Crossing, or Not, in the Quasisymmetric World

Nantel Bergeron
York University

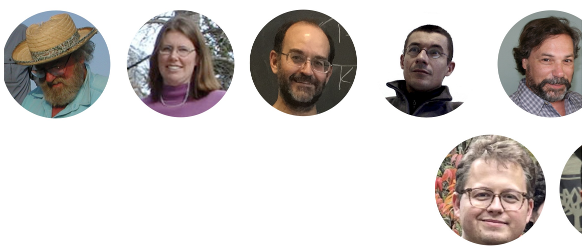
Joint work with:

and many others...

FPSAC, Seattle – July 2026

Crossi
in the Quasi

Nai
Y
J



an

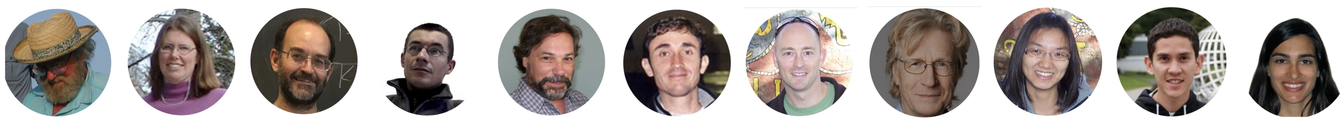
Aguiar, Marcelo André, Carlos A. M. Assaf, Sami H. Aval, Jean-Christophe Baker-Jarvis, Duff Barcelo, Helene Marie-Louise Benedetti Velásquez, Carolina Berg, Christopher James Bergeron, François Bhargava, Sandeep Billey, Sara Cosette Cartier, Noémie Ceballos, Cesar Chan, Kelvin Chen, Zhi[5] Choquette, Philippe Colmenarejo Hernando, Laura Cruz, Lilian J. Darewych, Jurij W. Dermenjian, Aram Descouens, François Diaconis, Persi W. Emami-Razavi, Mohsen Gagnon, Lucas Gao, Yun[1] Garsia, Adriano M. González D'León, Rafael S. Haiman, Mark D. Hendrickson, Anders O. F. Hivert, Florent Hohlweg, Christophe Howlett, Robert B. Hsiao, Samuel K. Hu, Nai Hong Isaacs, Irving Martin Jedwab, Andrea Johnson, Kenneth Walter Karaali, Gizem Kowalski, Marian Küstner, Josef Labbé, Jean-Philippe Lam, Thomas F. Lange, Carsten E. M. C. Lauve, Aaron Le, Tung[2] Lewis, Stephen David Li, Huilan Li, Shu Xiao Livernet, Muriel Loday, Jean-Louis Machacek, John M. Magaard, Kay Marberg, Eric Mootoo, Xavier Mykytiuk, Stefan Novelli, Jean-Christophe Nyman, Kathryn L. Pang, C. Y. Amy Patras, Frédéric Pilaud, Vincent Pon, Steven Ram, Arun Reutenauer, Christophe Rosas, Mercedes H. Saliola, Franco V. Serrano Herrera, Luis Guillermo Soltani, Farhad Sottile, Frank Sánchez-Ortega, Juana Taylor, Donald E. Tesler, Glenn Tevlin, Lenny Thibon, Jean-Yves Thiem, F. Nathaniel Edgar Thomas, Hugh Vargas, Yannic Venkateswaran, Vidya Vinroot, C. Ryan Vyas, Vedarth Wolfgang, H. Lewis Yan, Ning Zabrocki, Michael van Willigenburg, Stephanie J.

FPSAC, Seattle – July 2026

Crossing, or Not, in the Quasisymmetric World

Nantel Bergeron
York University

Joint work with:



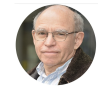
and many others. . .

FPSAC, Seattle – July 2026

Outline:

Λ
$x_1^2x_3$
$x_1x_2x_3$
$x_2^2x_3$
$x_1^2x_4$
$x_1x_2x_4$
$x_2^2x_4$
$x_1x_3x_4$
$x_2x_3x_4$
$x_3^2x_4$
$x_4x_5^2$
$\mathfrak{S}$
$x_3x_4x_5$
$x_3x_4x_5$
$x_2x_4x_5$
$x_1x_4x_5$
$x_3^2x_5$
$x_1x_3x_5$
$x_2^2x_5$
$x_1x_2x_5$
$x_1^2x_5$
$x_1^2x_2$

Quasisymmetric Polynomials

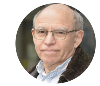


1970-2005

- $\mathbf{QSym}_{\boldsymbol{n}} \subseteq \mathbb{Q}[x_1, x_2, \ldots, x_n]$

$$f(x_1, \ldots, x_n) \in \mathrm{QSym}_n \quad \Longleftrightarrow \quad \boldsymbol{[x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}]f = [x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}]f} \quad \forall i_1 < i_2 < \ldots < i_k$$

$$f(x_1, x_2, x_3) = 2x_1^2x_2 + 2x_1^2x_3 + 2x_2^2x_3 + 3x_1x_2x_3 - x_1^3 - x_2^3 - x_3^3$$

 Quasisymmetric Polynomials 

1970-2005

- $\mathbf{QSym}_{\boldsymbol{n}} \subseteq \mathbb{Q}[x_1, x_2, \ldots, x_n]$

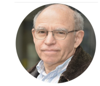
$$f(x_1, \ldots, x_n) \in \mathrm{QSym}_n \quad \Longleftrightarrow \quad \boldsymbol{[x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}]f = [x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}]f} \quad \forall i_1 < i_2 < \ldots < i_k$$

Hivert S_n **action** on $\mathbb{Q}[x_1, x_2, \ldots, x_n]$

$$\boldsymbol{\sigma * x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k} = x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}} \qquad \{j_1 < j_2 < \cdots < j_k\} = \{\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k)\}$$

$$f(x_1, \ldots, x_n) \in \mathrm{QSym}_n \quad \Longleftrightarrow \quad \boldsymbol{\sigma * f = f}, \ \forall \sigma$$

Quasisymmetric Polynomials



1970-2005

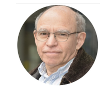
- $\mathbf{QSym}_n \subseteq \mathbb{Q}[x_1, x_2, \ldots, x_n]$

$$f(x_1, \ldots, x_n) \in \mathrm{QSym}_n \quad \Longleftrightarrow \quad [x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}]f = [x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}]f \quad \forall i_1 < i_2 < \ldots < i_k$$

Hivert S_n **action** on $\mathbb{Q}[x_1, x_2, \ldots, x_n]$

$$f(x_1, \ldots, x_n) \in \mathrm{QSym}_n \quad \Longleftrightarrow \quad \boldsymbol{\sigma * f = f}, \ \forall \sigma$$

$$\boxed{\mathbb{Q}[x_1, x_2, \ldots, x_n] \big/ \langle \mathrm{QSym}_n^+ \rangle \overset{\dim}{=} C_n} \ ?$$

 Quasisymmetric Polynomials 

1970-2005

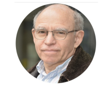
- **QSym**$_n \subseteq \mathbb{Q}[x_1, x_2, \ldots, x_n]$

$$f(x_1,\ldots,x_n) \in \mathrm{QSym}_n \iff \boldsymbol{[x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}]f = [x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}]f} \quad \forall i_1 < i_2 < \ldots < i_k$$

Hivert S_n **action** on $\mathbb{Q}[x_1, x_2, \ldots, x_n]$

$$f(x_1,\ldots,x_n) \in \mathrm{QSym}_n \iff \boldsymbol{\sigma * f = f},\ \forall \sigma$$

$$\boxed{\mathbb{Q}[x_1, x_2, \ldots, x_n] \Big/ \langle \mathrm{QSym}_n^+ \rangle \overset{\dim}{=} C_n} \quad ?$$

 Quasisymmetric Polynomials 

1970-2005

- $\mathbf{QSym}_n \subseteq \mathbb{Q}[x_1, x_2, \ldots, x_n]$

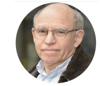
$$f(x_1, \ldots, x_n) \in \mathrm{QSym}_n \quad \Longleftrightarrow \quad [x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}] f = [x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}] f \quad \forall i_1 < i_2 < \ldots < i_k$$

Hivert S_n **action** on $\mathbb{Q}[x_1, x_2, \ldots, x_n]$

$$f(x_1, \ldots, x_n) \in \mathrm{QSym}_n \quad \Longleftrightarrow \quad \sigma * f = f, \ \forall \sigma$$

$$\boxed{\mathbb{Q}[x_1, x_2, \ldots, x_n] \Big/ \langle \mathrm{QSym}_n^+ \rangle \overset{\dim}{=} C_n}$$

Quasisymmetric Polynomials



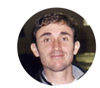
1970-2005

- $\mathbf{QSym}_n \subseteq \mathbb{Q}[x_1, x_2, \ldots, x_n]$

$$f(x_1, \ldots, x_n) \in \mathrm{QSym}_n \quad \Longleftrightarrow \quad [x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}]f = [x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}]f \quad \forall i_1 < i_2 < \ldots < i_k$$

Hivert S_n **action** on $\mathbb{Q}[x_1, x_2, \ldots, x_n]$

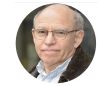
$$f(x_1, \ldots, x_n) \in \mathrm{QSym}_n \quad \Longleftrightarrow \quad \sigma * f = f, \ \forall \sigma$$






$$\boxed{\mathbb{Q}[x_1, x_2, \ldots, x_n] \big/ \langle \mathrm{QSym}_n^+ \rangle \overset{\dim}{=} C_n}$$

$$\begin{tikzcd}
\mathcal{H} \arrow[rr, dashed] \arrow[dr, "\zeta"'] & & \mathrm{Qsym} \arrow[dl, "\varphi_0"] \\
& \mathbb{Q} &
\end{tikzcd}$$

 Quasisymmetric Polynomials 

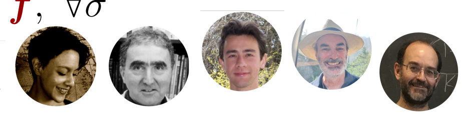
1970-2005

- $\mathbf{QSym}_{\boldsymbol{n}} \subseteq \mathbb{Q}[x_1, x_2, \ldots, x_n]$

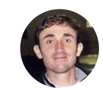
$$f(x_1, \ldots, x_n) \in \mathrm{QSym}_n \quad \Longleftrightarrow \quad \boldsymbol{[x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}]f = [x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}]f} \quad \forall i_1 < i_2 < \ldots < i_k$$

Hivert S_n **action** on $\mathbb{Q}[x_1, x_2, \ldots, x_n]$

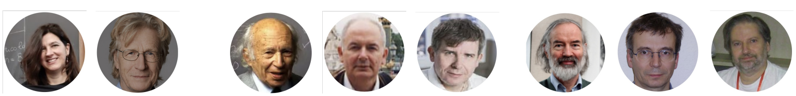
$$f(x_1, \ldots, x_n) \in \mathrm{QSym}_n \quad \Longleftrightarrow \quad \boldsymbol{\sigma * f = f}, \ \forall \sigma$$

Stefan 

$$\boxed{\mathbb{Q}[x_1, x_2, \ldots, x_n] \Big/ \langle \mathrm{QSym}_n^+ \rangle \overset{\dim}{=} C_n}$$



$$\begin{tikzcd}
\mathcal{H} \arrow[rr, dashed] \arrow[dr, "\zeta"'] & & \mathrm{Qsym} \arrow[dl, "\varphi_0"] \\
& \mathbb{Q} &
\end{tikzcd}$$



A Hidden Gem

2005 - 2024

$$\mathbb{Q}[x_1, x_2, \ldots, x_n] \Big/ \langle \mathrm{QSym}_n^+ \rangle \stackrel{\dim}{=} C_n$$

Invariants

?

Geometry

A Hidden Gem

2005 - 2024

$$\boxed{\mathbb{Q}[x_1, x_2, \ldots, x_n] \Big/ \langle \mathrm{QSym}_n^+ \rangle \overset{\dim}{=} C_n}$$

?

Invariants

Hivert $*$-action 

$\mathbf{Ker(*)} = \langle \sum_{\sigma \in S_3} (-1)^{\ell(\sigma)} \sigma \rangle \subset \mathbb{Q}S_n$

$\boldsymbol{TL_n} = \mathbb{Q}S_n \big/ \mathrm{Ker}(*)$ $\boxed{\boldsymbol{TL_n} \overset{\dim}{=} C_n}$

Geometry

A Hidden Gem

2005 - 2024

$$\boxed{\mathbb{Q}[x_1, x_2, \ldots, x_n] \Big/ \langle \mathrm{QSym}_n^+ \rangle \overset{\dim}{=} C_n}$$


?

Invariants

Geometry

Hivert $*$-action 

$\boldsymbol{TL_n} = \mathbb{Q}S_n / \mathrm{Ker}(*)$ $\qquad \boxed{\boldsymbol{TL_n} \overset{\dim}{=} C_n}$

$s_1 = (1\,2), \quad n = 2$

$s_1 * x_1 = x_2; \quad s_1 * x_2^2 = x_1^2$

$s_1 * x_1 x_2^2 = x_1 x_2^2 \neq x_1^2 x_2 = \big(s_1 * x_1\big)\big(s_1 * x_2^2\big)$

A Hidden Gem

2005 - 2024

$$\boxed{\mathbb{Q}[x_1, x_2, \ldots, x_n] \Big/ \langle \mathrm{QSym}_n^+ \rangle \overset{\dim}{=} C_n}$$

?

Invariants

Geometry

Hivert $*$-action 

$\boldsymbol{TL_n} = \mathbb{Q}S_n \big/ \mathrm{Ker}(*)$ $\boxed{\boldsymbol{TL_n} \overset{\dim}{=} C_n}$

Worse: $\langle \mathrm{QSym}_n^+ \rangle$ is **not** $*$-invariant

No action of TL_n on $\mathbb{Q}[x_1, x_2, \ldots, x_n] \Big/ \langle \mathrm{QSym}_n^+ \rangle$ 

A Hidden Gem

2005 - 2024

$$\boxed{\mathbb{Q}[x_1, x_2, \ldots, x_n] \Big/ \langle \mathrm{QSym}_n^+ \rangle \overset{\dim}{=} C_n}$$

 ?

Invariants

Geometry

$\boldsymbol{TL_n} = \mathbb{Q}S_n / \mathrm{Ker}(*)$ $\boxed{\boldsymbol{TL_n} \overset{\dim}{=} C_n}$

 ! [COVID idea] :  Try orbit harmonics!

Almost **impossible** to identify the right set of permutations 😟

A Hidden Gem

2005 - 2024

$$\boxed{\mathbb{Q}[x_1, x_2, \ldots, x_n] \Big/ \langle \mathrm{QSym}_n^+ \rangle \overset{\dim}{=} C_n}$$

Invariants

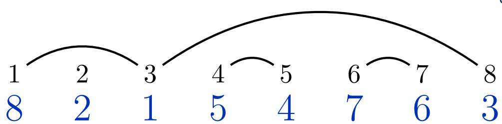
?

Geometry

$\boldsymbol{TL_n} = \mathbb{Q}S_n / \mathrm{Ker}(*)$

$\boxed{\boldsymbol{TL_n} \overset{\dim}{=} C_n}$

 ! [COVID idea] :  Try orbit harmonics!



A Hidden Gem

2005 - 2024

$$\boxed{\mathbb{Q}[x_1, x_2, \ldots, x_n]\Big/\langle \mathrm{QSym}_n^+\rangle \overset{\dim}{=} C_n}$$

?

Invariants

$\boldsymbol{TL_n} = \mathbb{Q}S_n/\mathrm{Ker}(*)$

$\boxed{\boldsymbol{TL_n} \overset{\dim}{=} C_n}$

! [COVID idea] :

Try orbit harmonics!

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 1 & 5 & 4 & 7 & 6 & 3 \end{array}$$

Geometry

Cohomology ?

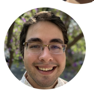

Quasisymmetry Mania

2024 - ...



Quasisymmetry Mania


2024 - ...

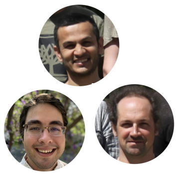




Revisiting quasisymmetry


Quasisymmetry Mania



2024 - ...



Revisiting quasisymmetry

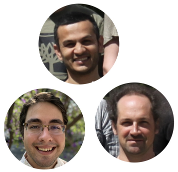
Operator $\boldsymbol{R_i} : f(x_1, \ldots, x_n) \to f(x_1, \ldots, x_{i-1}, \mathbf{0}, x_i, \ldots, x_{n-1})$

$$\boxed{f(x_1, \ldots, x_n) \in \mathrm{QSym}_n \quad \Longleftrightarrow \quad \boldsymbol{R_i f = R_{i+1} f},\ \forall i < n}$$

Quasisymmetry   Mania 

2024 - ...



Revisiting quasisymmetry

Operator $\boldsymbol{R_i} : f(x_1,\ldots,x_n) \to f(x_1,\ldots,x_{i-1},\mathbf{0},x_i,\ldots,x_{n-1})$

$$\boxed{f(x_1,\ldots,x_n) \in \mathrm{QSym}_n \quad \Longleftrightarrow \quad \boldsymbol{R_i f = R_{i+1} f},\ \forall i < n}$$

Now Geometry can start: Consider $\boldsymbol{T_i} = \frac{1}{x_i}(R_{i+1} - R_i)$, which is adjoint to a natural

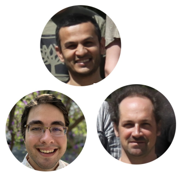
 geometric operator on flags.

Quasisymmetry Mania

2024 - ...



Revisiting quasisymmetry

Operator $\boldsymbol{T_i} = \frac{1}{x_i}(R_{i+1} - R_i)$ quasisymmetric divided difference

$$\boxed{f(x_1, \ldots, x_n) \in \mathrm{QSym}_n \iff \boldsymbol{T_i f = 0},\ \forall i < n}$$

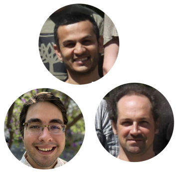
Compare this to the divided difference operator $\boldsymbol{\partial_i} = \frac{\mathrm{Id} - s_i}{x_i - x_{i+1}}$ with property

$$f(x_1, \ldots, x_n) \in \mathrm{Sym}_n \iff \boldsymbol{\partial_i f = 0},\ \forall i < n$$


Quasisymmetry Mania



2024 - ...



Revisiting quasisymmetry

Operator $\boldsymbol{T_i} = \frac{1}{x_i}(R_{i+1} - R_i)$ quasisymmetric divided difference

$$\boxed{f(x_1,\ldots,x_n) \in \mathrm{QSym}_n \iff \boldsymbol{T_i f = 0},\ \forall i < n}$$

Build analogue of Schubert polynomials using the $\boldsymbol{T_i}$ operator: Forest Polynomial $\boldsymbol{\mathfrak{P}_F(\mathbf{x}_n)}$:

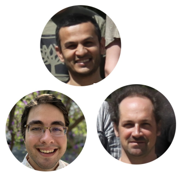
$$T_i \boldsymbol{\mathfrak{P}_F} = \begin{cases} \boldsymbol{\mathfrak{P}_{F/i}}(\mathbf{x}) & \text{if } i \in \mathrm{LTer} F \\ 0 & \text{otherwise.} \end{cases}$$

 Forest Polynomials are special  flagged P-partitions.

Quasisymmetry  Mania  

2024 - ...



Revisiting quasisymmetry

Operator $\boldsymbol{T_i} = \frac{1}{x_i}(R_{i+1} - R_i)$ quasisymmetric divided difference

$$\boxed{f(x_1,\ldots,x_n) \in \mathrm{QSym}_n \quad \Longleftrightarrow \quad \boldsymbol{T_i f = 0},\ \forall i < n}$$

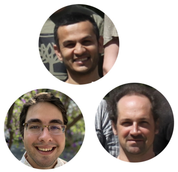
Forest Polynomial $\boldsymbol{\mathfrak{P}_F(\mathbf{x}_n)}$:

- $\{\boldsymbol{\mathfrak{P}_F}\}$ Basis of $\mathbb{Q}[x_1, x_2, \ldots, x_n]$
- $\{\boldsymbol{\mathfrak{P}_F} : F \text{ fully supported}\}$ Basis of $\mathbb{Q}[x_1, x_2, \ldots, x_n] \big/ \langle \mathrm{QSym}_n^+ \rangle$
- $\{F \text{ fully supported}\}$ index some homology class of $\mathrm{HHMP}_n = \bigcup X_w^{wc}$
- $\mathbb{Q}[x_1, x_2, \ldots, x_n] \big/ \langle \mathrm{QSym}_n^+ \rangle \hookrightarrow H^\bullet(\mathrm{HHMP}_n)$

Quasisymmetry   Mania 

2024 - ...



Revisiting quasisymmetry

Operator $\boldsymbol{T_i} = \frac{1}{x_i}(R_{i+1} - R_i)$ quasisymmetric divided difference

$$\boxed{f(x_1, \ldots, x_n) \in \mathrm{QSym}_n \quad \Longleftrightarrow \quad \boldsymbol{T_i f = 0},\ \forall i < n}$$

Forest Polynomial $\boldsymbol{\mathfrak{P}_F(\mathbf{x}_n)}$:

- $\{\mathfrak{P}_F\}$ Basis of $\mathbb{Q}[x_1, x_2, \ldots, x_n]$
- $\{\mathfrak{P}_F : F \text{ fully supported}\}$ Basis of $\mathbb{Q}[x_1, x_2, \ldots, x_n] \big/ \langle \mathrm{QSym}_n^+ \rangle$
- $\{F \text{ fully supported}\}$ index some homology class of $\mathrm{HHMP}_n = \bigcup X_w^{wc}$
- $\mathbb{Q}[x_1, x_2, \ldots, x_n] \big/ \langle \mathrm{QSym}_n^+ \rangle \hookrightarrow H^\bullet(\mathrm{HHMP}_n)$ Geometry!

Quasisymmetry Mania

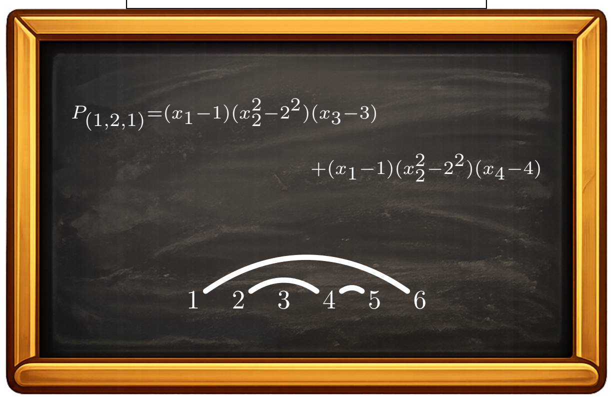
2024 - ...


Revisiting quasisymmetry

$$\boxed{f(x_1, \ldots, x_n) \in \mathrm{QSym}_n \quad \Longleftrightarrow \quad \boldsymbol{T_i f = 0}, \ \forall i < n}$$

Fields Institute

Ah! I bet we replace i by t_i



The vanishing polynomials looks like this

In which way exactly?


Quasisymmetry Mania



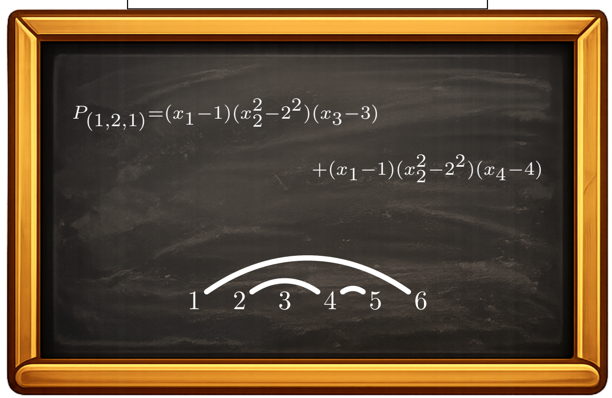
2024 - ...

Revisiting quasisymmetry

$$f(x_1,\ldots,x_n) \in \mathrm{QSym}_n \iff \boldsymbol{T_i f = 0},\ \forall i < n$$

Europe


Fields Institute




Equivariant Quasisymmetry

Double Forest Polynomials

Operator $\boldsymbol{R_i^-} : f(x_1, \ldots, x_n; t_1, \ldots, t_n) \to f(x_1, \ldots, x_{i-1}, \boldsymbol{t_i}, \boldsymbol{x_i}, \ldots, x_{n-1}; t_1, \ldots, t_n)$

Operator $\boldsymbol{R_i^+} : f(x_1, \ldots, x_n; t_1, \ldots, t_n) \to f(x_1, \ldots, x_{i-1}, \boldsymbol{x_i}, \boldsymbol{t_i}, \ldots, x_{n-1}; t_1, \ldots, t_n)$

$$\boldsymbol{E_i} = \frac{R_i^+ - R_i^-}{x_i - t_i}$$

$$\boxed{f(x_1, \ldots, x_n; T) \in \mathrm{EQSym}_n \quad \Longleftrightarrow \quad \boldsymbol{E_i f = 0}, \ \forall i < n}$$

$\boldsymbol{E_i} = \frac{R_i^+ - R_i^-}{x_i - t_i}$

Equivariant Quasisymmetry

Double Forest Polynomials

We want to find polynomials $\mathfrak{P}_{\boldsymbol{F}}(X;T)$ indexed by **Forests** that satisfy:

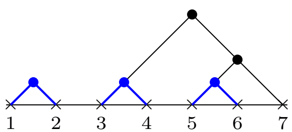
$$\mathfrak{P}_{\boldsymbol{F}}(t_1,\ldots,t_n;T) = \delta_{F=\emptyset} \quad \text{and} \quad E_i\mathfrak{P}_{\boldsymbol{F}} = \begin{cases} \mathfrak{P}_{\boldsymbol{F}/i}(X;t_1,\ldots,\widehat{\boldsymbol{t_i}},\ldots,t_n) & \text{if } i \in \mathrm{LTer}F \\ 0 & \text{otherwise.} \end{cases}$$

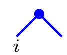
Equivariant Quasisymmetry

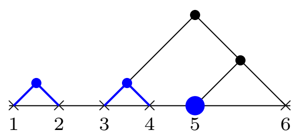
Double Forest Polynomials

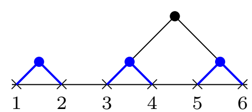
$\boldsymbol{E_i} = \frac{R_i^+ - R_i^-}{x_i - t_i}$

$$\mathfrak{P}_{\boldsymbol{F}}(t_1, \ldots, t_n; T) = \delta_{F=\emptyset} \quad \text{and} \quad E_i \mathfrak{P}_{\boldsymbol{F}} = \begin{cases} \mathfrak{P}_{\boldsymbol{F}/i}(X; t_1, \ldots, \widehat{\boldsymbol{t_i}}, \ldots, t_n) & \text{if } i \in \mathrm{LTer}(\boldsymbol{F}) \\ 0 & \text{otherwise.} \end{cases}$$

$\boldsymbol{F} =$ 

$\mathrm{LTer}(\boldsymbol{F}) = \{i :$  $\in \boldsymbol{F}\}$

$\boldsymbol{F}/5 =$ 

$=$ 

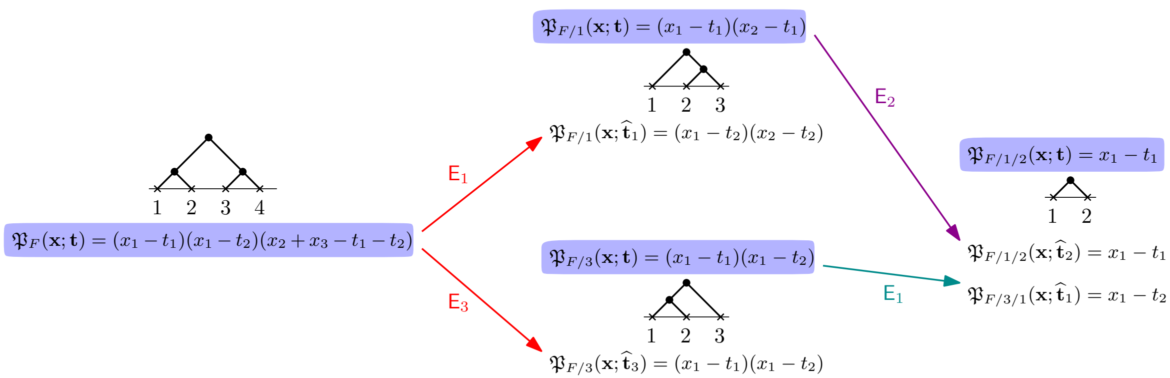
Equivariant Quasisymmetry

Double Forest Polynomials

$\boldsymbol{E_i} = \frac{R_i^+ - R_i^-}{x_i - t_i}$

$$\mathfrak{P}_{\boldsymbol{F}}(t_1,\ldots,t_n;T) = \delta_{F=\emptyset} \quad \text{and} \quad E_i \mathfrak{P}_{\boldsymbol{F}} = \begin{cases} \mathfrak{P}_{\boldsymbol{F}/i}(X;t_1,\ldots,\widehat{\boldsymbol{t_i}},\ldots,t_n) & \text{if } i \in \mathrm{LTer}(\boldsymbol{F}) \\ 0 & \text{otherwise.} \end{cases}$$

Is it well defined? ...



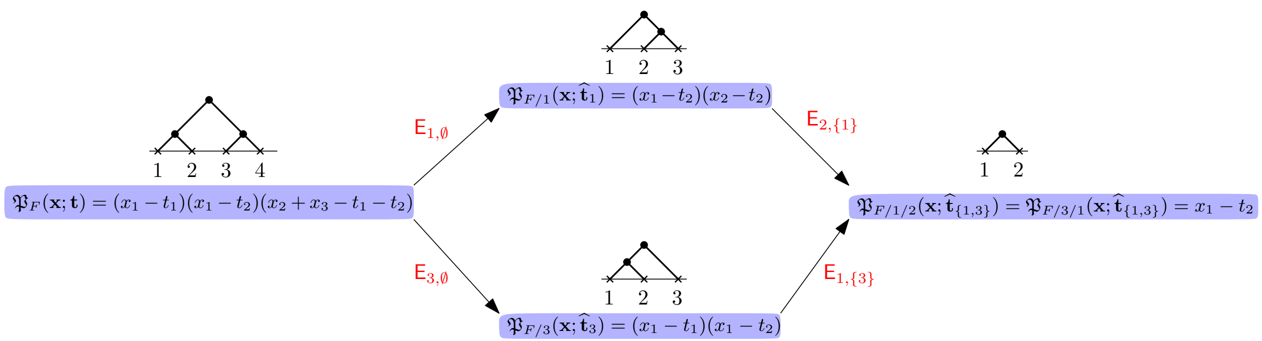
Equivariant Quasisymmetry

Double Forest Polynomials

$\boldsymbol{E_i} = \frac{R_i^+ - R_i^-}{x_i - t_i}$

$$\mathfrak{P}_{\boldsymbol{F}}(t_1,\ldots,t_n;T) = \delta_{F=\emptyset} \quad \text{and} \quad E_i\mathfrak{P}_{\boldsymbol{F}} = \begin{cases} \mathfrak{P}_{\boldsymbol{F}/i}(X;t_1,\ldots,\widehat{\boldsymbol{t_i}},\ldots,t_n) & \text{if } i \in \mathrm{LTer}(\boldsymbol{F}) \\ 0 & \text{otherwise.} \end{cases}$$

Yes!



Equivariant Quasisymmetry

Double Forest Polynomials

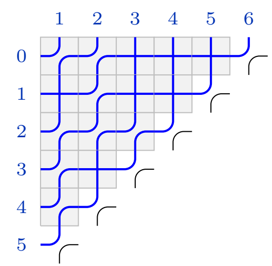
$\boldsymbol{E_i} = \frac{R_i^+ - R_i^-}{x_i - t_i}$

$$\mathfrak{P}_{\boldsymbol{F}}(t_1,\ldots,t_n;T) = \delta_{F=\emptyset} \quad \text{and} \quad E_i\mathfrak{P}_{\boldsymbol{F}} = \begin{cases} \mathfrak{P}_{\boldsymbol{F}/i}(X;t_1,\ldots,\widehat{\boldsymbol{t_i}},\ldots,t_n) & \text{if } i \in \mathrm{LTer}(\boldsymbol{F}) \\ 0 & \text{otherwise.} \end{cases}$$

RECALL:

Pipe Dream gives a way to compute double Schubert polynomials

$$\boldsymbol{\mathfrak{S}_w} = \sum_{D \in \mathrm{PD}(w)} \prod_{(i,j)\in D} (x_i - t_j)$$

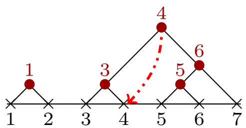


Equivariant Quasisymmetry

Double Forest Polynomials

$\boldsymbol{E_i} = \frac{R_i^+ - R_i^-}{x_i - t_i}$

$$\mathfrak{P}_{\boldsymbol{F}}(t_1, \ldots, t_n; T) = \delta_{F=\emptyset} \quad \text{and} \quad E_i \mathfrak{P}_{\boldsymbol{F}} = \begin{cases} \mathfrak{P}_{\boldsymbol{F}/i}(X; t_1, \ldots, \widehat{\boldsymbol{t_i}}, \ldots, t_n) & \text{if } i \in \mathrm{LTer}(\boldsymbol{F}) \\ 0 & \text{otherwise.} \end{cases}$$

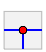
$\boldsymbol{F} =$ 

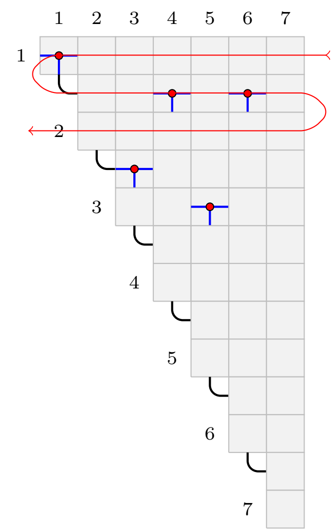
Tiles: 

Rules: Obtain $\boldsymbol{F}$ with the following procedure:

- exactly one  in column i for node i

Choose a **linear extension** of nodes of $\boldsymbol{F}$

place  respecting the chosen order (Here 14635)

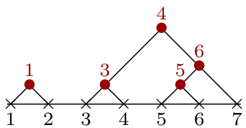


Equivariant Quasisymmetry

Double Forest Polynomials

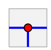
$\boldsymbol{E_i} = \frac{R_i^+ - R_i^-}{x_i - t_i}$

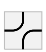
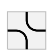
$$\mathfrak{P}_{\boldsymbol{F}}(t_1, \ldots, t_n; T) = \delta_{F=\emptyset} \quad \text{and} \quad E_i \mathfrak{P}_{\boldsymbol{F}} = \begin{cases} \mathfrak{P}_{\boldsymbol{F}/i}(X; t_1, \ldots, \widehat{\boldsymbol{t_i}}, \ldots, t_n) & \text{if } i \in \mathrm{LTer}(\boldsymbol{F}) \\ 0 & \text{otherwise.} \end{cases}$$

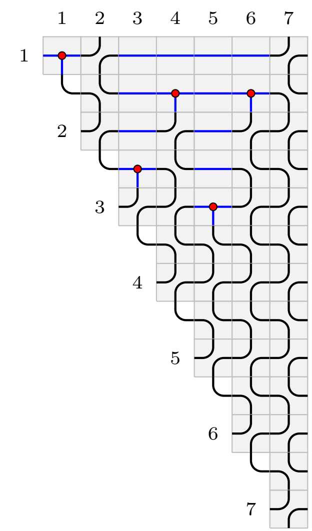
$\boldsymbol{F} =$ 

Tiles:    

Rules: Obtain $\boldsymbol{F}$ with the following procedure:

- exactly one  in column i for node i
- fill with  the column above each 
- fill the remaining spots with [**Vine Model**]

 in odd rows  in even rows and



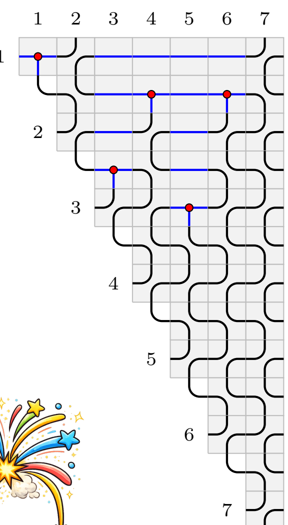
$\boldsymbol{E_i} = \frac{R_i^+ - R_i^-}{x_i - t_i}$

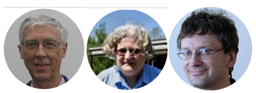
Equivariant Quasisymmetry

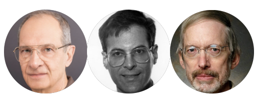
Double Forest Polynomials

$$\mathfrak{P}_{\boldsymbol{F}}(t_1,\ldots,t_n;T) = \delta_{F=\emptyset} \quad \text{and} \quad E_i\mathfrak{P}_{\boldsymbol{F}} = \begin{cases} \mathfrak{P}_{\boldsymbol{F}/i}(X;t_1,\ldots,\widehat{\boldsymbol{t_i}},\ldots,t_n) & \text{if } i \in \mathrm{LTer}(\boldsymbol{F}) \\ 0 & \text{otherwise.} \end{cases}$$

$$\boxed{\mathfrak{P}_{\boldsymbol{F}/i} = \sum_{V\in \mathrm{VD}(\boldsymbol{F})} \prod_{T\in V} \begin{cases} (x_i - t_j) & T \text{ in row } 2i-1 \text{ and col } j \\ (t_j - t_i) & T \text{ in row } 2i \text{ and col } j \end{cases}}$$



-   type formula for

-theory on  lattice Graph with  orientation

THEOREM: $\boxed{\mathbb{Q}[x_1,x_2,\ldots,x_n]\Big/\langle \mathrm{QSym}_n^+\rangle \cong H^\bullet(\mathrm{QF}\ell_n)}$

Open Problems

- (Graham)-combinatorial/geometric interpretation of

$$\mathfrak{P}_{\boldsymbol{F}}(X;T)\mathfrak{P}_{\boldsymbol{G}}(X;T) = \sum_{H} a_{F,G}^{H}(T)\mathfrak{P}_{\boldsymbol{H}}(X;T),$$

- Is there a graded (filtered?) action of TL_n on $\mathbb{Q}[x_1, x_2, \ldots, x_n]$ which gives a well defined action on $\mathbb{Q}[x_1, x_2, \ldots, x_n] \Big/ \langle \mathrm{QSym}_n^+ \rangle$?

- Various quasisymmetries:
  r-quasisymmetry;
  Diagonal quasisymmetry;
  Super quasysymmetry;
  Graph quasisymmetry

Open Problems

-  For $\iota : \mathrm{CF}\ell_c \hookrightarrow G/B$, find a combinatorial interpretation of

$$\iota^* \mathfrak{S}_v = \sum_{u \in \mathrm{NC}(W,c)} a_v^u \mathfrak{P}^c_{\boldsymbol{u}} \in H^\bullet(\mathrm{CF}\ell_c).$$

Geometry $[X_c^u] = \sum a_v^u [X^v]$ implies a_v^u positive. Also, are the $c_{u,v}^w$ positive in

$$\mathfrak{P}^c_{\boldsymbol{u}} \mathfrak{P}^c_{\boldsymbol{v}} = \sum_{w \in \mathrm{NC}(W,c)} c_{u,v}^w \mathfrak{P}^c_{\boldsymbol{w}} \in H^\bullet(\mathrm{CF}\ell_c)?$$

Open Problems

Venturing in the non-crossing world of quasisymmetry

• Invariants • Geometry • Hopf structures • Representation theory • Catalan • Positivity •

Open Problems

Venturing in the non-crossing world of quasisymmetry

• Invariants • Geometry • Hopf structures • Representation theory • Catalan • Positivity •