TREE METRICS & LOG-CONCAVITY FOR MATROIDS

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w/

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Motivation

Mason's conjectures:

M matroid of rank r on $[n]$

$I_k = \#$ independent sets of M w/ size k, $0 \leq k \leq r$

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E.g. $A \in K^{r \times n}$ full rank $\rightsquigarrow M$ rank r on $[n]$

$IN(M) =$ sets of linearly independent columns of A

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Mason's conjectures:

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$I_k = \#$ independent sets of M w/ size k, $0 \leq k \leq r$

(M1) $I_k^2 \geq I_{k-1} I_{k+1}$ log-concave

$\Uparrow$

(M2) $I_k^2 \geq \left(1+\frac{1}{k}\right) I_{k-1} I_{k+1}$ $(k!\, I_k)_k$

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(M3) $I_k^2 \geq \left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right) I_{k-1} I_{k+1}$ $\left(I_k / \binom{n}{k}\right)_k$

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(M1) $I_k^2 \geq I_{k-1} I_{k+1}$ realizable: Lenz '13, Huh-Katz '12; arbitrary: Adiprasito-Huh-Katz '18

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(M2) $I_k^2 \geq \left(1+\frac{1}{k}\right) I_{k-1} I_{k+1}$ Huh-Schröter-Wang '22

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(M3) $I_k^2 \geq \left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right) I_{k-1} I_{k+1}$

Lorentzian polynomials $\begin{cases} \text{Anari-Liu-Oveis Gharan-Vinzant '18} \\ \text{Brändén-Huh '20} \end{cases}$

1st Generalization: Polynomial Mason's

M matroid of rank r on $[n]$

$$I_k(x) = \sum_{\substack{|S|=k \\ \text{indepen.}}} x^S \in \mathbb{R}[x_1, \dots, x_n], \quad 0 \le k \le r$$

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Conjectured:

(PM1) $I_k(x)^2 \geq_m I_{k-1}(x) I_{k+1}(x)$) Dowling '80

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(PM2) $I_k(x)^2 \geq_m \left(1+\frac{1}{k}\right) I_{k-1}(x) I_{k+1}(x)$) Zhao '85

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False: $U_{2,2}$

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Eg: K field w. $\mathrm{val} : K \to \mathbb{R} \cup \{-\infty\}$

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$$A = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \end{array} \in \mathbb{Q}^{2 \times 3} \qquad \nu(13) = val_2\left(\det \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\right) = 1$$

2nd Generalization: Valuated Mason's

$v : IN(M) \longrightarrow \mathbb{R}$ valuated matroid

parameter $0<q<1$

$$I_{q;k} := \sum_{\substack{|S|=k \\ \text{indep.}}} q^{-v(S)} \quad , \quad 0 \leq k \leq r$$

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Giansiracusa - Rincón - Schleis - Ulirsch '24

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Conjectured

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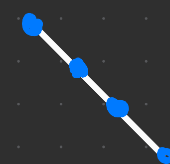
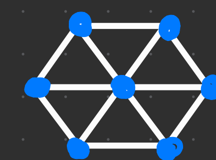
<u>$d>2$</u>: its support is M-convex





and every quadratic $\partial^\alpha f$ is Lorentzian

Lorentzian Polynomials & Log-concavity

(1) Bivariate Lorentzians:

$f(x,y) = a_0 x^n + a_1 x^{n-1} y + \dots + a_n y^n$ Lorentzian

$\Longleftrightarrow \quad \left(a_i / \binom{n}{i} \right)_{i=0}^{n}$ has no internal zeros & is log-concave.

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(2) Specialization:

Lorentzian polynomials are preserved by non-negative change of coordinates

Main Theorem: ACDEHW '26

A function $u: 2^{[n]} \to \mathbb{R} \cup \{-\infty\}$ satisfies the gross substitutes property iff for all $0 < q \leq 1$

$$Z_{q;u}(x) = \sum_{S \subseteq [n]} q^{-u(S)} x_0^{n-|S|} x^S \in \mathbb{R}_{\geq 0}[x_0, \dots, x_n]$$

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Examples:
- log-indicator of $ID(M)$
- matroid rank functions
- valuated matroids

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Corollaries: (VM3) & (PM2) hold!

Tree metrics

T a rooted ultrametric tree with n leaves

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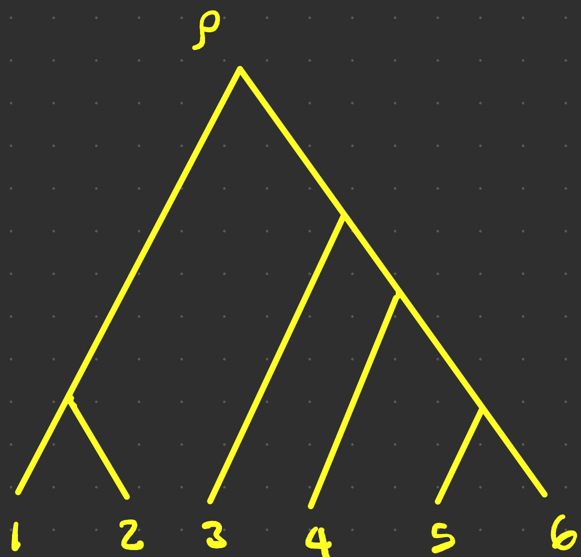

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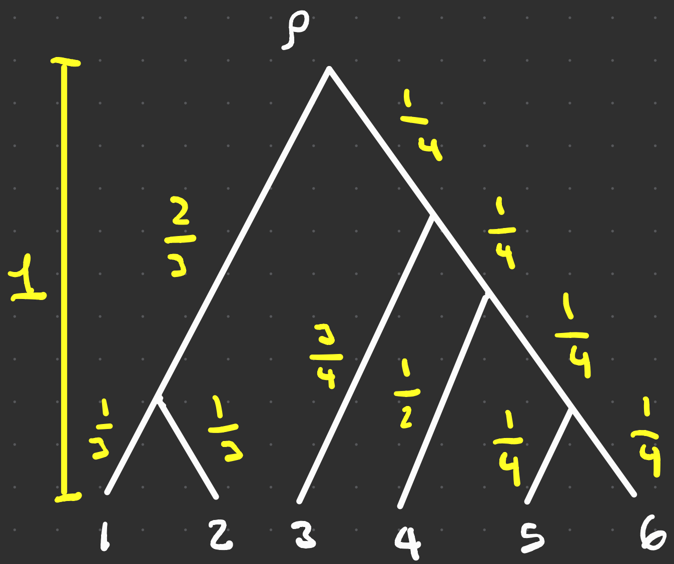

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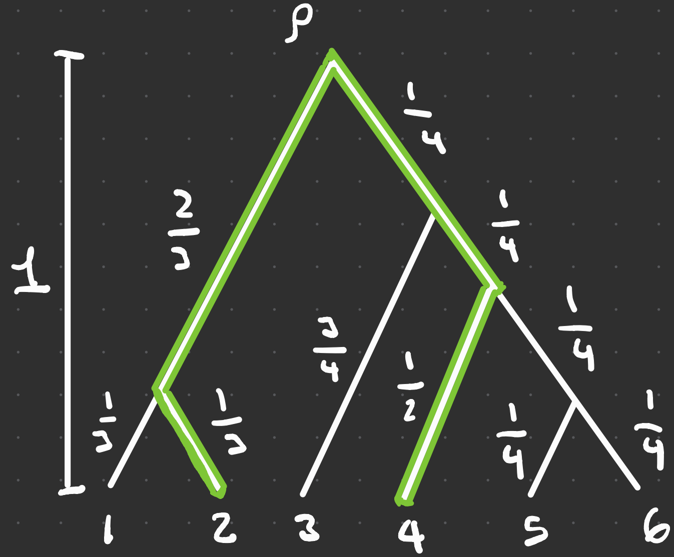

$$D_T := (d_{ij})_{ij}$$

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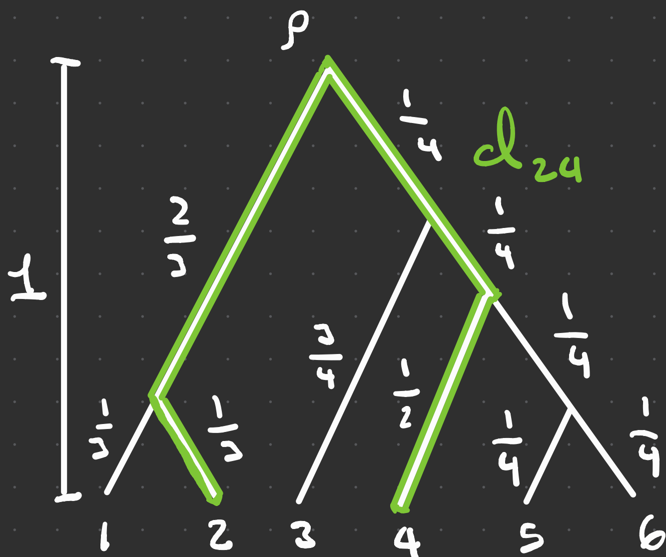

$$D_T := (d_{ij})_{ij} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 2/3 & 2 & 2 & 2 & 2 \\ 2/3 & 0 & 2 & 2 & 2 & 2 \\ 2 & 2 & 0 & 3/2 & 3/2 & 3/2 \\ 2 & 2 & 3/2 & 0 & 1 & 1 \\ 2 & 2 & 3/2 & 1 & 0 & 1/2 \\ 2 & 2 & 3/2 & 1 & 1/2 & 0 \end{pmatrix} \end{array}$$

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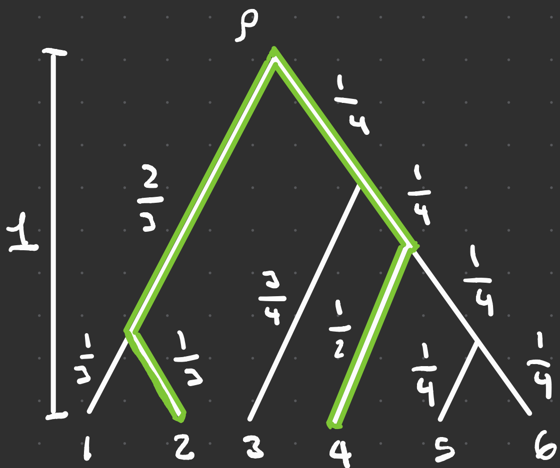

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Eigenvalues: $7.93, -0.5, -0.66, -1.14, -1.8, -3.82$

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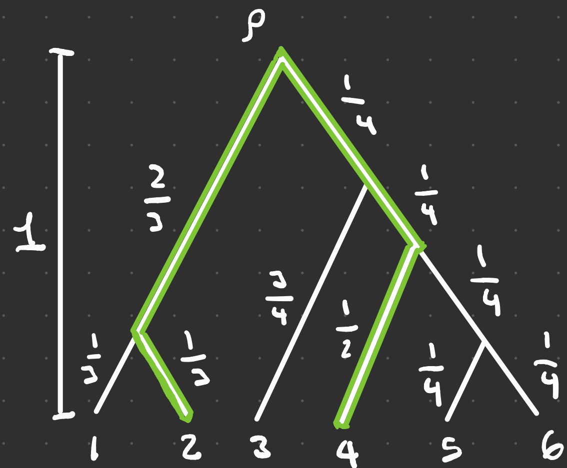

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Graham - Pollack '71: At most one positive eigenvalue

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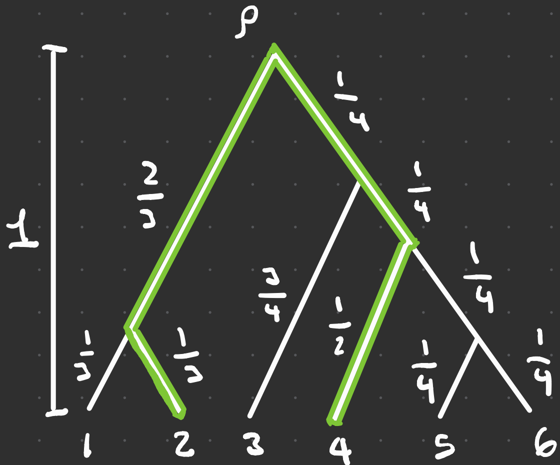

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Schoenberg '35: $D_T|_{\mathbb{1}_n^\perp}$ is neg. semidefinite.

Tree metrics

$$2\left(1-\frac{1}{6}\right)\mathbb{1}_{6\times 6}-D_T=\begin{pmatrix} 5/3 & 1 & -1/3 & -1/3 & -1/3 & -1/3 \\ 1 & 5/3 & -1/3 & -1/3 & -1/3 & -1/3 \\ -1/3 & -1/3 & 5/3 & 1/6 & 1/6 & 1/6 \\ -1/3 & -1/3 & 1/6 & 5/3 & 2/3 & 2/3 \\ -1/3 & -1/3 & 1/6 & 2/3 & 5/3 & 7/6 \\ -1/3 & -1/3 & 1/6 & 2/3 & 7/6 & 5/3 \end{pmatrix}$$

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4, 2.21, 1.47, 1.13, $2/3$, $1/2$

Thm (ACDEHW): T ultrametric tree of radius 1

$$\Longrightarrow \quad 2\left(1-\frac{1}{n}\right)\mathbb{1}_{n\times n} - D_T \text{ is PSD}$$

and $2\left(1-\frac{1}{n}\right)$ is the smallest constant wrto this property.

Thank You!