Ehrhart series of alcoved polytopes

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Plan

Background on Ehrhart theory and hypersimplices

More general alcoved setting

Application to $\Delta_{2,n}$

Background on Ehrhart theory and hypersimplices

Ehrhart theory

Let $P \subset \mathbb{R}^n$ be a rational convex polytope

- *Ehrhart quasipolynomial* [Ehrhart '62]:
  $E(P, t) = \#(t \cdot P) \cap \mathbb{Z}^n$ for $t \in \mathbb{Z}_{\geq 0}$
- *Ehrhart series*: $\mathrm{Ehr}(P, z) = \sum_{t=0}^{\infty} E(P, t) z^t$ is a rational function in z
- h^*-polynomial: when P is a lattice polytope, the numerator of $\mathrm{Ehr}(P, z) = \frac{h^*(P,z)}{(1-z)^{n+1}}$; $\deg h^* \leq n$ and h^* has nonnegative integer coefficients [Stanley '80]

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Motivating example: hypersimplices

Hypersimplex $\Delta_{k,n} = \mathrm{conv}(e_I \mid I \in \binom{[n]}{k})$

- Katzman ('05) computed the Ehrhart polynomial of the hypersimplices. Ferroni ('21) showed that the hypersimplices are Ehrhart positive.
- Nan Li ('11) computed the h^*-polynomial of the half-open hypersimplices.
- Nick Early ('17) conjectured that the h^*-polynomial of hypersimplices are counted by *hypersimplicial decorated ordered set partitions* (OSP), and Donghyun Kim ('20) proved it.

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h^* polynomials of hypersimplices

Definition (Ocneanu, Early)

- A *decorated ordered set partition* $(\mathbf{S}, \mathbf{r}) = ((S_1)_{r_1}, \ldots, (S_d)_{r_d})$ of type (k, n) consists of a cyclically ordered set partition $(S_1, \ldots, S_d)$ of $[n]$ and a d-tuple of positive integers $(r_1, \ldots, r_d)$ that sum up to k.
- A decorated ordered set partition is *hypersimplicial* if $r_i \leq |S_i| - 1$ for all i.
- The *winding vector* of a decorated ordered set partition is an n-tuple of integers $(l_1, \ldots, l_n)$ such that $l_i = r_k + \cdots + r_{\ell-1}$ if $i \in S_k$ and $i + 1 \in S_\ell$.
- The *winding number* is equal to $(l_1 + \cdots + l_n)/k$.

We denote the set of hypersimplicial decorated ordered set partitions of type (k, n) by $\mathrm{OSP}(\Delta_{k,n})$.

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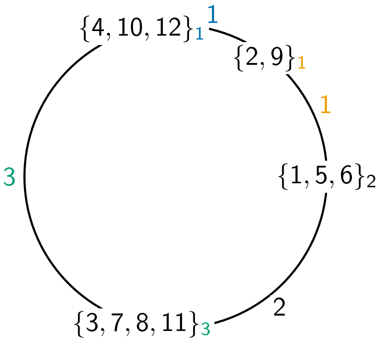


$$(\mathbf{S}, \mathbf{r}) = (\{1,5,6\}_2, \{3,7,8,11\}_3, \{4,10,12\}_1, \{2,9\}_1)$$

Winding vector: (6,3,3,2,0,2,0,4,6,4,3,2)

Winding number: 35/7=5

h^* polynomials of hypersimplices

Theorem (Conjectured by Early, Proof by Kim)

The number of hypersimplicial decorated ordered set partitions of type (k, n) and winding number d is equal to the d-th entry in the h^* vector of $\Delta_{k,n}$.

winding number	$\mathrm{OSP}(\Delta_{2,4})$
0	$(\{1234\}_2)$
1	$(\{12\}_1\{34\}_1)$
1	$(\{14\}_1\{23\}_1)$
2	$(\{13\}_1\{24\}_1)$

The h^*-polynomial of the octahedron $\Delta_{2,4}$ is $1 + 2z + z^2$.

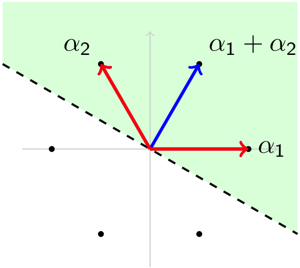
Question

Is there a bijective proof of the OSP formula for hypersimplices?

More general alcoved setting

Brief overview of Coxeter arrangements

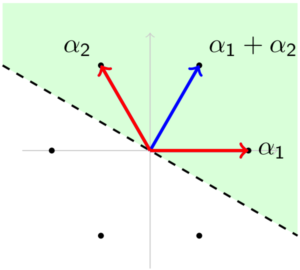
A root system is a finite subset $\Phi \subset \mathbb{R}^n$ satisfying some geometric conditions.



Important data for our uses:

1. Basis of simple roots $\alpha_1, \ldots, \alpha_n \in \Phi^+$ that cannot be written as sum of two positive roots
2. Fundamental coweights $\omega_1, \ldots, \omega_n$ such that $(\omega_i, \alpha_j) = \delta_{ij}$
3. Unique highest root $\Theta \in \Phi^+$ that maximizes $(\omega_1 + \ldots + \omega_n, \Theta)$. We take $\alpha_0 = -\Theta$, $a_0 = 1$, and define all other a_i via $\Theta = a_1\alpha_1 + \ldots + a_n\alpha_n$.

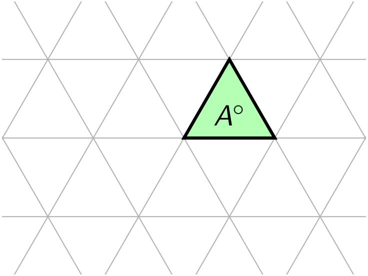
Type A_2 root system lives in the hyperplane $\{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$



Important data for our uses:

1. Simple roots $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3$
2. Fundamental coweights $\omega_1 = \frac{1}{3}(2, -1, -1), \omega_2 = \frac{1}{3}(1, 1, -2)$
3. Highest root $\Theta = \alpha_1 + \alpha_2 = e_1 - e_3$. $a_0 = a_1 = a_2 = 1$

Coxeter arrangements



- Hyperplanes $H_{\alpha,k} = \{\lambda \in \mathbb{R}^n \mid (\lambda, \alpha) = k\}$, where $\alpha \in \Phi^+, k \in \mathbb{Z}$
- The regions of the affine Coxeter arrangements are simplices called *alcoves*
- The fundamental alcove A° is the convex hull of $0, \omega_1/a_1, \ldots, \omega_n/a_n$

Alcoved polytopes

Definition (Lam, Postnikov 2012)

An alcoved polytope is a convex union of closed alcoves, ie. its defining hyperplanes are of the form $H_{\alpha,k}$.

Many examples:

- hypersimplices
- order polytopes
- positroid polytopes
- generalized associahedra

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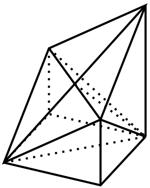

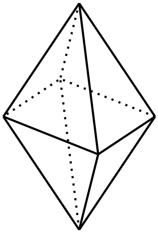

Formula for the Ehrhart series of alcoved polytopes

Let P be an alcoved polytope

- Let Γ_P be the dual graph of the alcoved triangulation of P, with edge weights $\mathrm{wt}((A, A'))$ given by ℓ_i's (defined later)
- Fix an arbitrary alcove A_0 in P, we define a partial order $\prec_{A_0}$ on Γ_P with minimal element A_0 (defined later)

Proposition (B.–Jiang '24)

$$\mathrm{Ehr}(P, z) = \frac{\sum_{alcove\ A \subset P} z^{\mathrm{wt}(A)}}{\prod_{i=0}^{n}(1 - z^{\ell_i})}$$

where $\mathrm{wt}(A) = \sum_{A \prec\cdot_{A_0} A'} \mathrm{wt}((A, A'))$ *is the sum of the weights of the edges between* A *and the alcoves it covers.*

- For Type A and integer "sliced polytopes," this result is documented in (Benedetti, Knauer, Valencia Porras 2023), (Valencia Porras 2021).

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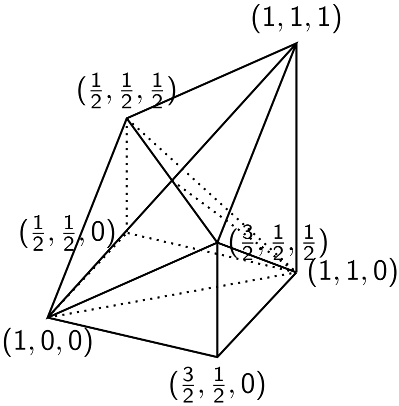
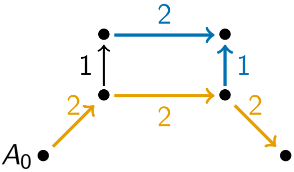




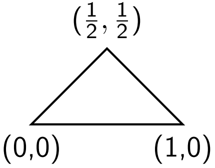
An alcoved polytope P for $\Phi = B_3$. The arrows indicate cover relations in the poset $\prec_{A_0}$ where A_0 is chosen to be the lower left alcove. The Ehrhart series is

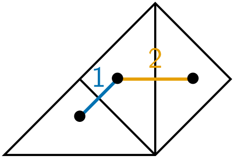
$$\mathrm{Ehr}(P,z) = \frac{1 + z + 3z^{2} + z^{2+1}}{(1-z)^2(1-z^2)^2}.$$

Defining edge weights

Let P be an alcoved polytope. We construct a weighted graph Γ_P such that

- Let ℓ_i be the smallest positive integer such that $\ell_i \omega_i / a_i$ is an integer point
- vertices: closed alcoves $A \subset P$
- edges: (A, A') if A and A' share a common facet
- edge weights: $\mathrm{wt}((A, A')) = \ell_i$ if the facet $F = A \cap A'$ is transformed to a facet F° of the fundamental alcove A° under the action of the affine Weyl group such that ω_i / a_i is the vertex of A° that does not belong to F°





Breadth-first search order

Let $\Gamma = (V, E)$ be an undirected graph, and $v_0 \in V$ be a vertex of Γ.

- The *breadth-first search order of* Γ *with root* v_0 is the partial order $\prec_{v_0}$ on V such that $u \prec_{v_0} v$ if and only if there is a shortest path from v_0 to v passing through u, for $u, v \in V$
- For alcoved polytopes, if we take v_0 to be the fundamental alcove, this is weak order of affine Weyl group

Lemma

Any linear extension of the breadth-first search order is a shelling order of the alcoved triangulation complex of P.

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Application to $\Delta_{2,n}$

- Recall $\Delta_{k,n} = \mathrm{conv}(e_I \mid I \in \binom{[n]}{k})$
- $\Delta_{k,n}$ can be realized as an alcoved polytope under the linear transformation $y_i = x_1 + \cdots + x_i$
- The volume of $\Delta_{k,n}$ is the Eulerian number $A(k-1, n-1)$

The relation between $OSP(\Delta_{2,n})$ and $h^*(\Delta_{2,n})$

- For hypersimplices of type A, our main result simplifies to

$$h^*(\Delta_{k,n}, z) = \sum_{\text{alcove } A \subset P} z^{\mathrm{cover}(A)}$$

  where $\mathrm{cover}(A)$ is the number of elements A covers in the breadth-first order of $\Gamma_{k,n}$ with arbitrary root alcove A_0.
- Early and Kim's formula:

$$h^*(\Delta_{k,n}, z) = \sum_{(\mathbf{S},\mathbf{r}) \in \mathrm{OSP}(\Delta_{k,n})} z^{\mathrm{wind}((\mathbf{S},\mathbf{r}))}$$

  where $\mathrm{wind}((\mathbf{S}, \mathbf{r}))$ is the winding number of $(\mathbf{S}, \mathbf{r})$.

Is there a bijective proof showing the equivalence of these two formulas?

Relation between $OSP(\Delta_{2,n})$ and $h^*(\Delta_{2,n})$

We found a bijection when $k = 2$.

- $\mathrm{OSP}(\Delta_{2,n}) \setminus ([n]_2)$ consists of pairs of a nonempty proper subset $T \subsetneq [n]$ that is not a singleton and its complement T^c, both decorated by 1
- Let $A, A' \subseteq \Delta_{2,n}$ be two adjacent alcoves. The hyperplane containing the facet $A \cap A'$ is defined by $y_j - y_i = 1$ for some $i \not\equiv j \pm 1 \bmod n$
- We associate to the facet $A \cap A'$ the pair of intervals $([i+1, j]_1, [j+1, i]_1) \in \mathrm{OSP}(\Delta_{2,n})$ with winding number 1, which is an edge labeling of the dual graph $\Gamma_{k,n}$ of the alcoved triangulation of $\Delta_{k,n}$. We'll also call this edge label $\{i, j\}$.

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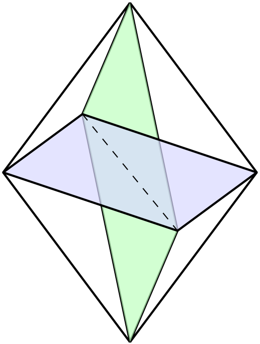
Example



$$\begin{tikzcd}
\bullet \arrow[r, "((12)_1(34)_1)"] & \bullet \\
\bullet \arrow[u, "((14)_1(23)_1)"] \arrow[r, "((12)_1(34)_1)"'] & \bullet \arrow[u, "((14)_1(23)_1)"']
\end{tikzcd}$$

Defining a bijection

We want a bijection between the subset of $\mathrm{OSP}(\Delta_{2,n})$ with winding number d and the set of alcoves that cover d other alcoves in the breadth-first search order with root A_0.

- We assign A_0 with the unique OSP $([n]_2)$ with winding number 0.
- The set of alcoves that cover d alcoves through d edges correspond to d OSPs with winding number 1.
- Define a map ψ from the set of d OSPs with winding number 1 to the set of OSPs with winding number d as follows:
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Theorem (B.–Jiang '24)

Applying the map ψ to the set of incoming edge labels of an alcove gives a bijection from the set of alcoves A with $\mathrm{cover}(A) = d$ and $\mathrm{OSP}(\Delta_{2,n})$ with winding number d.

Defining a bijection

We want a bijection between the subset of $\mathrm{OSP}(\Delta_{2,n})$ with winding number d and the set of alcoves that cover d other alcoves in the breadth-first search order with root A_0.

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Applying the map ψ to the set of incoming edge labels of an alcove gives a bijection from the set of alcoves A with $\mathrm{cover}(A) = d$ *and* $\mathrm{OSP}(\Delta_{2,n})$ *with winding number d.*

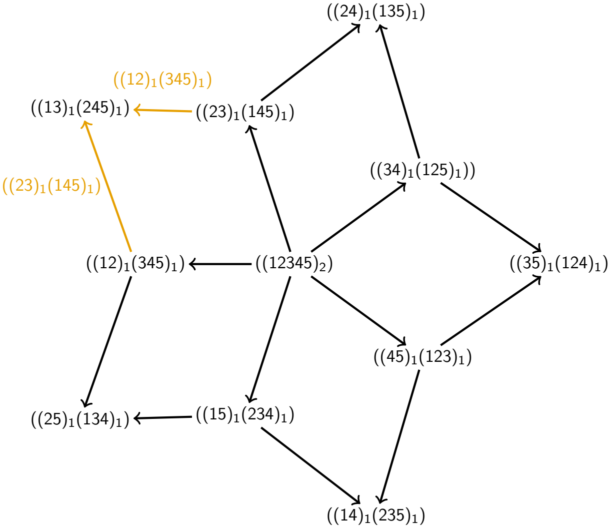

Figure: $\Delta_{2,5}$ with A_0 at the center. The arrows indicate cover relations in $\prec_{A_0}$, pointing in increasing directions. The orange arrows are facets representing the cover relations of the alcove labeled by $((13)_1(245)_1)$.

Proof overview

1. Characterize the possible sets of incoming edge labels. These must be *crossing*, meaning that if $\{i_1, j_1\}$ and $\{i_2, j_2\}$ are incoming edge labels for some alcove A, then the corresponding arcs in the circle cross.
2. Show that the map from sets of pairwise crossing edge labels to decorated ordered set partitions is injective.
3. Show that the map from alcoves to sets of incoming edge labels is injective.

Once we have these facts we are done, since we know from (Ocneanu 2013) that the number of decorated ordered set partitions of type (k, n) is equal to the Eulerian number $A(k-1, n-1)$.

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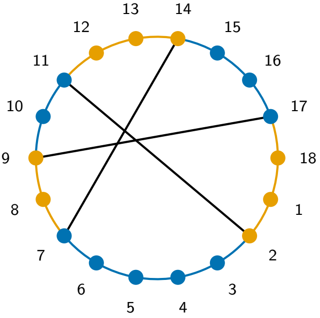
What sets of incoming edge labels appear?

Incoming edge labels must be crossing:



Applying ψ maps to the dOSP (*orange*$_1$, *blue*$_1$).

Labelling alcoves in $\Delta_{k,n}$

Theorem (Lam, Postnikov 2005)

- *alcoves of $\Delta_{k,n}$ are in bijection with permutations $w \in \mathcal{S}_n$ modulo cycle shifts (ie. $[w_1, \dots, w_n] \sim [w_n, w_1, \dots, w_{n-1}]$) such that if we take the representative of w with $w_n = n$, w^{-1} has $k-1$ descents*
- *write $(w) = (w_1, \dots, w_n)$ to denote the corresponding long cycle in $\mathcal{S}_n$, $\Delta_{(w)}$ denotes the corresponding alcove*
- *$\Delta_{(u)}$ and $\Delta_{(w)}$ are adjacent in $\Gamma_{k,n}$ if and only if there exists $i \in [n]$ such that $u_i - u_{i+1} \neq \pm 1 (mod\ n)$ and the cycle (w) is obtained from (u) by switching the positions of u_i and u_{i+1}.*

Locally determining edge orientations from permutations

Definition

Let $w \in \mathcal{S}_n$. We say that $i \in [n]$ is a *left cyclic descent* if $i < n$ and $w^{-1}(i) > w^{-1}(i+1)$ or if $i = n$ and $w^{-1}(1) < w^{-1}(n)$. We write $\mathrm{cDes}_L(w)$ for the set of left cyclic descents of w.

Definition

For $w \in \mathcal{S}_n$, define $w^{(c)}$ to be the rotation of w ending in c.

Locally determining edge orientations from permutations

Definition

Let $i \in [n]$. The *i-order* $<_i$ on the set $[n]$ is the total order

$$i <_i i+1 <_i \cdots <_i n <_i 1 <_i \cdots <_i i-2 <_i i-1.$$

The i^{th} *Gale order* on $\binom{[n]}{k}$ is the partial order $\leq_i$ defined as follows: for any two k-subsets $S = \{s_1 <_i \cdots <_i s_k\} \subseteq [n]$ and $T = \{t_1 <_i \cdots <_i t_k\} \subseteq [n]$, we have $S \leq_i T$ if and only if $s_j \leq_i t_j$ for all $j \in [k]$.

Lemma (B.–Jiang '24)

Let A be an alcove in $\Delta_{2,n}$. For any $\{i, j\}$ in the incoming edge label set of A, such that $w_A^{-1}(i) + 1 \equiv w_A^{-1}(j) \pmod{n}$, then we have $\mathrm{cDes}_L(w_{A_0}^{(j)}) <_j \mathrm{cDes}_L(w_A^{(j)})$ and $\mathrm{cDes}_L(w_{A_0}^{(i)}) >_i \mathrm{cDes}_L(w_A^{(i)})$ in the Gale order on $\binom{[n]}{2}$ with respect to $<_j$ and $<_i$.

Example

- Consider $\Delta_{2,15}$,
  $(w_{A_0}) = (1, 2, 3, 4, 5, 10, 6, 7, 11, 8, 12, 9, 13, 14, 15)$.
- Say alcove A has incoming edge label $\{1, 8\}$
- $\mathrm{cDes}_L(12, 9, 13, 14, 15, 1, 2, 3, 4, 5, 10, 6, 7, 11, 8) = \{8, 11\}$
- Either $(w_A) = (\ldots, 1, 8)$ or $(w_A) = (1, \ldots, 8)$. The first has $\mathrm{cDes}_L(w_A^{(8)}) = \{1, 8\}$, while the second has $\mathrm{cDes}_L(w_A^{(8)}) = \{15, 8\}$
- $11 <_8 15 <_8 1$, so we must be in the first case

Further directions

- Can our results be extended to $\Delta_{k,n}$?
- Can we generalize the dOSP picture for the hypersimplex to other alcoved polytopes?

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References

Thank you!

C. Benedetti, K. Knaur, J. Valencia Porras, "On lattice path matroid polytopes: alcoved triangulations and snake decompositions," arXiv: 2303.10458 (2023).

E. Bullock, Y. Jiang, "The Ehrhart series of alcoved polytopes," arXiv: 2412.02787 (2024).

N. Early., "Conjectures for Ehrhart h^*-vectors of Hypersimplices and Dilated Simplices," arXiv: 1710.09507 (2017).

N. Early., "From weakly separated collections to matroid subdivisions," Combinatorial Theory 2, no. 2 (2022).

D. Kim, "A combinatorial formula for the Ehrhart h^*-vector of the hypersimplex," Journal of Combinatorial Theory, Series A, **173** (2020).

T. Lam, A. Postnikov, "Alcoved polytopes II," arXiv: 1202.4015 (2012).

A. Ocneanu, "On the inner structure of a permutation: Bicolored Partitions and Eulerians, Trees and Primitives," arXiv: 1304.1263 (2023).

J. Valencia Porras, "Ehrhart theory of lattice path matroid polytopes," Universidad de los Andes (2021).