A combinatorial formula for Interpolation Macdonald polynomials

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joint work with
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Overview

Macdonald polynomials (homogeneous polynomials)

- Introduced by Macdonald in 1988, they generalize Schur, Hall-Littlewood and Jack polynomials.
- Have combinatorial interpretations in terms of tableaux (Haglund–Haiman–Loehr '05), alcove walks (Ram–Yip '11), vertex models (Borodin–Wheeler '19), multiline queues (Corteel–Mandelshtam–Williams '23).

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Interpolation Macdonald polynomials (inhomogeneous polynomials)

- Introduced by Knop and Sahi in 1996,
- further studied by Lassalle '98, Okounkov '97, Lascoux–Rains–Warnaar '08, Olshansky '17, Chen–Sahi '26 ...
- In the Jack limit, these polynomials are positive in the monomial basis (Naqvi–Sahi–Sergel '23) and are related to the enumeration of maps on non-orientable surfaces (BD–Dołęga '23).
- **Today:** A combinatorial formula in terms of *signed multiline queues*.
- **Application:** A probabilistic model (analogue of PushTASEP).

Notation

Fix an integer $n \geq 1$. We consider the space of polynomials in n variables $x_1, \ldots, x_n$ with coefficients $\mathbb{Q}(q,t)$.

We say that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a **partition** (with n parts) if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$.

We say that $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$ is a **composition** of size k (with n parts) if $\mu_1 + \cdots + \mu_n = k$. The integer $k = |\mu|$ is the size of μ.

We denote by $S_n(\mu)$ the set of compositions that can be obtained by permuting the parts of μ.

For a composition μ, write $x^\mu := x_1^{\mu_1} \ldots x_n^{\mu_n}$.

Notation

Given a composition $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n$, we define

$$\widetilde{\mu} := \left(q^{\mu_1} t^{-k_1(\mu)}, \dots, q^{\mu_n} t^{-k_n(\mu)}\right),$$

where

$$k_i(\mu) := \#\{j : j < i \text{ and } \mu_j > \mu_i\} + \#\{j : j > i \text{ and } \mu_j \geq \mu_i\}.$$

Example: When $\mu = (4, 2, 0, 1, 4)$ we have

$$\widetilde{\mu} = (q^4 t^{-1}, q^2 t^{-2}, t^{-4}, q t^{-3}, q^4).$$

For a polynomial $f(x_1, \dots, x_n)$ and $\mu \in \mathbb{N}^n$, define $f(\widetilde{\mu}) = f(\widetilde{\mu}_1, \dots, \widetilde{\mu}_n)$.

g^*: will denote an interpolation polynomial (inhomogeneous),
g : the homogeneous polynomial obtained from g by taking the top degree part.

Interpolation polynomials

Theorem (Knop, Sahi 1996)

For any partition λ, there exists a unique ***symmetric polynomial*** $P^*_\lambda(x_1,\dots,x_n;q,t)$ *such that:*

- $\deg(P^*_\lambda)=|\lambda|$,
- $[x^\lambda]P^*_\lambda=1$
- $P^*_\lambda(\widetilde{\kappa})=0$, *for any* ***partition*** κ *satisfying* $|\kappa|\leq|\lambda|$ *and* $\kappa\neq\lambda$.

Moreover, the top homogeneous part of P^*_λ *is the Macdonald polynomial* P_λ.

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For any composition $\mu\in\mathbb{N}^n$, there exists a unique ***polynomial*** $f^*_\mu(x_1,\dots,x_n;q,t)$ *such that:*

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- $f^*_\mu(\widetilde{\nu})=0$, *for any* ***composition*** ν *satisfying* $|\nu|\leq|\mu|$ *and* $\nu\notin S_n(\mu)$.

Moreover, the top homogeneous part of f^*_μ is the ASEP polynomial f_μ introduced by Cantini–de Gier–Wheeler '15.

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- This theorem is a variant of the theorem of Knop and Sahi defining interpolation nonsymmetric polynomials E^*_μ.
- We have a triangular change of basis

$$E^*_\mu \longleftrightarrow f^*_\mu,$$

which is the same as the change of basis between their homogeneous analogues:
$$E_\mu \longleftrightarrow f_\mu.$$

Interpolation polynomials

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Proposition

For any partition λ,

$$P^*_\lambda(x_1, \ldots, x_n; q, t) = \sum_{\mu \in S_n(\lambda)} f^*_\mu(x_1, \ldots, x_n; q, t).$$

Interpolation polynomials

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Moreover, the top homogeneous part of f^*_μ is the ASEP polynomial f_μ.

Example: $n = 1$ and $\mu = (k)$. We want

$$\begin{aligned} f^*_{(k)}(x) &= x^k + a_{k-1}x^{k-1} + \cdots + a_0, \\ f^*_{(k)}(q^m) &= 0 \qquad\qquad \text{for } 0 \leq m < k, \end{aligned}$$

We then have

$$f^*_{(k)}(x) = (x-1)(x-q)\dots(x-q^{k-1}).$$

Nonsymmetric interpolation polynomials

Example: $n = 2$ and $\mu = (0, 2)$.

$$\begin{aligned}
&f^*_{(0,2)}(x_1, x_2) = x_2^2 + a x_1 x_2 + b x_1 + c x_2 + d, && \\
&f^*_{(0,2)}(q/t, q) = 0 && \nu = (1,1) \\
&f^*_{(0,2)}(q, 1/t) = 0 && \nu = (1,0) \\
&f^*_{(0,2)}(1/t, q) = 0 && \nu = (0,1) \\
&f^*_{(0,2)}(1/t, 1) = 0 && \nu = (0,0)
\end{aligned}$$

We then have

$$f^*_{(0,2)}(x_1, x_2) = x_2^2 + \frac{1-t}{1-qt} x_1 x_2 + q \frac{1-t}{1-qt} x_1 + \frac{1+qt-qt^2-q^2t^2}{t(1-qt)} x_2 + \frac{q(1-qt)}{t(1-qt^2)}.$$

In particular,

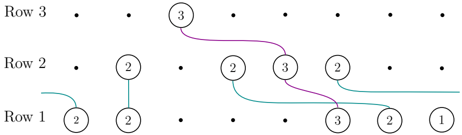
$$f_{(0,2)}(x_1, x_2) = x_2^2 + \frac{1-t}{1-qt} x_1 x_2.$$

Multiline queues

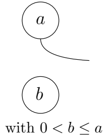
A ball system is an $L \times n$ array (for some $L, n \geq 1$), with rows labeled from bottom to top as $1, 2, \ldots, L$, and columns labeled from left to right from 1 to n, in which each of the Ln positions is either empty or occupied by a ball labeled by $a > 0$.

A multiline queue (MLQ) is a ball system such that:

- each ball in row $r > 1$ is paired with a ball in the row below, using the shortest strand traveling (weakly) from left to right, allowing the strand to wrap around if necessary.
- a ball in a strand of height k is labeled by k,
- it does not contain the forbidden configuration.



A multiline queue.



The forbidden configuration for multiline queues.

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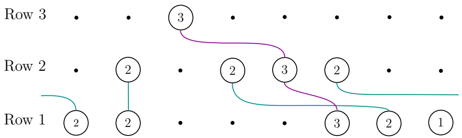


A multiline queue of type $(2, 2, 0, 0, 0, 3, 2, 1)$.

The type of a multiline queue is the composition μ obtained by reading the labels in row 1.

- n = #columns in the multiline queue = #parts of μ.
- L = #rows in the multiline queue = the size of the maximal part in μ.

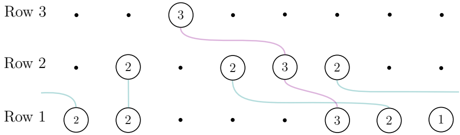
Weights of Multiline queues

Pairing order in each layer:

We read the balls in row r in decreasing order of their label; within a fixed label a:

- we start by placing the trivial a-pairings,
- we pair the remaining balls labeled a from right to left.

As we read the balls in this order, we imagine placing the strands pairing the balls one by one.



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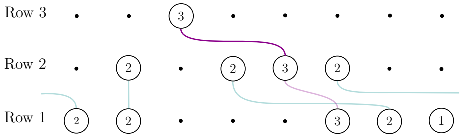

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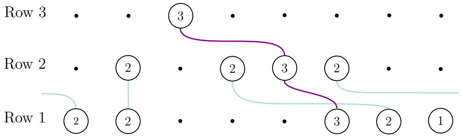

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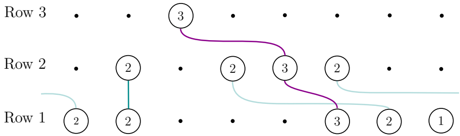

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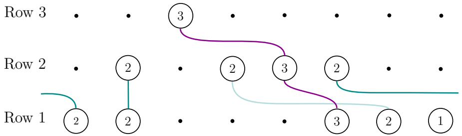

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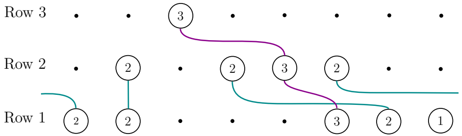

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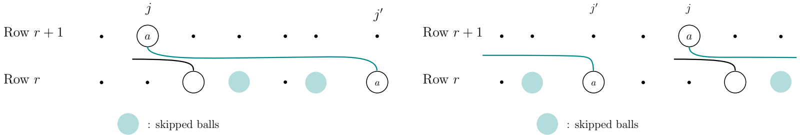
Weights of Multiline queues

- A ball in column i has the weight x_i.
- Each nontrivial pairing p, connecting balls labeled a, between rows $r+1$ and r, and columns j and j', has weight $\mathrm{wt}_{\mathrm{pair}}(p)$:

$$\mathrm{wt}_{\mathrm{pair}}(p) = \begin{cases} \frac{(1-t)t^{\mathrm{skip}(p)}}{1-q^{a-r}t^{\mathrm{free}(p)}} & \text{if } j' > j \\ \frac{(1-t)t^{\mathrm{skip}(p)}}{1-q^{a-r}t^{\mathrm{free}(p)}} \cdot q^{a-r} & \text{if } j' < j. \end{cases}$$

free(p) : balls not yet paired in row r,
skip(p): free balls which have been skipped by p.



Case $j < j'$ Case $j > j'$

- The total weight of a multiline queue Q is denoted $\mathrm{wt}(Q)$:

$$\mathrm{wt}(Q) = \prod_{\text{balls } B} \mathrm{wt}(B) \prod_{\text{pairings } p} \mathrm{wt}_{\mathrm{pair}}(p).$$

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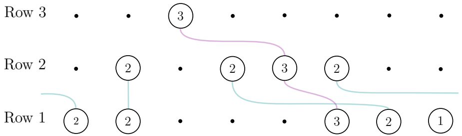

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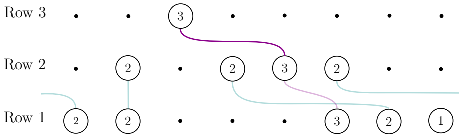

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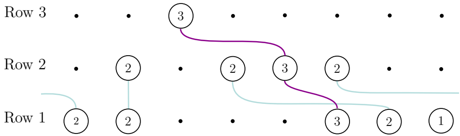

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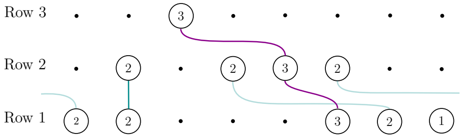

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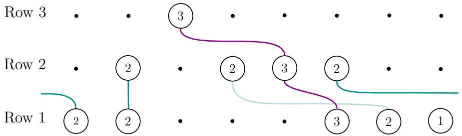

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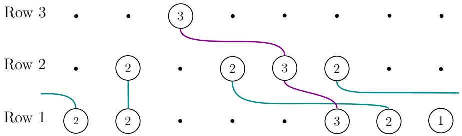

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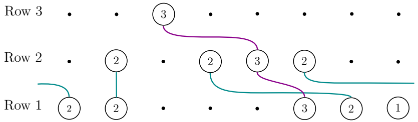
The multiline queue formula for Macdonald polynomials



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Theorem (Corteel–Mandelshtam–Williams '23)

For any composition μ, we have

$$f_\mu(x_1, \ldots, x_n; q, t) = \sum_{\substack{\textit{multiline-queues} \\ Q \textit{ of type } \mu}} \mathrm{wt}(Q).$$

As a consequence, for any partition λ,

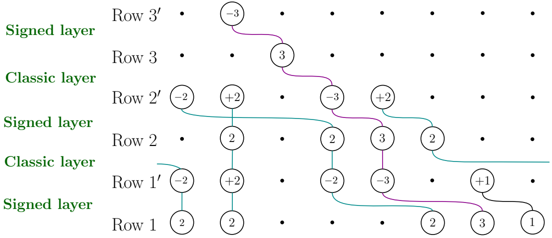
$$P_\lambda(x_1, \ldots, x_n; q, t) = \sum_{\mu \in S_n(\lambda)} \sum_{\substack{\textit{multiline-queues} \\ Q \textit{ of type } \mu}} \mathrm{wt}(Q).$$

Signed Multiline queues

An enhanced ball system is a $2L \times n$ array ($L, n \geq 1$), with rows labeled from bottom to top as $1, 1', 2, 2', \ldots, L, L'$, and columns labeled from left to right from 1 to n, in which each of the $2Ln$ positions is either empty or occupied by a ball. A ball in row r is labeled by $a > 0$, and a ball in row r' is labeled $\pm a$, where $a > 0$.

A signed multiline queue (SMLQ) is an enhanced ball system, satisfying the conditions:

- each strand starts at row 1 and ends at row r' for some $r \geq 1$. In such a strand, all balls are labeled r, $+r$ or $-r$.
- after forgetting the signs, classic layers correspond to layers from classic MLQ.
- each ball in row r' is paired with a ball in row r, using the shortest strand traveling (weakly) from left to right, without wrapping around.
- it does not contain any forbidden configuration.



A signed multiline queue of type $(2, 2, 0, 0, 0, 3, 2, 1)$.

Signed Multiline queues

An enhanced ball system is a $2L \times n$ array ($L, n \geq 1$), with rows labeled from bottom to top as $1, 1', 2, 2', \ldots, L, L'$, and columns labeled from left to right from 1 to n, in which each of the $2Ln$ positions is either empty or occupied by a ball. A ball in row r is labeled by $a > 0$, and a ball in row r' is labeled $\pm a$, where $a > 0$.

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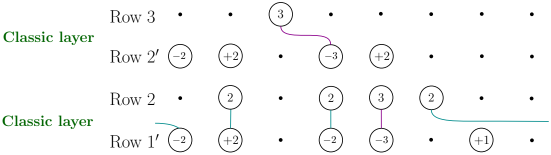

Classic layers of a signed multiline queue.

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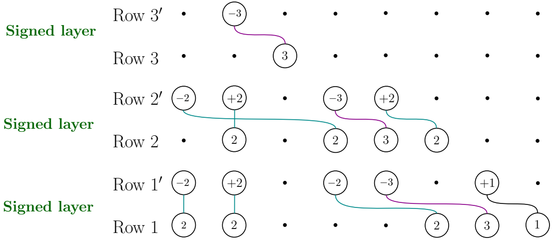

Signed layers of a signed multiline queue.

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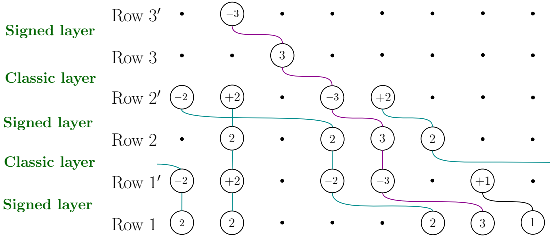

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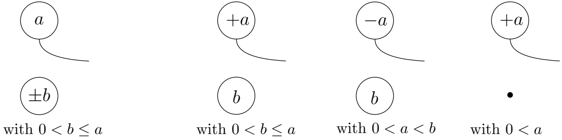
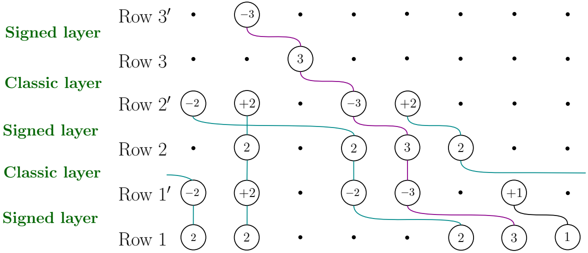


The forbidden configurations for signed multiline queues.

Weights of Signed Multiline Queues

- Only signed balls have weights. A ball in column i, row r', has the weight

$$\begin{cases} x_i & \text{if it is positive,} \\ \frac{-q^{r-1}}{t^{n-1}} & \text{if it is negative.} \end{cases}$$



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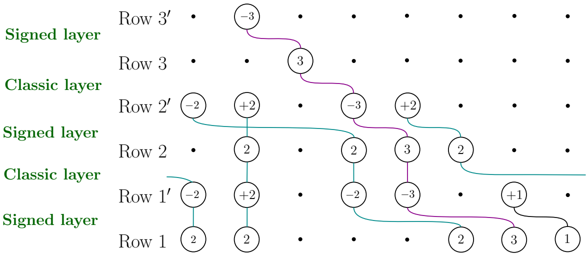

A signed multiline queue with ball weight: $x_2^2 x_5 x_7 \left(\frac{-1}{t^7}\right)^3 \left(\frac{-q}{t^7}\right)^2 \left(\frac{-q^2}{t^7}\right)$

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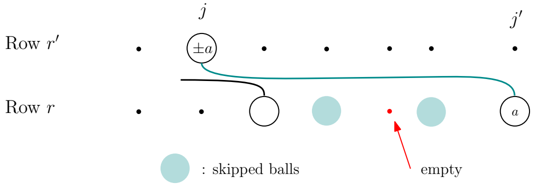
- Each nontrivial pairing p has a weight $\mathrm{wt}_{\mathrm{pair}(p)}$ (we use the same order to place pairings as before)
- Weights of pairings in classic layers are defined as before,
- The weight of a nontrivial pairing in a signed layer is given by

$$\mathrm{wt}_{\mathrm{pair}}(p) = \begin{cases} (1-t)t^{\mathrm{skip}(p)+\mathrm{empty}(p)} & \text{if } p \text{ is connected to a positive ball} \\ -(1-t)t^{\mathrm{skip}(p)+\mathrm{empty}(p)} & \text{if } p \text{ is connected to a negative ball.} \end{cases}$$



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Remark: A SMLQ contributes to the top homogeneous part if and only if all its signed balls are positive.

Signed Multiline queues

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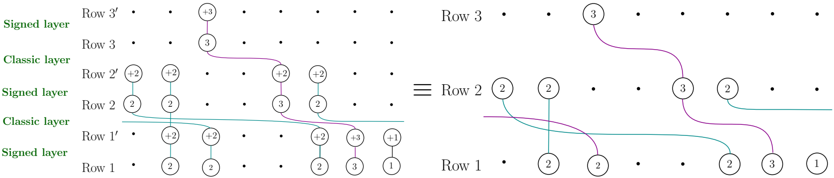

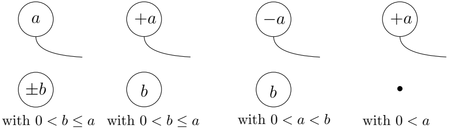

The forbidden configuration for signed multiline queues.

Main result

Theorem (BD–Williams)

For any composition μ, we have

$$f^*_\mu(x_1,\ldots,x_n;q,t) = \sum_{\substack{\textit{signed multiline-queues} \\ Q \textit{ of type } \mu}} \mathrm{wt}(Q).$$

As a consequence, for any partition λ,

$$P^*_\lambda(x_1,\ldots,x_n;q,t) = \sum_{\mu\in S_n(\lambda)} \sum_{\substack{\textit{signed multiline-queues} \\ Q \textit{ of type } \mu}} \mathrm{wt}(Q).$$

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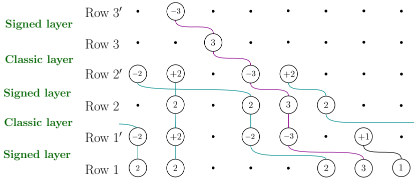
We also give a tableau formula for f^*_μ and P^*_λ.

Some proof ideas

Step 1: Write a combinatorial decomposition



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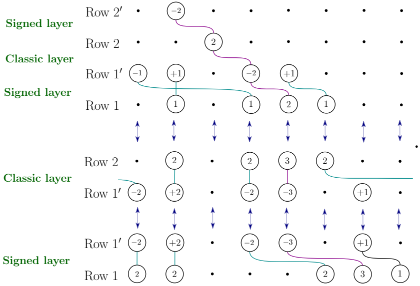

Step 2: A composition μ is said to be **packed** if no zero part is on the left of a nonzero part.

Example: $\mu = (1, 3, 2, 0, 0)$; **Non-example:** $\mu = (1, 3, 0, 2, 0)$.

Use the interpolation theory to establish a factorization of the polynomials f^*_μ when μ is packed.

Step 3: Apply the Hecke operators to deduce a recurrence in the non-packed case: **Example:** $T_3 \cdot f^*_{(1,3,2,0,0)} = f^*_{(1,3,0,2,0)}$.

Some proof ideas: Action of Hecke algebra

For $1 \leq i \leq n-1$, define the operator

$$T_i := t - \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(1 - s_i).$$

These operators satisfy the relations of the Hecke algebra of type A_{n-1}

$$\begin{array}{ll} (T_i - t)(T_i + 1) = 0 & \text{for } 1 \leq i \leq n-1 \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} & \text{for } 1 \leq i \leq n-2 \\ T_i T_j = T_j T_i & \text{for } |i-j| > 1. \end{array}$$

Proposition

For any composition μ, we have

$$T_i f^*_\mu = \begin{cases} f^*_{s_i\mu} & \text{if } \mu_i > \mu_{i+1}, \\ t f^*_\mu & \text{if } \mu_i = \mu_{i+1}, \\ t f^*_{s_i\mu} - (1-t) f^*_\mu & \text{if } \mu_i < \mu_{i+1} \end{cases}.$$

Some proof ideas: Algebraic recurrence

Define $\mathrm{Pack}(k,n)$ as the set of compositions $\mu \in \mathbb{N}^n$ such that $\mu_i \neq 0$ if and only if $1 \leq i \leq k$.

Example: $(2,3,1,0,0) \in \mathrm{Pack}(3,5)$.

Theorem

If $\mu \in \mathrm{Pack}(k,n)$, then the polynomial f^*_μ is divisible by $\prod_{1\leq i\leq k}\left(x_i - \frac{1}{t^{n-1}}\right)$. Moreover,

$$f^*_\mu(x_1,\dots,x_n) = \prod_{1\leq i\leq k}\left(x_i - \frac{1}{t^{n-1}}\right) \sum_{\substack{\nu\in\mathbb{N}^n \\ |\nu|=|\mu|-k}} K^\nu_\mu(q,t)q^{|\nu|} f^*_\nu\left(\frac{x_1}{q},\dots,\frac{x_n}{q}\right).$$