

Rowmotion and Echelonmotion

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Rene Marczinzik

Adrien Segovia

David E Speyer

Hugh Thomas

Nathan Williams

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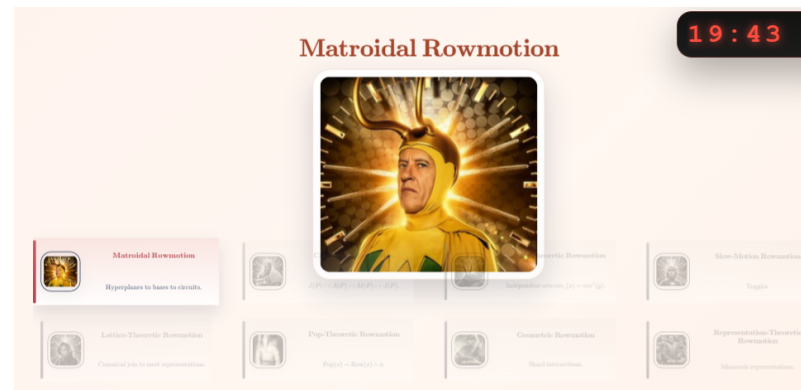
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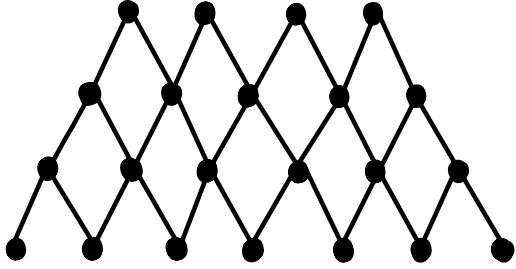
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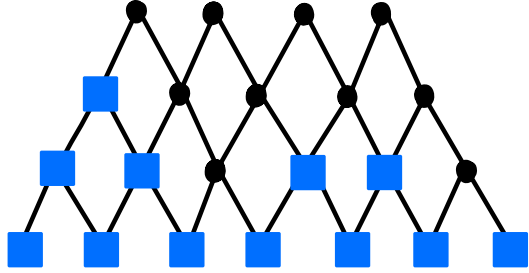
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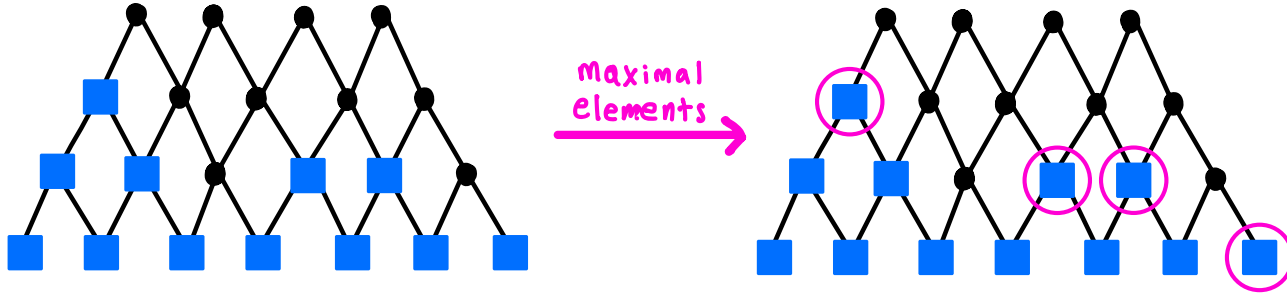
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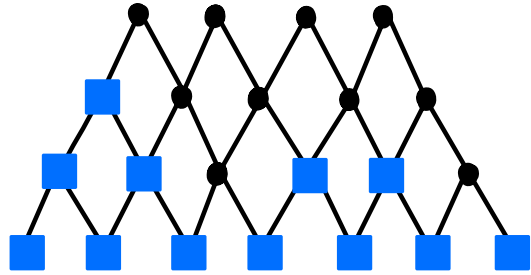
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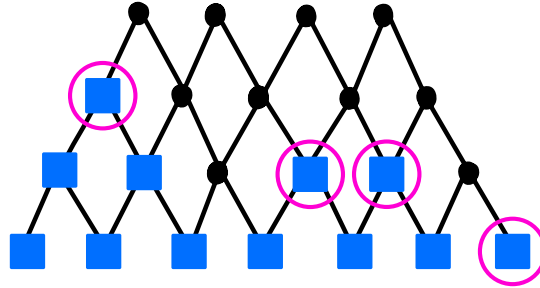
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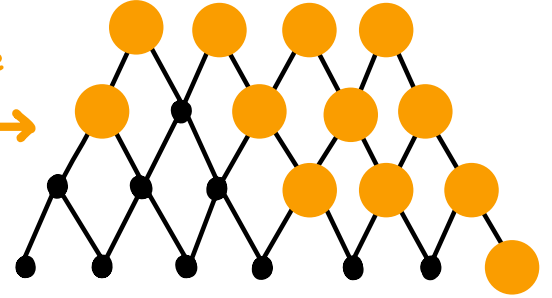
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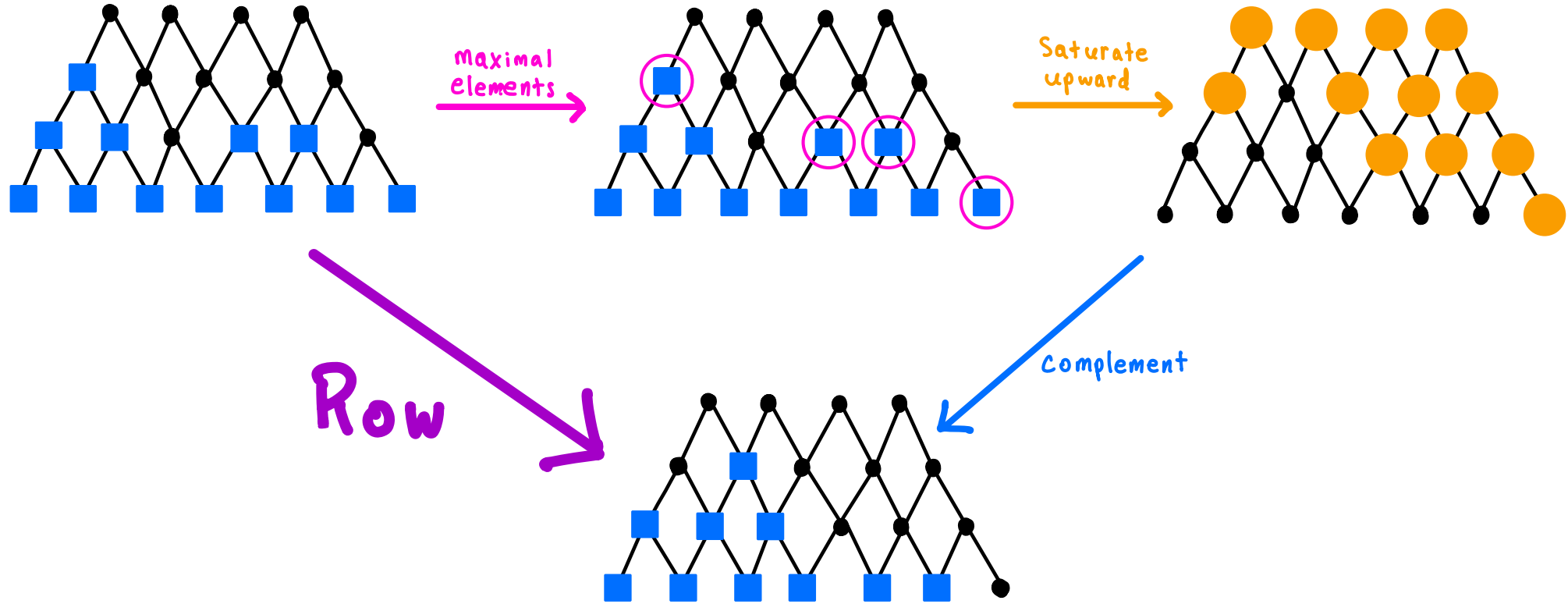
maximal
elements



Saturate
upward



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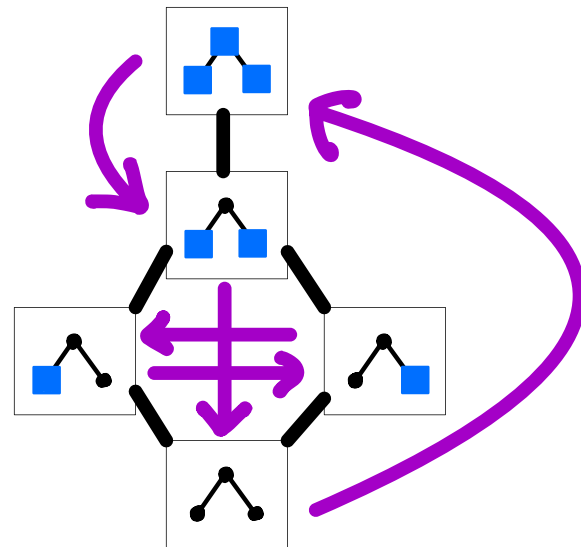


This defines the rowmotion bijection $\text{Row}: \mathcal{J}(Q) \rightarrow \mathcal{J}(Q)$.

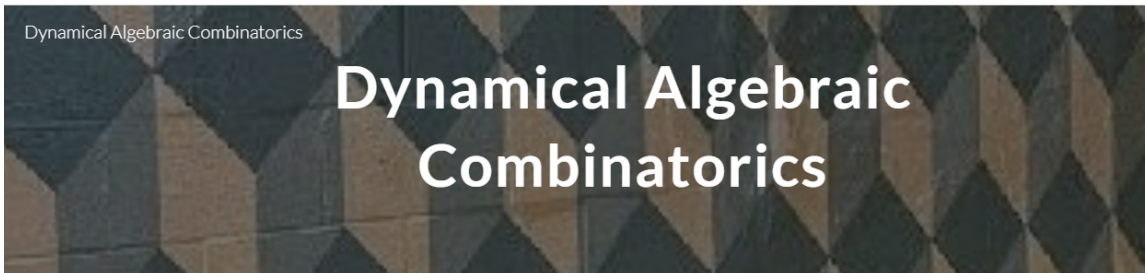
Dynamical Algebraic Combinatorics

Conferences and recorded talks

- [Special Session on Dynamical Algebraic Combinatorics](#), AMS Spring Central Section Meeting, Fargo, ND, April 18-19, 2026
- [Special Session on Dynamical Algebraic Combinatorics](#), AMS Spring Eastern Section Meeting, Hartford, CT, April 5-6, 2025
- [Special Session on Enumerative and Dynamical Combinatorics](#), AMS Spring Southeastern Sectional Meeting, Clemson, SC, March 8-9, 2025.
- [Dynamical Algebraic Combinatorics hybrid BIRS workshop](#), Kelowna, BC, Canada, Nov. 1-6, 2021. *[Talk recordings](#)
- [Dynamical Algebraic Combinatorics online BIRS workshop](#), Oct. 25-30, 2020. *[Talk recordings](#)
- [Special Session on Dynamical Systems and Algebraic Combinatorics](#), AMS Spring Central and Western Joint Sectional Meeting, Honolulu,



Order the
downward-closed
sets by inclusion.



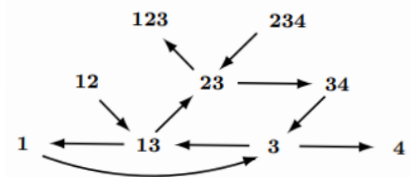
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MINUSCULE REVERSE PLANE PARTITIONS VIA QUIVER REPRESENTATIONS

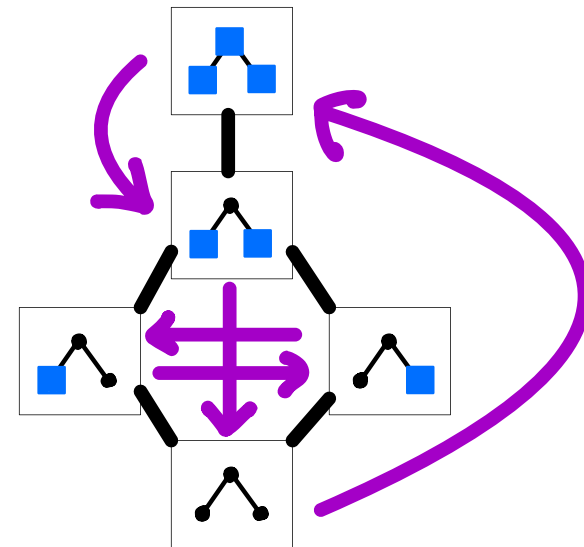
ALEXANDER GARVER, REBECCA PATRIAS, AND HUGH THOMAS

ON THE INTERACTION OF THE COXETER TRANSFORMATION AND THE ROWMOTION BIJECTION

RENÉ MARCZINZIK, HUGH THOMAS, AND EMINE YILDIRIM



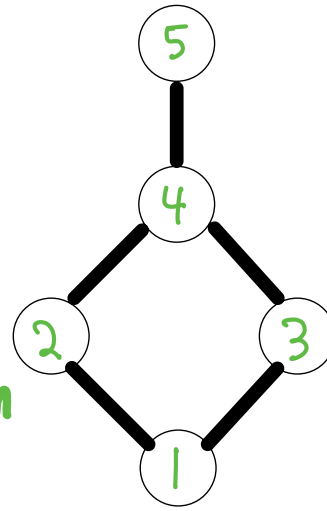
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“A quiver is a directed graph with pretensions.”

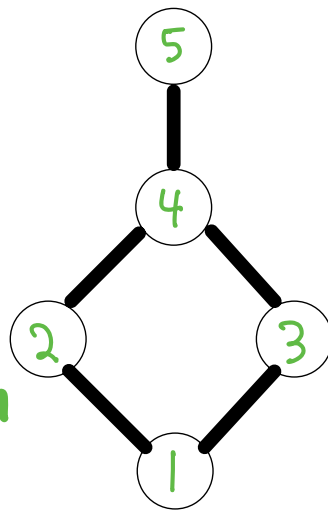
Let R be a poset with $|R|=n$.

Linear
extension
 ψ



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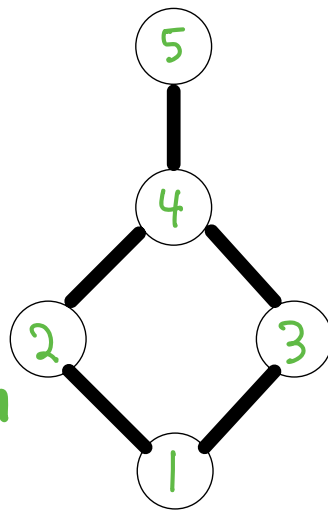
The Cartan matrix C
is given by

$$C_{ij} = \begin{cases} 1 & \text{if } i \geq j \text{ in } R \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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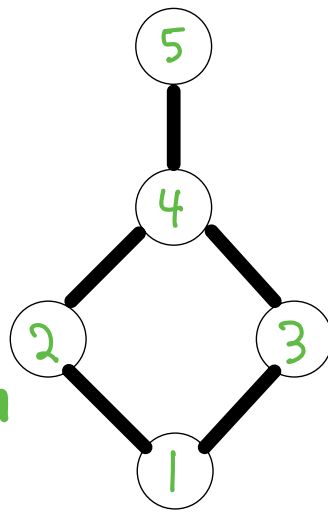
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$$5 \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$\uparrow$
3

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↑ 2

Theorem (Bruhat, 1954):

Every invertible $n \times n$ matrix M over $\mathbb{C}$ can be factored as $M = UV$, where U and V are upper-triangular and P is a permutation matrix. Moreover, P is unique.

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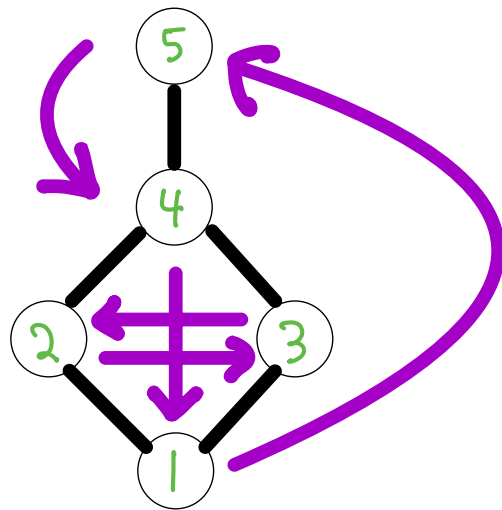
$\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$
 C U P V

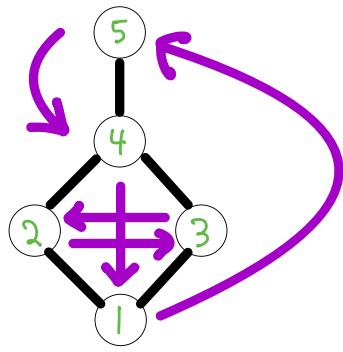
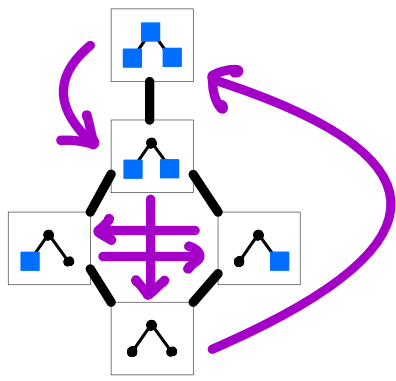
Using ψ , view P as a bijection $\text{Ech}_\psi: R \rightarrow R$,
called **echelonmotion**.

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Example:

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$



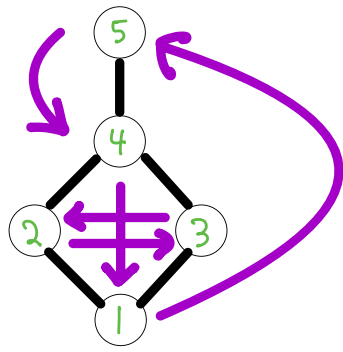
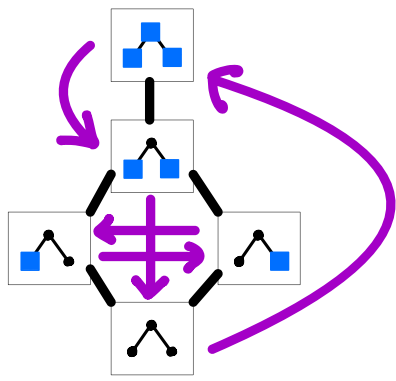


Recall: R is distributive if $R \cong J(Q)$ for some Q .

Theorem (Klász – Marczinzik – Thomas, 2025):

If $R \cong J(Q)$ is a distributive lattice and ψ is any linear extension of R , then $\text{Ech}_\psi = \text{Row}$.

So $C \rightsquigarrow P \rightsquigarrow \text{Row}!$



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Definition (DJMSSTW, 2026+):

Say a poset R is echelon-independent if Ech_ψ is independent of ψ .

Semidistributive Lattices

A lattice R is semidistributive if
for all $x, y, z \in R$,

$$x \vee y = x \vee z \implies x \vee y = x \vee (y \wedge z)$$

&

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Semidistributive Lattices

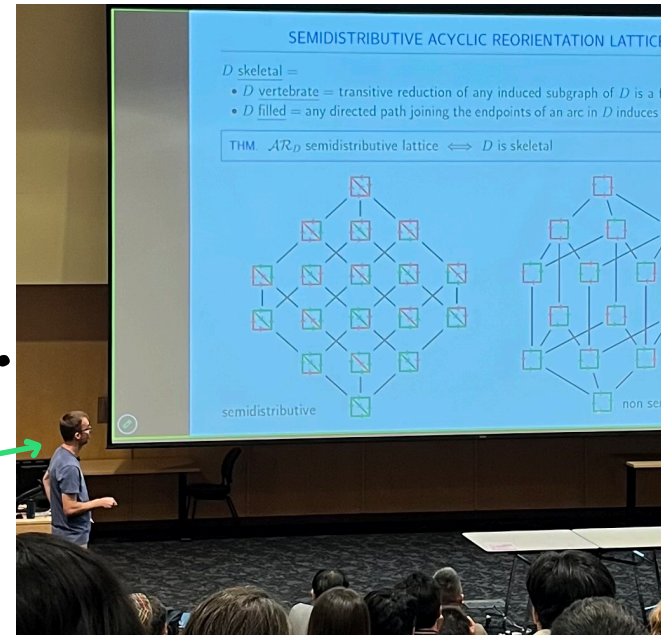
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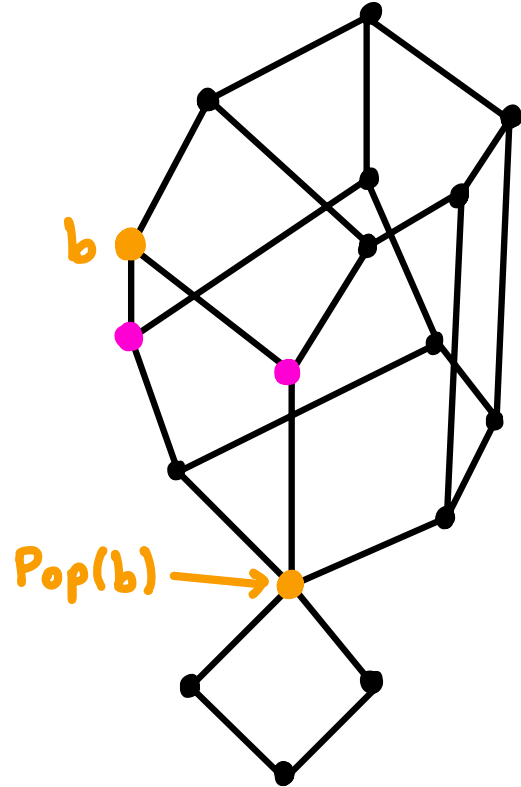
Vincent Pilaud



Let R be a semidistributive lattice.

For $b \in R$, let $\text{Pop}(b) = \bigwedge \{a \in R : a \leq b\}$.

$\bigwedge$ = meet = greatest lower bound



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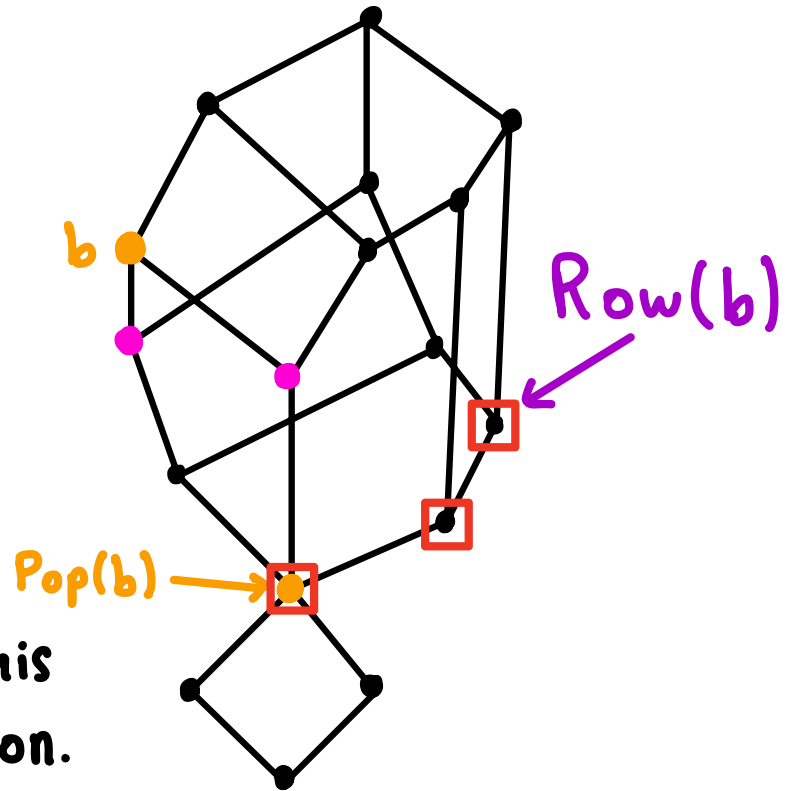
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$\bigwedge = \text{meet} = \text{greatest lower bound}$

The set $\{z \in R : b \wedge z = \text{Pop}(b)\}$ has a unique top element, which we define to be $\text{Row}(b)$.

not obvious, but not hard

When $R \cong J(Q)$ is distributive, this agrees with the earlier definition.



Theorem (DJMSSSTW, 2026+):

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Open Problem: Characterize echelon-independent
posets.

↖ not just lattices

Trim Lattices

"Definition"* (Thomas, 2006):

A lattice is trim if it's like it's distributive but not necessarily graded.

* Actual Definition: A lattice is trim if it is extremal and left-modular.

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"Definition"* (Thomas, 2006):

A lattice is trim if it's like it's distributive but not necessarily graded.

Thomas—Williams defined rowmotion "in slow motion" on trim lattices.

Theorem (DJMSSTW, 2026+):

Every trim lattice has a linear extension ψ such that $\text{Ech}_\psi = \text{Row}$.

* Actual Definition: A lattice is trim if it is extremal and left-modular.

Eulerian Posets

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Theorem (DJMSSTW, 2026+):

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Recall that R is Eulerian if it is graded with rank function $\text{rk}: R \rightarrow \mathbb{Z}$ and

$$\mu(x, y) = (-1)^{\text{rk}(y) - \text{rk}(x)}$$

for all $x \leq y$.

Proof: Define the diagonal matrix D by $D_{t,t} = (-1)^{\text{rk}(t)}$.

Check that $C^{-1} = DCD$. Factor $C = UPV$.

Then $C^{-1} = DUPVD$.

|| $\leftarrow$ uniqueness in Bruhat

Also $C^{-1} = V^{-1}P^{-1}U^{-1}$. So $P^{-1} = P \implies \text{Ech}_\psi^{-1} = \text{Ech}_\psi$. $\blacksquare$

Modular Lattices

Examples include

- the lattice of submodules of a module,
- the lattice of subspaces of a vector space over $\mathbb{F}_q$,
- distributive lattices.

Modular Lattices

Theorem (Dilworth, 1954):

Let R be a modular lattice.

For every $d \geq 0$,

$$|\{x \in R : x \text{ has down-degree } d\}| \\ = |\{x \in R : x \text{ has up-degree } d\}|.$$

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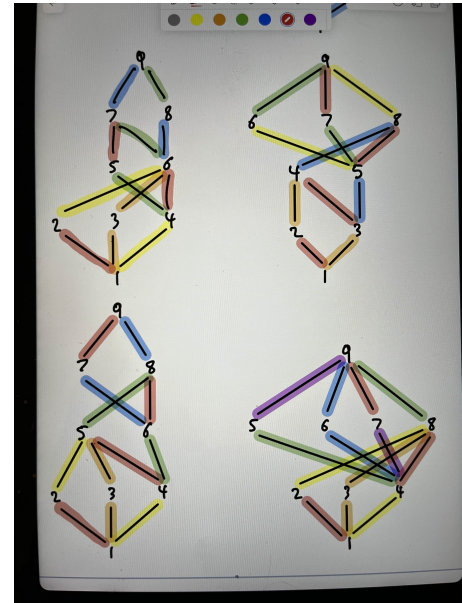
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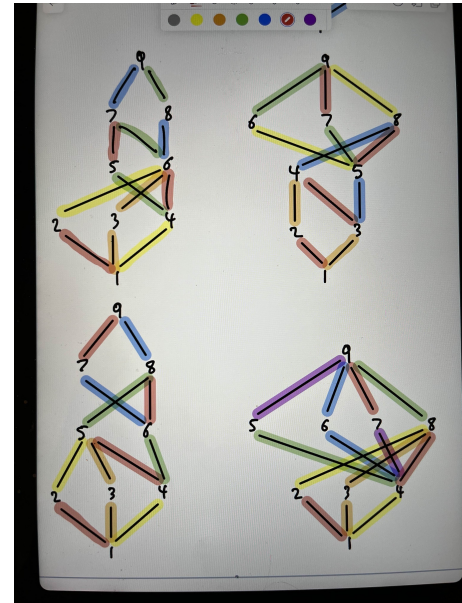
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Conjecture (DJMSSTW, 2026+):

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I want you to find a counterexample to Conjecture 8.5 from this paper.

Thought for 83m 16s >

I don't have a valid counterexample to Conjecture 8.5.



So I can't honestly give you a counterexample. The negative searches suggest that, if Conjecture 8.5 is false, the first counterexample is likely not in the small standard modular-lattice families I tested, or it is very sparse among linear extensions.

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Yes. The conjecture follows from a simple intertwining relation between the Cartan matrix and two upper-triangular “cover-counting” matrices. I'll write the proof in the notation of the paper: $W = W^{L,\sigma}$ is the Cartan matrix, $P = P^{L,\sigma}$ is the permutation matrix in its Bruhat decomposition, and $P_{\sigma(y),\sigma(x)} = 1$ means $\text{Ech}_\sigma(x) = y$.  2507.18230v1 (1)

Theorem

Let L be a finite modular lattice and let σ be any linear extension of L . Then, for every $x \in L$,

$$\left| \text{Cov}_L^\uparrow(\text{Ech}_\sigma(x)) \right| = \left| \text{Cov}_L^\downarrow(x) \right|.$$

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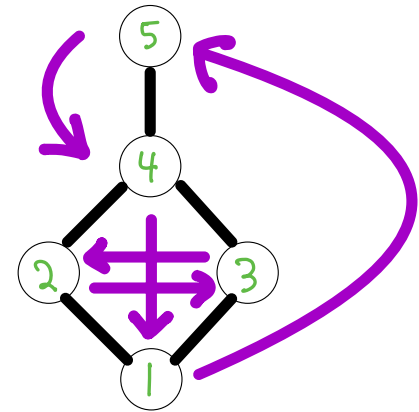
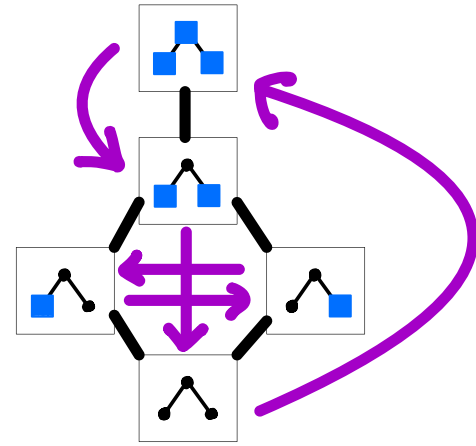
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THANK
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$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$