A modular (q,t)-Nekrasov–Okounkov formula and wreath Macdonald polynomials

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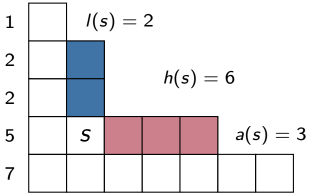
The Nekrasov–Okounkov formula

Discovered by Nekrasov and Okounkov in 2003 as a special case of Nekrasov's conjecture, it is the formal power series expansion

$$\prod_{k\geqslant 1}(1-T^k)^{z-1}=\sum_{\lambda}T^{|\lambda|}\prod_{\square\in\lambda}\left(1-\frac{z}{h(\square)^2}\right).$$

Here the product is over the hook lengths of λ:



$$a(s) = \#\text{ cells to the right of } s$$
$$l(s) = \#\text{ cells above } s$$
$$h(s) = a(s) + l(s) + 1$$

The formula

$$\prod_{k\geqslant 1}(1-T^k)^{z-1}=\sum_{\lambda} T^{|\lambda|}\prod_{\square\in\lambda}\left(1-\frac{z}{h(\square)^2}\right)$$

was independently discovered by

1. Westbury (2004) as a hook-length formula for the D'Arcais polynomials, and
2. Han (2008) from a combinatorial perspective.

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All three discoveries point out that if $z=r^2$ where $r\geqslant 2$, then the Nekrasov–Okounkov formula reduces to a specialisation of the Macdonald identity for $\mathrm{A}_{r-1}^{(1)}$ in the form

$$\prod_{k\geqslant 1}(1-T^k)^{r^2-1}=\sum_{\lambda \text{ an } r\text{-core}} T^{|\lambda|}\prod_{\square\in\lambda}\left(1-\frac{r^2}{h(\square)^2}\right),$$

where an r-core is a partition with no hook of length r.

Han further proved the first "modular" Nerkasov–Okounkov formula by showing that for any integer $r \geqslant 2$,

$$\sum_{\lambda} T^{|\lambda|} S^{|\mathcal{H}_r(\lambda)|} \prod_{h \in \mathcal{H}_r(\lambda)} \left(1 - \frac{z}{h^2}\right) = \frac{(T^r; T^r)_\infty^r}{(T; T)_\infty (ST^r; ST^r)_\infty^{r - z/r}},$$

where

$$\mathcal{H}_r(\lambda) := \{h(\Box) : \Box \in \lambda \text{ and } h(\Box) \equiv 0 \bmod r\},$$

and

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For example with $r = 4$ and $\lambda = (7,4,3,3)$:

3	2	1				
4	3	2				
6	5	4	1			
10	9	8	7	3	2	1

This is an immediate consequence of the core–quotient construction or Littlewood decomposition.

Theorem (Nakayama, Robinson, Littlewood, etc.): For each $r \geqslant 2$ there is a bijection

$$\begin{aligned} \{\text{partitions}\} &\longrightarrow \{r\text{-cores}\} \times \{\text{partitions}\}^r \\ \lambda &\longmapsto \big(r\text{-core}(\lambda), (\lambda^{(0)}, \ldots, \lambda^{(r-1)})\big). \end{aligned}$$

such that

$$|\lambda| = |r\text{-core}(\lambda)| + r\big(|\lambda^{(0)}| + \cdots + |\lambda^{(r-1)}|\big).$$

Idea: remove r-hooks (as r-ribbons) until only the r-core remains, recording them in the r-quotient.

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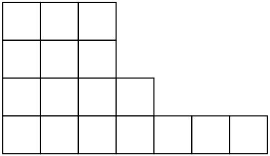

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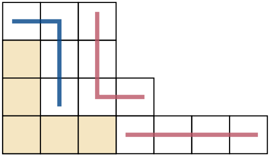

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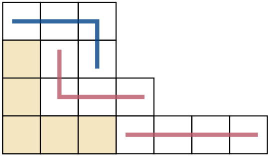

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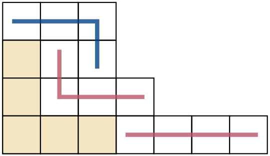

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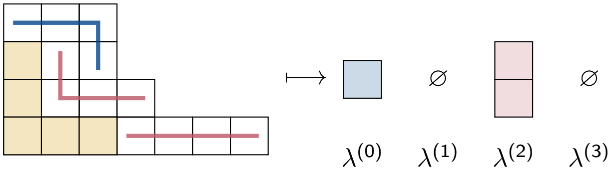

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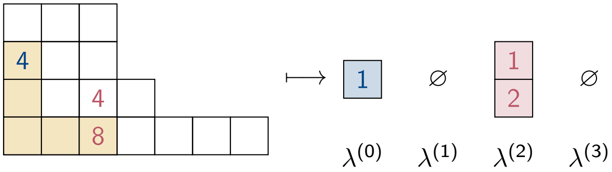

As multisets

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Assuming the Nekrasov–Okounkov formula, this immediately implies

$$\begin{aligned}
\sum_{\substack{\lambda \\ r\text{-core}(\lambda)=\alpha}} T^{(|\lambda|-|\alpha|)/r} \prod_{h\in\mathcal{H}_r(\lambda)} \left(1-\frac{z}{h^2}\right)
&= \sum_{\lambda^{(0)},\ldots,\lambda^{(r-1)}} \prod_{i=0}^{r-1} T^{|\lambda^{(i)}|} \prod_{h\in\mathcal{H}(\lambda^{(i)})} \left(1-\frac{z}{r^2h^2}\right) \\
&= \left(\sum_{\mu} T^{\mu} \prod_{h\in\mathcal{H}(\mu)} \left(1-\frac{z}{r^2h^2}\right)\right)^r \\
&= (T;T)_\infty^{z/r-r},
\end{aligned}$$

which is equivalent to Han's modular analogue after summing over all r-cores.

(q, t)-analogues

Define

$$(a_1, \ldots, a_n; q_1, \ldots, q_m) := \prod_{i=1}^{n} \prod_{j_1, \ldots, j_m \geqslant 0} \left(1 - a_i q_1^{j_1} \cdots q_m^{j_m}\right).$$

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In connection with mixed Hodge polynomials of character varieties, Hausel and Rodriguez Villegas (2008) conjectured the following (q,t)-Nekrasov–Okounkov formula:

$$\sum_{\lambda} T^{|\lambda|} \prod_{\square\in\lambda} \frac{(1-uq^{a(\square)}t^{l(\square)+1})(1-u^{-1}q^{a(\square)+1}t^{l(\square)})}{(1-q^{a(\square)}t^{l(\square)+1})(1-q^{a(\square)+1}t^{l(\square)})}$$

$$= \frac{(uqT,u^{-1}tT;q,t,T)_\infty}{(T,tT;q,t,T)_\infty}.$$

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It has two proofs: one by Carlsson and Rodriguez Villegas and one by Rains and Warnaar. Both are based on Macdonald polynomial theory.

Conjecture (Walsh and Warnaar, 2020): For any r-core α,

$$\sum_{\substack{\lambda \\ r\text{-core}(\lambda)=\alpha}} T^{(|\lambda|-|\alpha|)/r} \prod_{\substack{\square\in\lambda \\ h(\square)\equiv 0 \bmod r}} \frac{(1-uq^{a(\square)}t^{l(\square)+1})(1-u^{-1}q^{a(\square)+1}t^{l(\square)})}{(1-q^{a(\square)}t^{l(\square)+1})(1-q^{a(\square)+1}t^{l(\square)})}$$

$$= \frac{1}{(T;T)_\infty^r} \prod_{i=1}^{r} \frac{(uq^i t^{r-i}T, u^{-1}q^{r-i}t^iT; q^r, t^r, T)_\infty}{(q^i t^{r-i}T, q^{r-i}t^iT; q^r, t^r, T)_\infty}.$$

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Theorem (SAF and Wen): The conjecture is true for $r \geqslant 3$.

Caveat: The case $r = 2$ would require results for the quantum toroidal algebra $U_{\mathfrak{q},\mathfrak{d}}(\ddot{\mathfrak{sl}}_r)$ analogous to work of Wen for $r \geqslant 3$.

Wreath Macdonald polynomials

Introduced by Haiman in 2003, they are multi-symmetric functions in r sets of variables $X^\bullet := (X^{(0)}, \ldots, X^{(r-1)})$, i.e.,

$$H^\bullet_\lambda(X^\bullet; q, t) = H^\bullet_\lambda \in \Lambda^{\otimes r}_{\mathbb{Q}(q,t)}.$$

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Let $\sigma^{-1} X^{(i)} = X^{(i-1)}$ read modulo r. Then the wreath Macdonald polynomial $H^\bullet_\lambda[X^\bullet; q, t]$ is the unique element of $\Lambda^{\otimes r}_{\mathbb{Q}(q,t)}$ satisfying

1. $H^\bullet_\lambda[(1 - q\sigma^{-1})X^\bullet; q, t]$ lies in the span of

$$\{s_{\mu^{(\mathbf{0})}}[X^{(0)}] \otimes \cdots \otimes s_{\mu^{(r-\mathbf{1})}}[X^{(r-1)}] : \mu \geqslant \lambda \text{ and } r\text{-core}(\mu) = r\text{-core}(\lambda)\},$$

2. $H^\bullet_\lambda[(1 - t^{-1}\sigma^{-1})X^\bullet; q, t]$ lies in the span of

$$\{s_{\mu^{(\mathbf{0})}}[X^{(0)}] \otimes \cdots \otimes s_{\mu^{(r-\mathbf{1})}}[X^{(r-1)}] : \mu \leqslant \lambda \text{ and } r\text{-core}(\mu) = r\text{-core}(\lambda)\},$$

3. the coefficient of $h_n[X^{(0)}]$ in the multi-Schur expansion of $H^\bullet_\lambda[X^\bullet; q, t]$ is one where $n = \sum_{i=0}^{r-1} |\lambda^{(i)}|$.

$$
\begin{aligned}
H^{\bullet}_{(7,2,1)} = {} & \frac{t^3}{q^3}(s_\varnothing \otimes s_\varnothing \otimes s_{(1,1,1)}) + \frac{q^4 t + t^3}{q^6}(s_\varnothing \otimes s_\varnothing \otimes s_{(2,1)}) + \frac{t}{q^5}(s_\varnothing \otimes s_\varnothing \otimes s_{(3)}) \\
& + \frac{q^4 t + q^2 t^2 + t^3}{q^5}(s_\varnothing \otimes s_{(1)} \otimes s_{(1,1)}) + \frac{q^5 t + q^2 t + t^2}{q^6}(s_\varnothing \otimes s_{(1)} \otimes s_{(2)}) \\
& + \frac{q^5 t + q^3 t^2 + t^2}{q^5}(s_\varnothing \otimes s_{(1,1)} \otimes s_{(1)}) + \frac{t^2}{q}(s_\varnothing \otimes s_{(1,1,1)} \otimes s_\varnothing) \\
& + \frac{q^4 + q^2 t + t^2}{q^5}(s_\varnothing \otimes s_{(2)} \otimes s_{(1)}) + \frac{q^4 + t^2}{q^4}(s_\varnothing \otimes s_{(2,1)} \otimes s_\varnothing) \\
& + \frac{1}{q^3}(s_\varnothing \otimes s_{(3)} \otimes s_\varnothing) + \frac{q^4 t + q^2 t^2 + t^3}{q^4}(s_{(1)} \otimes s_\varnothing \otimes s_{(1,1)}) \\
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& + q^2 t^2(s_{(1,1,1)} \otimes s_\varnothing \otimes s_\varnothing) + \frac{q^4 + q^2 t + t^2}{q^3}(s_{(2)} \otimes s_\varnothing \otimes s_{(1)}) \\
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\end{aligned}
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For each r-core α the set

$$\left\{ H^\bullet_\lambda[X^\bullet; q, t] : r\text{-core}(\lambda) = \alpha \right\}$$

forms a basis for $\Lambda^{\otimes r}_{\mathbb{Q}(q,t)}$ which is homogeneous of degree $\sum_{i=0}^{r-1} |\lambda^{(i)}|$.

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Bezrukavnikov and Finkelberg (2014): Proof of existence and an analogue of Macdonald positivity: the multi-Schur coefficients $K^{\bullet}_{\lambda\mu}(q, t) \in \mathbb{N}[q, t, 1/q, 1/t]$.

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Romero and Wen (2025): Introduce ∇_α for each r-core α and use it to prove an analogue of the Tesler identity of (Garsia, Haiman and Tesler), Macdonald–Koornwinder evaluation symmetry and much more.

An idea of the proof

The numerator of the hook-product in the (q,t)-Nekrasov–Okounkov formula was already known in the physics literature as a special case of the so-called Nekrasov factor:

$$N_{\lambda,\mu}(u) := \prod_{\square\in\lambda}\left(1-uq^{-a_\mu(\square)}t^{l_\lambda(\square)+1}\right)\prod_{\square\in\mu}\left(1-uq^{a_\lambda(\square)+1}t^{-l_\mu(\square)}\right),$$

where the generalised arm- and leg-lengths are given by

$$a_\lambda(\square) := \lambda_i - j \quad \text{and} \quad l_\lambda(\square) := \lambda'_j - i,$$

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$$N_{(2),(1,1,1)}(u) = (1-ut)(1-uqt)(1-uq^2/t^2)(1-u/t)(1-u).$$

$$(a_\mu, l_\lambda+1) \qquad \begin{array}{|c|c|}\hline 0,1 & -1,1 \\ \hline\end{array} \qquad\qquad \begin{array}{|c|}\hline 0,0 \\ \hline 0,1 \\ \hline 2,2 \\ \hline\end{array} \qquad (a_\lambda+1, l_\mu)$$

Carlsson and Rodriguez Villegas define the operator

$$\Gamma(u) := \Omega\left[\frac{(u^{-1}-1)X}{(1-t)(1-q)}\right]\mathcal{T}[(1-uqt)X]$$

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Following Carlsson, Nerkasov and Okounkov, they have that

$$\left\langle \Gamma(u)\tilde{H}_\lambda, \tilde{H}_\mu \right\rangle_* = (-u)^{-|\mu|}q^{n(\mu')}t^{n(\mu)}N_{\lambda,\mu}(u).$$

Note that this generalises orthogonality under this (modified) scalar product. The (q,t)-Nekrasov–Okounkov formula follows by taking the graded trace of this vertex operator.

For a partition pair (λ, μ) and a cell $\square$ in either of them, define the mixed hook-length by

$$h_{\lambda\mu}(\square) = a_\lambda(\square) + l_\mu(\square) + 1.$$

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The wreath Nekrasov factor is

$$N^{\bullet}_{\lambda,\mu}(u) := \prod_{\substack{\square\in\lambda \\ h_{\mu\lambda}(\square)\equiv 0 \bmod r}} (1 - uq^{-a_\mu(\square)}t^{l_\lambda(\square)+1}) \times \prod_{\substack{\square\in\mu \\ h_{\lambda\mu}(\square)\equiv 0 \bmod r}} (1 - uq^{a_\lambda(\square)+1}t^{-l_\mu(\square)}).$$

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For example with $r = 3$,

3	2
5	4

2	1	0	
3	2	1	-1

$$N^{\bullet}_{(2,2),(4,3)}(\square) = (1 - u)(1 - ut/q^2)(1 - uq^2/t).$$

We introduce an operator $\mathsf{W}(u)$, the wreath analogue of the Carlsson–Nekrasov–Okounkov operator $\Gamma(u)$, defined by way of matrix plethysm.

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Theorem (SAF and Wen):

$$\langle H_{\lambda}^{\bullet\dagger}, W(u) H_{\mu}^{\bullet} \rangle_{*} = u^{-\sum_{i=0}^{r-1} |\mu^{(i)}|} N_{\lambda\mu}^{\bullet}(u).$$

This gives a new proof of the quadratic norm evaluation for wreath Macdonald polynomials due to Orr and Shimozono.

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A corollary is that, for any r-core α,

$$\mathsf{W}(u) H_\lambda^\bullet = \sum_{\substack{\mu \\ r\text{-core}(\mu)=\alpha}} u^{-\sum_{i=0}^{r-1}|\mu^{(i)}|} \frac{N_{\lambda,\mu}^\bullet(u)}{N_{\mu,\mu}^\bullet(1)} H_\mu^\bullet.$$

Taking the graded trace of the operator $\mathsf{W}(u)$ gives a proof of the Walsh–Warnaar conjecture for $r \geqslant 3$.

Elliptic Nekrasov–Okounkov formulae

There is an elliptic analogue of the Nekrasov–Okounkov formula. Denote the modified theta function by

$$\theta(z;p) := (z;p)_\infty(p/z;p)_\infty.$$

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Then

$$\sum_{\lambda} T^{|\lambda|} \prod_{\square\in\lambda} \frac{\theta(ut_1^{-a_\lambda(\square)-1}t_2^{l_\lambda(\square)}, u^{-1}t_1^{-a_\lambda(\square)}t_2^{l_\lambda(\square)+1};p)}{\theta(t_1^{-a_\lambda(\square)-1}t_2^{l_\lambda(\square)}, t_1^{-a_\lambda(\square)}t_2^{l_\lambda(\square)+1};p)}$$
$$= \prod_{m\geqslant 0}\prod_{k\geqslant 1}\prod_{\ell,n_1,n_2\in\mathbb{Z}} \frac{1}{(1-p^m T^k u^\ell t_1^{n_1}t_2^{n_2})^{c(km,\ell,n_1,n_2)}}$$

where the numbers $c(km,\ell,n_1,n_2)$ come from the expansion

$$\frac{\theta(ut_1^{-1},u^{-1}t_2;p)}{\theta(t_1^{-1},t_2;p)} = \sum_{m\geqslant 0}\sum_{\ell,n_1,n_2\in\mathbb{Z}} c(m,\ell,n_1,n_2)p^m u^\ell t_1^{n_1}t_2^{n_2}.$$

This recovers the (q,t)-NO formula for $(t_1,t_2)=(q^{-1},t)$ and $p\to 0$.

This elliptic formula comes as an easy consequence of the equivariant Dijkgraaf–Moore–Verlinde–Verlinde (DMVV) formula for the Hilbert scheme of points, as pointed out by Bryan to Rains and Warnaar.

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Complementing their modular (q,t)-conjecture, Walsh and Warnaar further conjecture that the modular sum:

$$\sum_{\substack{\lambda \\ r\text{-core}(\lambda)=\alpha}} T^{(|\lambda|-|\alpha|)/r} \times \prod_{\substack{\square\in\lambda \\ h_\lambda(\square)\equiv 0 \bmod r}} \frac{\theta(u t_1^{-a_\lambda(\square)-1} t_2^{l_\lambda(\square)}, u^{-1} t_1^{-a_\lambda(\square)} t_2^{l_\lambda(\square)+1}; p)}{\theta(t_1^{-a_\lambda(\square)-1} t_2^{l_\lambda(\square)}, t_1^{-a_\lambda(\square)} t_2^{l_\lambda(\square)+1}; p)}$$

is independent of the chosen core α. They were able to prove this conjecture for the coefficient of T^n with $0 \leqslant n \leqslant 2$, the last case requiring heavy use of addition formulas for theta functions.

the end