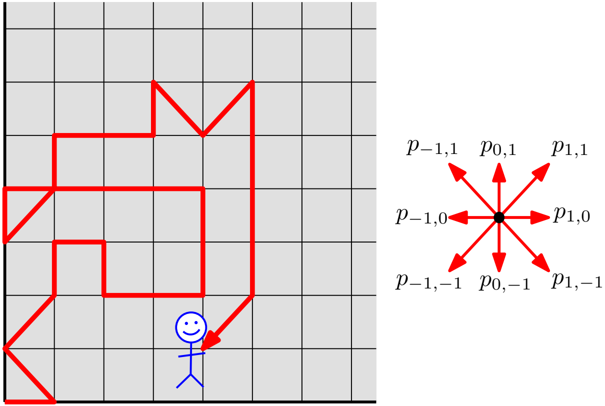
Quadrant walks

Model: Fix $p_{i,j} \geq 0$ for $i, j \in \{-1, 0, 1\}$.
Quadrant walks: Consider walks starting at $(0,0)$, staying in the positive quadrant with steps in $\{(i,j) : i, j \in \{-1, 0, 1\}\}$.
Step weights: each step in direction (i,j) gets weight $p_{i,j}$
Walk weight: Product of step weights



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Generating function: $\mathsf{Q}(t,x,y) := \sum_{n,a,b=0}^{\infty} q_{n,a,b} t^n x^a y^b.$

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Classification: $\mathsf{Q}(t,x,y)$ has been fully classified into the heirarchy Rational/Algebraic/etc.

For a series (or a function) $F(t)$, the following properties satisfy

$$\text{Rational} \Rightarrow \text{Algebraic} \Rightarrow \text{D-finite} \Rightarrow \text{D-Algebraic} :$$

- **Rational:** $F(t) = \frac{P(t)}{Q(t)}$ for polynomials $P(t)$ and $Q(t)$.
- **Algebraic:** $P(F(t), t) = 0$ for some non-zero polynomial $P(x)$.
- **D-finite:** $F(t)$ satisfies some non-trivial linear differential equation. E.g.

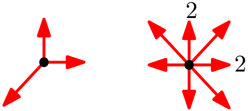
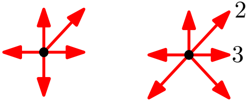
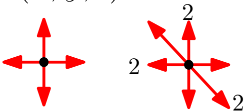
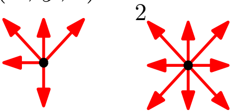
$$t^3 F''(t) + t^2 F'(t) + (t+1)F(t) - 1 = 0$$

- **D-algebraic:** $F(t)$ satisfies some non-trivial algebraic differential equation. E.g.

$$t^2 F'(t) + F''(t)F(t) + tF(t) = 0$$

For multivariate functions/series: Classification considered separately with respect to each variable

COMPLETE CLASSIFICATION $\mathsf{Q}(t, x, y)$

	Finite Group	Infinite Group
Decoupling	$\mathsf{Q}(x, y, t)$ Algebraic 	$\mathsf{Q}(x, y, t)$ D-alg. 
Non-Decoupling	$\mathsf{Q}(x, y, t)$ D-finite 	$\mathsf{Q}(x, y, t)$ non-D-alg. 

[Bousquet-Mélou, Mishna, Denisov, Wachtel, Rechnitzer, Bostan, Chyzak, Van Hoeij, Kauers, Pech, Dreyfus, Hardouin, Roques, Singer, Fayolle, Raschel, Kurkova, Bernardi, Elvey Price]

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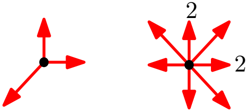
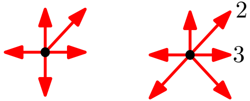
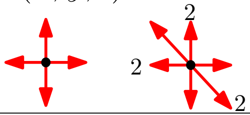
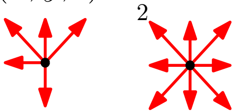
Decoupling	$\mathsf{Q}(x, y, t)$ Algebraic 	$\mathsf{Q}(x, y, t)$ D-alg. 
Non-Decoupling	$\mathsf{Q}(x, y, t)$ D-finite 	$\mathsf{Q}(x, y, t)$ non-D-alg. 

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Remaining question: When is the group finite?

Model: Fix reals $p_{i,j} \geq 0$ for $i, j \in \{-1, 0, 1\}$.
Inventory: The *Inventory* of the model is

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Group definition: There are involutions $\varphi_1, \varphi_2 : (\mathbb{R}^+)^2 \to (\mathbb{R}^+)^2$ defined by

$$\varphi_1([x,y]) = \left[\frac{A_{-1}(y)}{A_1(y)x}, y\right] \quad \text{and} \quad \varphi_2([x,y]) = \left[x, \frac{B_{-1}(x)}{B_1(x)y}\right],$$

which both fix $\chi(x,y)$. The *group of the model* is the group G generated by φ_1 and φ_2.

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Question: In which cases is G finite?

GROUP EXAMPLES

Example 1: in the case $\chi(x,y) = x + y + \frac{1}{x} + \frac{1}{y}$, the transformations are

$$\varphi_1([x,y]) = \left[\frac{1}{x}, y\right] \qquad \text{and} \qquad \varphi_2([x,y]) = \left[x, \frac{1}{y}\right].$$

These generate a group of size 4:

$$\begin{tikzcd}
(x,y) \arrow[rr, leftrightarrow, "\varphi_1"] \arrow[d, leftrightarrow, "\varphi_2"] & & \left(\frac{1}{x}, y\right) \arrow[d, leftrightarrow, "\varphi_2"] \\
\left(x, \frac{1}{y}\right) \arrow[rr, leftrightarrow, "\varphi_1"] & & \left(\frac{1}{x}, \frac{1}{y}\right)
\end{tikzcd}$$

Example 2: in the case

$$\chi(x,y) = \left(\frac{1}{x} + \frac{y}{x} + 2y + xy + 2x + \frac{x}{y} + \frac{1}{y}\right),$$

the involutions φ_1 and φ_2 generate a group of size 10:

$$\begin{tikzcd}
\left(x, \frac{x}{y(1+x)}\right) \arrow[r, leftrightarrow, "\varphi_2"] \arrow[d, leftrightarrow, "\varphi_1"] & (x,y) \arrow[r, leftrightarrow, "\varphi_1"] & \left(\frac{y}{x(1+y)}, y\right) \arrow[r, leftrightarrow, "\varphi_2"] & \left(\frac{y}{x(1+y)}, \frac{1}{x+y+xy}\right) \arrow[d, leftrightarrow, "\varphi_2"] \\
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Note: In most cases the group is infinite.

Previous classification results:

- The group is infinite or has order at most 12 [Jiang, Tanakoli, Zhao, 2021][Hardouin, Singer, 2021]
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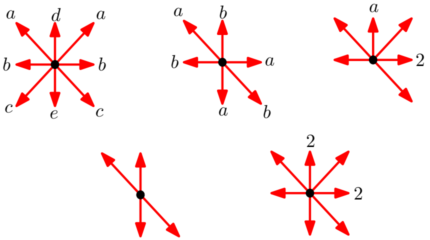
This work (part 1): complete classification of finite group models:



Along with reflections, rotations and transformations $\chi(x, y) \to \alpha\chi(\beta x, \gamma y)$ of the above.

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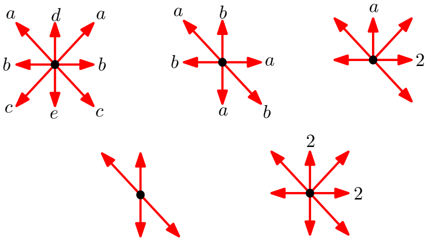

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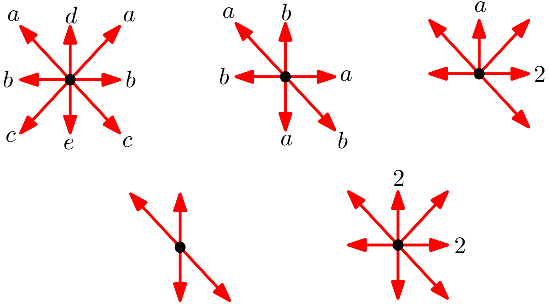
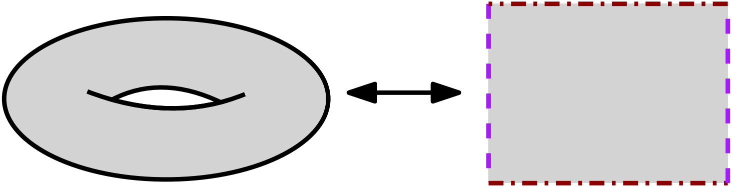
Part 2: Higher dimension

Part 1: Dimension 2: Full classification of finite group cases



Part 1a: Possible orders of group via elliptic parametrisation

Origin of elliptic parametrisation: [Fayolle, Iasnogorodski, 1979], [Fayolle, Ianogorodski, Malyshev, 1999], [Raschel, 2010], [Bernardi, Bousquet-Mélou, Raschel, 2021]



GROUP ON REIMANN SURFACE

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Consequence: Each $g \in G$ fixes the surface

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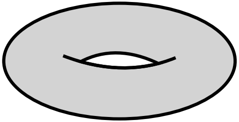

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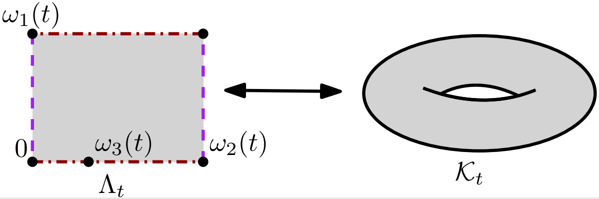
Claim: For $t > 0$ small, $\mathcal{K}_t$ is conformally equivalent to a flat torus

$$\Lambda_t = \mathbb{C}/(\omega_1(t)\mathbb{Z} + \omega_2(t)\mathbb{Z}).$$



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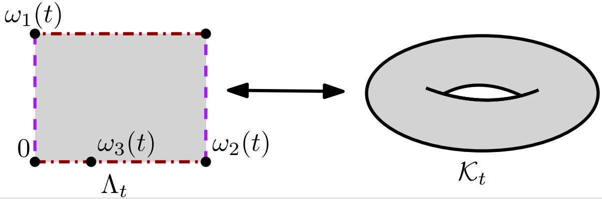
$$z \mapsto -z \qquad \text{and} \qquad z \mapsto \omega_3(t) - z.$$

on Λ_t. Note: $\omega_1(t) \in i\mathbb{R}$ and $\omega_2(t), \omega_3(t) \in \mathbb{R}$.



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Finiteness: G is finite if and only if $\frac{\omega_3(t)}{\omega_2(t)} \in \mathbb{Q}$ is fixed.

Order of group: If $\frac{\omega_3(t)}{\omega_2(t)} = \frac{m}{n}$ then G has order $2n$.

From elliptic parameterisation:

Elliptic nome $q(t) = e^{-i\pi \frac{\omega_2(t)}{\omega_1(t)}}$ and $q(t)^{r(t)} = e^{-i\pi \frac{\omega_3(t)}{\omega_1(t)}}$ related to t by

$$\sum_{j\geq 4} k_j t^{\frac{j}{2}} = q - 8q^2 + 44q^3 + \cdots$$

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RHS: Explicit series using Jacobi elliptic functions

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$$r(0) \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{3}{4}, \frac{3}{5}, \frac{3}{7}, \frac{3}{8}, \frac{4}{7}, \frac{5}{7}, \frac{5}{8}\right\}.$$

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Recall: $r(t) = \frac{m}{n}$ fixed $\iff$ G is finite.

possible finite group orders: $2n = 4, 6, 8, 10, 14, 16$

Finite group possibilities summary

Recall: $r(t) = \frac{m}{n}$ is a fixed rational number if and only if the group G is finite. In this case $|G| = 2n$

Previous results:

- If G is finite then $|G| \leq 12$, that is $n \leq 6$ [Jiang, Tanakoli, Zhao, 2021][Hardouin, Singer, 2021]
- Classification already complete for $|G| \leq 8$, that is $n \leq 4$ [Kauers, Yatchak, 2015]

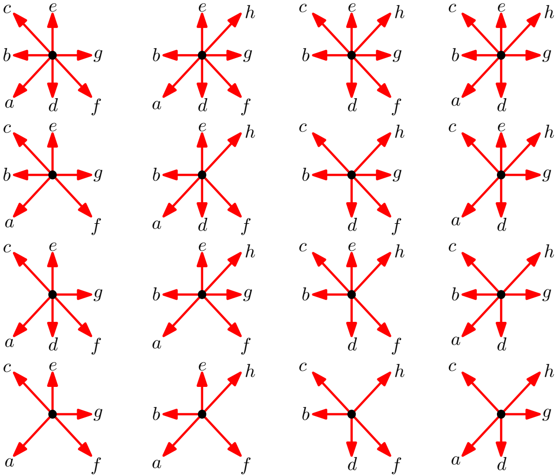
Our result (so far): if G is finite,

$$r \in \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{3}{4}, \frac{3}{5}, \frac{3}{7}, \frac{3}{8}, \frac{4}{7}, \frac{5}{7}, \frac{5}{8} \right\},$$

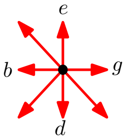
that is $|G| = 2n \in \{4, 6, 8, 10, 14, 16\}$, so $|G| = 12$ impossible.

Remaining case: $|G| = 10$.

Part 1b: groups of order 10



Possible models: After symmetries only one case ($r = \frac{2}{5}$).



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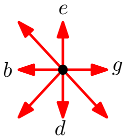

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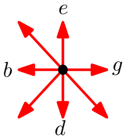

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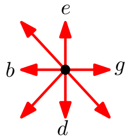

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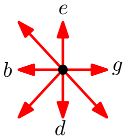

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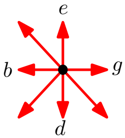

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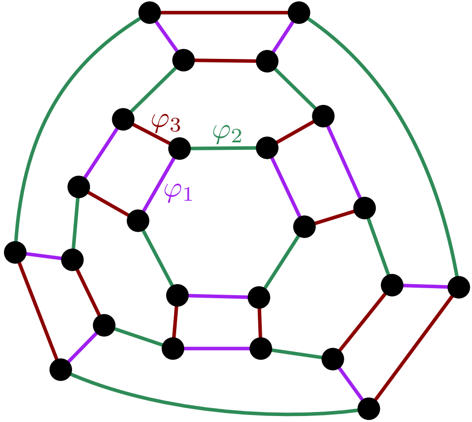
Comparing initial terms only solution is $(b, d, e, g) = (2, 2, 1, 1)$.

Part 2: Group in higher dimension



Group in general dimension

Model: fix $w(i_1, \ldots, i_d) \geq 0$ for $i_1, \ldots, i_d \in \{-1, 0, 1\}$

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Inventory of the model:

$$\begin{aligned}
\chi(x_1, \ldots, x_d) &= \sum_{i_1, \ldots, i_d \in \{-1,0,1\}} w(i_1, \ldots, i_d) x_1^{i_1} \cdots x_d^{i_d} \\
&= x_1 A_1(x_2, \ldots, x_d) + B_1(x_2, \ldots, x_d) + x_1^{-1} C_1(x_2, \ldots, x_d) \\
&= \cdots \\
&= x_d A_d(x_1, \ldots, x_{d-1}) + B_d(x_1, \ldots, x_{d-1}) + x_d^{-1} C_d(x_1, \ldots, x_{d-1}),
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Group: G of transformations $g:(\mathbb{R}^+)^d\to(\mathbb{R}^+)^d$ generated by

$$\left\{\begin{array}{ccc}\varphi_1([x_1,\ldots,x_d]) & = & \left[x_1^{-1}\frac{C_1(x_2,\ldots,x_d)}{A_1(x_2,\ldots,x_d)},x_2,\ldots,x_d\right],\\ & \cdots & \\ \varphi_d([x_1,\ldots,x_d]) & = & \left[x_1,\ldots,x_{d-1},x_d^{-1}\frac{C_d(x_1,\ldots,x_{d-1})}{A_d(x_1,\ldots,x_{d-1})}\right],\end{array}\right.$$

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which fix χ
Useful for counting walks in $(\mathbb{Z}^+)^d$.

Reflection group

From [Gohier, Humbert, Raschel 25]:
Fixed point: There is a unique vector $\Lambda \in (\mathbb{R}^+)^d$ fixed by each $g \in G$.
Group homomorphism:

$$J : \left| \begin{array}{ccc} G & \to & GL_d(\mathbb{R}) \\ g & \mapsto & \mathrm{Jac}_\Lambda\, g \end{array} \right.$$

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New results:

Theorem: $\ker(J)$ is trivial or infinite (in fact it is *torsion free*)

Corollary: If G is finite then it is isomorphic to $\mathrm{Im}(J)$.

FINITE GROUP CLASSIFICATION FOR $d \geq 3$

Method: G finite $\implies$ every pair of generators generates a finite group. Possible models classified in part 1.

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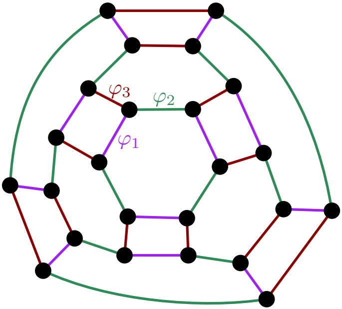
Problem: Lots of cases!
First steps: Let r_{ij} be the order of $\phi_i\phi_j$. We have classified groups where $(r_{12}, r_{13}, r_{23}) = (3, 2, 3)$ or $(3, 2, 4)$.



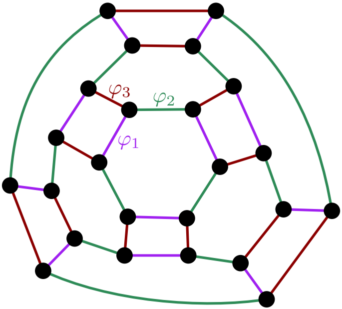
FINITE GROUPS

Groups with $(r_{12}, r_{13}, r_{23}) = (3, 2, 3)$:

$$\chi_S(x,y,z) = \frac{1}{yz} + \frac{2}{y} + \frac{z}{y} + \frac{x}{z} + x + \frac{y}{xz} + \frac{y}{x} + \frac{c}{z},$$

and

$$\chi_S(x,y,z) = a\left(x + \frac{y}{x} + \frac{z}{y} + \frac{1}{z}\right) + b\left(\frac{1}{x} + \frac{x}{y} + \frac{y}{z} + z\right) + c\left(y + \frac{1}{y} + \frac{xz}{y} + \frac{y}{xz} + \frac{z}{x} + \frac{x}{z}\right)$$



Groups with $(r_{12}, r_{13}, r_{23}) = (3, 2, 3)$**:**

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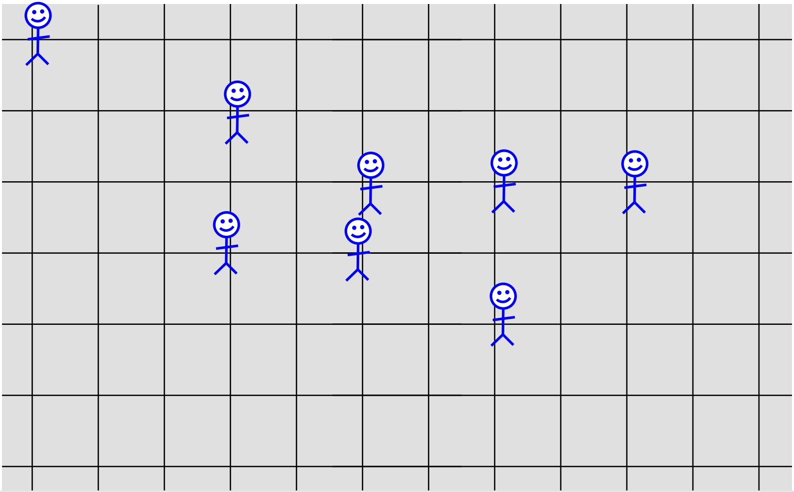

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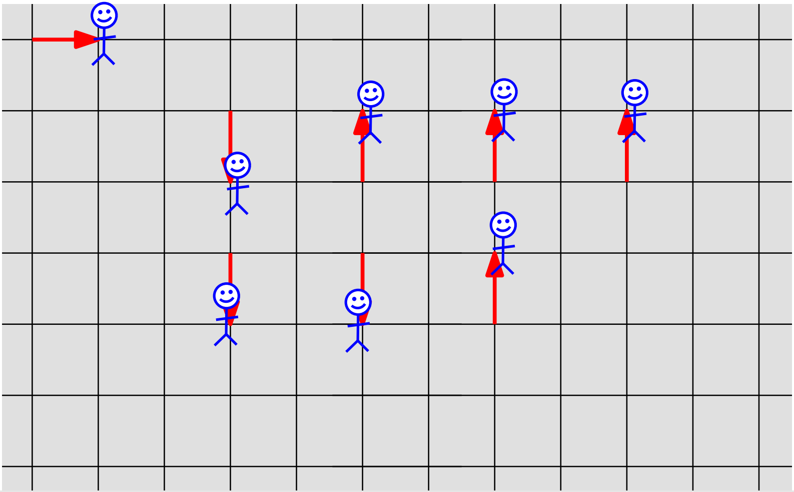

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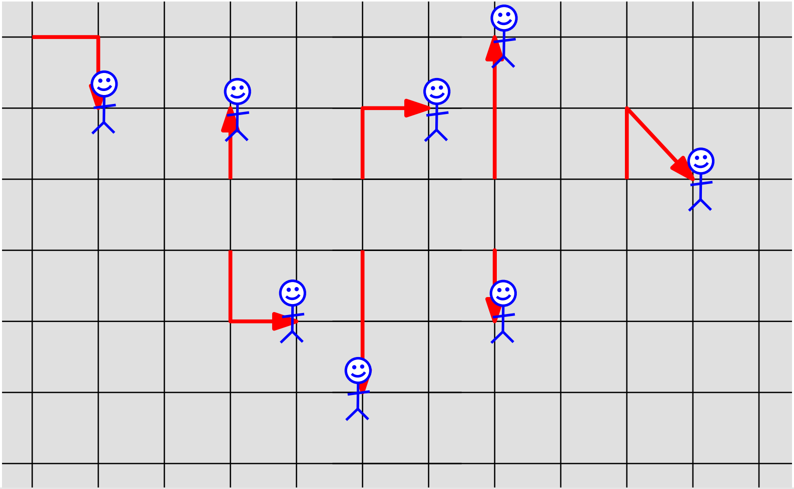

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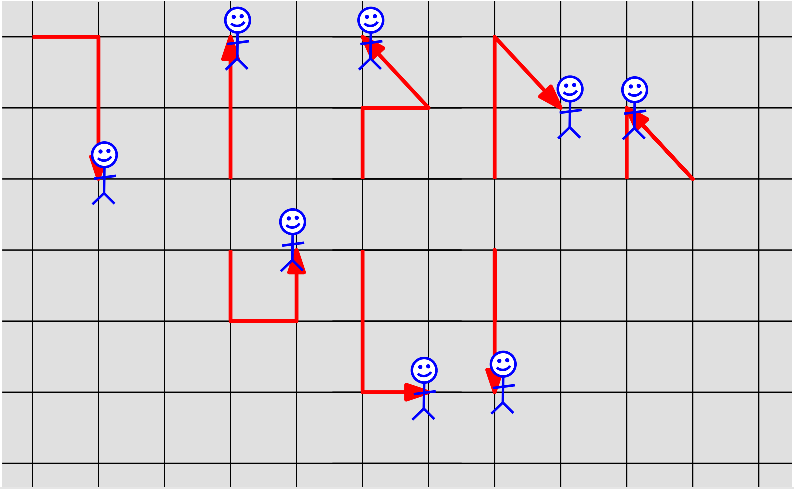

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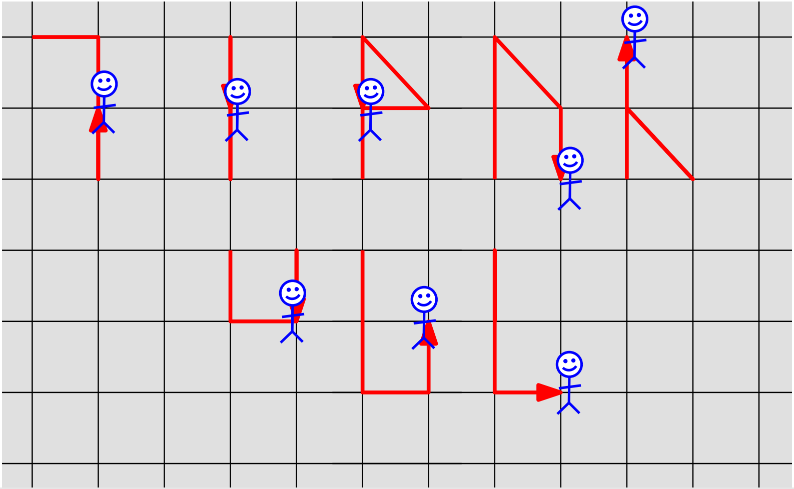

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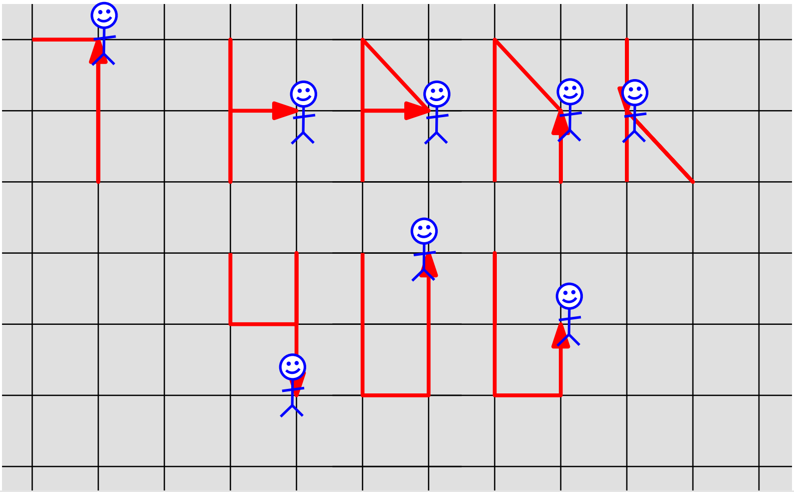

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