Séminaire Lotharingien de Combinatoire **XX** (2026)
Article #42, 12 pp.

Proceedings of the 38th Conference on Formal Power Series and Algebraic Combinatorics (Seattle)

Structure constants of Peterson Schubert calculus

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Abstract. We give an explicit, positive, and type-uniform formula for all equivariant structure constants of the Peterson Schubert calculus in arbitrary Lie types, using only the Cartan matrix of the corresponding root system Φ. As an application, we derive a type-uniform formula for the mixed Φ-Eulerian numbers.

Keywords: Peterson varieties, Peterson Schubert calculus, structure constants, Cartan matrices, mixed Eulerian numbers

1 Introduction

The Peterson variety is a remarkable subvariety of the flag variety, introduced by Dale Peterson to realize the quantum cohomology rings of all (Langlands dual) partial flag varieties geometrically. It can also be used to describe the homology of the affine Grassmannian (see [13]) and it is closely related to the Toeplitz matrices in type A [16].

There are interesting works studying the structure constants of the cohomology ring of the Peterson varieties. Let Pet_G denote the Peterson variety inside the flag variety G/B (see (2.2) for a precise definition). It admits an action of a one-dimensional sub-torus S of the maximal torus $T \subset G$. Let $\sigma_{v_I} \in H_S^*(G/B)$ be a Schubert class indexed by some Coxeter element $v_I \in W$ for $I \subset \Delta$, where W is the Weyl group and Δ is the set of simple roots. Let $\iota^*(\sigma_{v_I})$ be the equivariant pullback of σ_{v_I} along the inclusion $\iota : \mathrm{Pet}_G \to G/B$. The class $p_I := \frac{\iota^*(\sigma_{v_I})}{m(v_I)}$ is independent of the choice of Coxeter elements after dividing by a certain intersection multiplicity $m(v_I)$, known as the Peterson Schubert class. In [8], Goldin, Mihalcea, and Singh showed that $\{p_I\}_{I\subset\Delta}$ is an $H_S^*(pt)$-basis of $H_S^*(\mathrm{Pet}_G;\mathbb{Z})$ and the structure constants of multiplication $c_{I,J}^K \in H_S^*(pt) \cong \mathbb{Z}[t]$, with respect to the basis $\{p_I\}_{I\subset\Delta}$, defined by

$$p_I \cdot p_J = \sum_K c_{I,J}^K p_K, \tag{1.1}$$

are polynomials in t with non-negative integer coefficients.

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A natural open question (asked in [11, P.43] for Lie type A) is the following

Question 1.1. *Is there a positive formula for all structure constants $c_{I,J}^K$ in (1.1)?*

In type A, Goldin–Gorbutt [7] found positive combinatorial formulae for the structure constants by reducing the general I, J, K's to the consecutive ones successively, while Abe–Horiguchi–Kuwata–Zeng [1] found a different combinatorial model computing these structure constants in non-equivariant cohomology. Some years earlier, in type A, Harada–Tymoczko [11] gave a positive equivariant Monk formula for the product of a degree-2 Peterson Schubert class with an arbitrary Peterson Schubert class, and Bayegan–Harada [2] gave an equivariant Giambelli formula to express the Peterson Schubert class p_I in terms of the ring generators $p_{\{i\}}$'s. Drellich [6] gave the equivariant Monk rule and Giambelli formula for all Lie types.

In this extended abstract, we give a simple, explicit, positive, and type-uniform formula for all structure constants in (1.1) of the Peterson Schubert calculus in arbitrary Lie types, which completely solves the above question in a type-uniform way. Our formula is a concise algebraic formula, determined only by the Cartan matrix. As far as we know, this is the first example of an explicit, complete, and type-uniform positive algebraic formula in general Schubert calculus. We hope that our method can shed some new light on related structure constants problems in Schubert calculus. We derive this formula from an algebraic viewpoint, which is different from the above-mentioned work. We now present our main results.

Let $C_\Delta := \big(\langle \alpha_i, \alpha_j \rangle\big)_{i,j\in\Delta} = (c_{ij})_{i,j\in\Delta}$ be the Cartan matrix associated with the semisimple Lie algebra $\mathfrak{g}$ of G. Harada–Horiguchi–Masuda [10] gave an explicit and beautiful Borel-type presentation of the equivariant cohomology ring of the Peterson varieties in all Lie types; see Theorem 2.3. Using the Giambelli formula (see (2.8)) and the intersection multiplicity formula (see (2.7)) due to Drellich[6] and Goldin-Singh[9], we have

Theorem 1.2 (Peterson Schubert monomials)**.** *Under the isomorphism (2.5), the Peterson Schubert class p_I, $I \subset \Delta$, is represented by the monomial*

$$\frac{\det(C_I)}{|W_I|}\prod_{i\in I}\varpi_i, \tag{1.2}$$

where $\det(C_I)$ is the determinant of the Cartan sub-matrix C_I determined by $I \subset \Delta$ and $|W_I|$ is the order of the parabolic subgroup W_I of the Weyl group W determined by $I \subset \Delta$.

Using the above Peterson Schubert monomials and the Harada–Horiguchi–Masuda presentation (2.5), we can expand the product of two monomials into the monomial basis representing the Peterson Schubert classes, and compute all the structure constants.

To state our main theorem, we first need to introduce some notation. For any $K \subset \Delta$, we use $[C_K^{-1}]_{j,k}$ to denote the (j,k)-entry of the matrix C_K^{-1}, which is the inverse of the

Cartan sub-matrix C_K determined by K. Suppose $|\Delta| = n$, we use the notation $(\quad)_{J,K}$ to denote a $2^n \times 2^n$ matrix, with rows indexed by $J \subset \Delta$ and columns indexed by $K \subset \Delta$, and the ordering of the subsets of Δ is the lexicographical order.

Theorem 1.3 (Structure constants of equivariant Peterson Schubert calculus)**.** *With the notation as above. Suppose $I = \{i_1, ..., i_l\} \subset \Delta$. Then the structure constant $c^K_{I,J}$, defined in* (1.1)*, is given by*

$$c^K_{I,J} = \left[D_{i_1} \cdots D_{i_\ell}\right]_{J,K} \frac{\det(C_I)\det(C_J)\,|W_K|}{|W_I|\,|W_J|\det(C_K)},$$

where $D_i := (d^K_{i,J})_{J,K\subset\Delta}$ and

$$d^K_{i,J} = \begin{cases} \frac{[C_K^{-1}]_{i,s}}{[C_K^{-1}]_{s,s}}, & \text{if } i \in K, |K| = |J|+1, \text{ and } K \setminus J = \{s\}, \\ 2t\sum_{k\in K}[C_K^{-1}]_{i,k}, & \text{if } i \in K \text{ and } K = J, \\ 0, & \text{otherwise .} \end{cases} \tag{1.3}$$

By letting $t = 0$ in (1.3), we get a simple formula for all non-equivariant structure constants in the Peterson Schubert calculus for $H^*(\mathrm{Pet}_G; \mathbb{Z})$.

Theorem 1.4. *Suppose $I = \{i_1, ..., i_l\} \subset \Delta$. Then the non-equivariant structure constant $n^K_{I,J}$, defined to be the constant term of $c^K_{I,J}$, is given by*

$$n^K_{I,J} = \left[M_{i_1} \cdots M_{i_\ell}\right]_{J,K} \frac{\det(C_I)\det(C_J)\,|W_K|}{|W_I|\,|W_J|\det(C_K)},$$

where $M_i := (m^K_{i,J})_{J,K\subset\Delta}$ and

$$m^K_{i,J} = \begin{cases} \frac{[C_K^{-1}]_{i,s}}{[C_K^{-1}]_{s,s}}, & \text{if } i \in K, |K| = |J|+1, \text{ and } K \setminus J = \{s\}, \\ 0, & \text{otherwise .} \end{cases} \tag{1.4}$$

It is well known that the entries of the inverse of any Cartan matrix are all non-negative, hence the entries of all the matrices $(d^K_{i,J})_{J,K}$ in Theorem 1.3 are polynomials in t with non-negative coefficients. Since the entries in the product of such matrices remain polynomials in t with non-negative coefficients, our formula is indeed a positive formula, which provides a new purely algebraic proof of the positivity of equivariant structure constants of the Peterson Schubert calculus. We also provide a simple criterion for when these structure constants are non-zero; see Corollary 3.9 and Remark 3.10.

The relationship between the (non-equivariant) Peterson Schubert calculus and the mixed Eulerian numbers (which were introduced and studied by Postnikov [15]) has been actively explored very recently. These numbers compute the mixed volumes of Φ-hypersimplices, see Section 4 for the precise definitions. The origins of this connection lie in the realization of the Peterson variety as a flat degeneration of a smooth projective toric variety, the W-permutohedral variety. In [12], Horiguchi provided a combinatorial model introduced in [1] for the computation of the mixed Φ-Eulerian numbers when Φ is of type A and derived a type-by-type computation for the mixed Φ-Eulerian numbers for general Lie types by iteratively applying the (non-equivariant version of) Monk formula from Drellich in [6]. As an application of our main theorem, we derive a type-uniform formula for the mixed Φ-Eulerian numbers in arbitrary Lie type.

Theorem 1.5. *Let Φ be an irreducible root system of rank n. Let $c_1, \ldots, c_n$ be non-negative integers with $c_1 + \cdots + c_n = n$. The mixed Φ-Eulerian number $A^{\Phi}_{c_1,\ldots,c_n}$ can be computed using the following formula:*

$$A^{\Phi}_{c_1,\ldots,c_n} = \frac{|W_\Phi|}{\det(C_\Phi)} \left[M_1^{c_1} \cdots M_n^{c_n}\right]_{\emptyset,\Delta}.$$

where M_i is the matrix defined in (1.4). The notation $[\quad]_{\emptyset,\Delta}$ denotes the entry in the row indexed by $\emptyset$ and the column indexed by Δ.

Compared with the computations in [12], our formula for the mixed Φ-Eulerian numbers is quite simple and direct, which avoids the need to discuss the changes in the Lie types case-by-case when considering sub-root systems as in [12, Section 7].

2 Preliminaries

2.1 Flag varieties and Schubert varieties

Let G be a complex simply connected semisimple algebraic group of rank n, $B \subset G$ be a Borel subgroup, and $B^- \subset G$ be the opposite Borel. Write $T := B \cap B^-$ for the maximal torus common to both Borel subgroups. Let $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{b} = \mathrm{Lie}(B)$, and $\mathfrak{h} = \mathrm{Lie}(T)$ be the corresponding Lie algebras. Denote by Φ the root system determined by (G, T), and by $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ the simple roots specified by (B, T). We abuse the notation by letting Δ also denote the index set of simple roots. The Weyl group $W := \langle s_i \mid i \in \Delta \rangle$ is a finite Coxeter group. For any subset $I \subset \Delta$, we have the standard parabolic subgroup $P_I \subset G$, whose Weyl group is $W_I = \langle s_i \mid i \in I \rangle$, a parabolic subgroup of W.

Recall that the flag variety G/B admits the Bruhat decomposition

$$G/B = \bigsqcup_{w \in W} BwB/B, \tag{2.1}$$

where each BwB/B is an affine space, which is the *Schubert cell* $X_w^\circ := BwB/B$.

For every $w \in W$, we set

$$X_w := \overline{X_w^\circ} = \overline{BwB/B},$$

the Schubert variety in G/B. It follows from (2.1) that the equivariant cohomology classes $\sigma_w \in H_T^*(G/B;\mathbb{Z})$, which are Kronecker dual to the fundamental class $[X_w]$, form a basis $\{\sigma_w \mid w \in W\}$ of $H_T^*(G/B;\mathbb{Z})$ as a module over $H_T^*(pt)$, called the *equivariant Schubert classes*.

2.2 Peterson varieties and some geometric constructions

Recall that we have the Cartan decomposition of $\mathfrak{g}$ into root spaces

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha.$$

Fix a regular nilpotent element $e \in \mathfrak{b}$. The *Peterson variety* Pet_G is a closed subvariety of the flag variety G/B, defined by

$$\mathrm{Pet}_G := \left\{ gB \in G/B \;\middle|\; \mathrm{Ad}_{g^{-1}}(e) \in \mathfrak{b} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_{-\alpha} \right\}. \tag{2.2}$$

It is known that the Peterson variety Pet_G is always irreducible with complex dimension $n := \operatorname{rank} G$, but it is singular and even non-normal in general. The Peterson varieties also form a special class of a larger family of subvarieties of the flag variety, namely the *Hessenberg varieties* introduced in [5].

It is known that there is a one-dimensional sub-torus $S \subset T$ acting on Pet_G and we set $H_S^*(pt) \cong \mathbb{Z}[t]$, where t is a certain character of S. By intersecting the Peterson variety Pet_G with Schubert cells in the Bruhat decomposition (2.1) of the flag variety G/B, we have the following

Proposition 2.1 (See [8, Proposition 3.3])**.** *The following are equivalent:*

1. $\mathrm{Pet}_G \cap X_w^\circ \neq \emptyset$;
2. *$w = w_I$ for some $I \subset \Delta$, where w_I is the longest element of W_I.*

It is known that the set-theoretic intersection $\mathrm{Pet}_{G,I}^\circ := \mathrm{Pet}_G \cap X_{w_I}^\circ$ is an affine space of dimension $|I|$, called a *Peterson cell*. The Bruhat decomposition (2.1) induces an S-stable affine paving

$$\mathrm{Pet}_G = \bigsqcup_{I\subset\Delta} \mathrm{Pet}_{G,I}^\circ\,. \tag{2.3}$$

For $I \subset \Delta$, we set

$$\mathrm{Pet}_{G,I} := \overline{\mathrm{Pet}_{G,I}^\circ} = \mathrm{Pet}_G \cap X_{w_I}, \qquad \Omega_I := \mathrm{Pet}_G \cap \Omega_{w_I}, \tag{2.4}$$

where $\mathrm{Pet}_{G,I}$ is known as the *Peterson Schubert variety* in the literature. By the above affine paving (2.3), we have the following theorem.

Proposition 2.2. *For every $I \subset \Delta$, we have the fundamental class $[\mathrm{Pet}_{G,I}]$ in $H^S_{2|I|}(\mathrm{Pet}_G;\mathbb{Z})$, and the collection $\{[\mathrm{Pet}_{G,I}] \mid I \subset \Delta\}$ forms a basis of the total equivariant homology $H^S_*(\mathrm{Pet}_G;\mathbb{Z})$.*

2.3 Cohomology ring of Peterson varieties and Peterson Schubert classes

Harada–Horiguchi–Masuda [10] gave the following explicit and beautiful Borel-type presentation of the equivariant cohomology ring of the Peterson varieties in all Lie types.

Theorem 2.3 ([10], Theorem 4.1)**.** *Suppose that rank of G is n. The S-equivariant cohomology ring of the Peterson variety* Pet_G *has the following presentation*

$$H_S^*(\mathrm{Pet}_G;\mathbb{Q}) \cong \frac{\mathbb{Q}[\varpi_1,\ldots,\varpi_n,t]}{\left(\sum_{j=1}^n \langle \alpha_i,\alpha_j\rangle \varpi_i\varpi_j - 2t\varpi_i \mid 1\le i\le n\right)}. \tag{2.5}$$

In the above Harada–Horiguchi–Masuda presentation (2.5), the variable ϖ_i corresponds to the equivariant first Chern class $c_1(L_{\varpi_i}) \in H_S^2(\mathrm{Pet}_G;\mathbb{Q})$ of the line bundle L_{ϖ_i} on Pet_G associated with the fundamental weight ϖ_i.

For $I\subset\Delta$, recall that an element $v\in W$ is called a *Coxeter element* for I if $v=s_{\alpha_1}\cdots s_{\alpha_k}$ for some enumeration $\alpha_1,\ldots,\alpha_k$ of I. Let $\iota:\mathrm{Pet}_G\hookrightarrow G/B$ denote the closed embedding. In [8], a basis $\{p_I\}_{I\subset\Delta}$ of $H_S^*(\mathrm{Pet}_G;\mathbb{Z})$ dual to the basis $\{[\mathrm{Pet}_{G,I}]\}_{I\subset\Delta}$ of $H_*^S(\mathrm{Pet}_G;\mathbb{Z})$ is constructed. We call p_I an (integral equivariant) *Peterson Schubert class*.

Theorem 2.4 ([8, Theorem 4.3 and Corollary 4.4])**.** *For each $I\subset\Delta$, fix a Coxeter element v_I, and consider the pull-back $\iota^*\sigma_{v_I}\in H_S^*(\mathrm{Pet}_G;\mathbb{Q})$. Then the classes*

$$\left\{ p_I := \frac{\iota^*\sigma_{v_I}}{m(v_I)} \in H_S^*(\mathrm{Pet}_G;\mathbb{Q}) \,\middle|\, I\subset\Delta \right\} \tag{2.6}$$

form a $H_S^*(pt)$-basis of $H_S^*(\mathrm{Pet}_G;\mathbb{Z})$, which is dual to the basis $\{[\mathrm{Pet}_{G,I}]\}_{I\subset\Delta}$ of $H_*^S(\mathrm{Pet}_G;\mathbb{Z})$, where $m(v_I)$ is the multiplicity of the unique point w_I in $\Omega_{w_I}\cap\mathrm{Pet}_{G,I}$.

In [8, Remark 7.7], a type-independent formula for the intersection multiplicity $m(v_I)$ was conjectured, which was proved in [9].

Theorem 2.5 (Intersection multiplicity formula, [9, Theorem 1.3])**.** *Suppose I is a connected Dynkin diagram. Let v_I be a Coxeter element of I, and let $R(v_I)$ denote the number of reduced expressions for v_I. We have*

$$m(v_I) = \frac{R(v_I)\,|W_I|}{|I|!\det(C_I)}, \tag{2.7}$$

where W_I is the Weyl group determined by I, and C_I is the Cartan matrix determined by I.

Next, we recall the equivariant Giambelli formula for the Peterson variety in [6] and [9], which expresses each $\iota^*\sigma_{v_I}$ as a polynomial with $\mathbb{Q}[t]$-coefficients in $\iota^*\sigma_{s_i}, i\in\Delta$.

Theorem 2.6 (Equivariant Giambelli formula for Peterson varieties, [9, Theorem 1.2])**.** *Suppose I is a connected Dynkin diagram. Let v_I be a Coxeter element for I, let $R(v_I)$ be the number of reduced words for v_I, then we have*

$$\iota^*\sigma_{v_I} = \frac{R(v_I)}{|I|!}\prod_{i\in I}\iota^*\sigma_{s_i}. \tag{2.8}$$

3 Structure constants of Peterson Schubert calculus

For $I \subset \Delta$, we define

$$\varpi_I := \prod_{i \in I} \varpi_i \in H_S^{2|I|}(\mathrm{Pet}_G; \mathbb{Q}). \tag{3.1}$$

Firstly, by the Giambelli formula (2.8) and the intersection multiplicity formula (2.7), we can identify the Peterson Schubert classes with our *Peterson Schubert monomials* in Theorem 1.2 under the Harada–Horiguchi–Masuda presentation (2.5).

Let $d_{I,J}^K$ denote the structure constant of $\{\varpi_I\}_{I \subset \Delta}$, determined by

$$\varpi_I \varpi_J = \sum_{K \subset \Delta} d_{I,J}^K \varpi_K.$$

Corollary 3.1. *The relation between the structure constants $c_{I,J}^K$ and $d_{I,J}^K$ is*

$$c_{I,J}^K = \frac{\det(C_I)\det(C_J)|W_K|}{|W_I||W_J|\det(C_K)} d_{I,J}^K.$$

For any $K \subset \Delta$, let $A_K := 2E - C_K$, known as the *Coxeter adjacency matrix* in the literature, where E is the identity matrix. Let $B_K := \frac{1}{2}A_K$, and $B_K^{\widehat{s}}$ be B_K with the entries in the row indexed by s being zeros. We have the following theorem.

Theorem 3.2. *Suppose $I = \{i_1, ..., i_l\} \subset \Delta$. Then the matrix $(d_{I,J}^K)_{J,K \subset \Delta}$ of structure constants is equal to the product of matrices $(d_{i_1,J}^K)_{J,K \subset \Delta} \cdots (d_{i_l,J}^K)_{J,K \subset \Delta}$, where*

$$d_{i,J}^K = \begin{cases} 1, & \text{if } K = J \bigsqcup \{i\}, \\ b_{i,s}^{K,\widehat{s}}, & \text{if } i \in J, |K| = |J| + 1, \text{ and } K \setminus J = \{s\}, \\ 2t \sum_{k \in K} [C_K^{-1}]_{i,k}, & \text{if } i \in K \text{ and } K = J, \\ 0, & \text{otherwise}. \end{cases} \tag{3.2}$$

Here $b_{i,s}^{K,\widehat{s}}$ denotes the entry in the row indexed by i and the column indexed by s of $B_K^{\widehat{s}} \left(E - B_K^{\widehat{s}}\right)^{-1}$.

A subset $K \subset \Delta$ is called *connected* if the induced Dynkin diagram with the set of vertices K is connected. Proving the above theorem relies on the following lemmas.

Lemma 3.3. *Let $J, K \subset \Delta$, $i \in J$, $K = J \bigsqcup \{s\}$. Then for any $m \geq 1$ and $k \in K$, we have*

$$[(B_K^{\widehat{s}})^m]_{i,k} = \sum_{a_1 \in J \setminus \{i\}} \sum_{a_2 \in J \setminus \{a_1\}} \cdots \sum_{a_{m-1} \in J \setminus \{a_{m-2}\}} b_{i,a_1} b_{a_1,a_2} \cdots b_{a_{m-1},k},$$

where $b_{i,j}$ is the (i,j)-entry of $B_\Delta = E - \frac{1}{2}C_\Delta$.

Lemma 3.4. *Let $K \subset \Delta$. Then for any $s \in K$, all eigenvalues of the matrix $B_K^{\widehat{s}}$ have absolute values less than 1. As a consequence, the matrix $E - B_K^{\widehat{s}}$ is invertible and we have*

$$B_K^{\widehat{s}} \left(E - B_K^{\widehat{s}}\right)^{-1} = \sum_{m=1}^{\infty} \left(B_K^{\widehat{s}}\right)^m. \tag{3.3}$$

The above Lemma 3.4 basically follows from the classical Perron–Frobenius theorem.

Finally, we can use Sherman–Morrison formula [17] to prove the following lemma. Combined with Corollary 3.1 and Theorem 3.2, it proves our main Theorem 1.3.

Lemma 3.5. *Let $J, K \subset \Delta$, $i \in J$, $K = J \bigsqcup \{s\}$. Then we have*

$$b_{i,s}^{K,\widehat{s}} = \frac{[C_K^{-1}]_{i,s}}{[C_K^{-1}]_{s,s}},$$

where $b_{i,s}^{K,\widehat{s}}$ is defined in (3.2).

Now we give some examples of computations using our formula.

Example 3.6. *For Lie type B_3, the structure constants matrix of $I = \{2\}$ is*

$$\left(d_{I,J}^K\right)_{J,K\subset[3]} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2t & 0 & \frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2t & 0 & 0 & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2t & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2t \end{pmatrix} \begin{matrix} J \\ \emptyset \\ \{1\} \\ \{2\} \\ \{3\} \\ \{1,2\} \\ \{1,3\} \\ \{2,3\} \\ \{1,2,3\} \end{matrix}$$

Then we have $d_{\{2\},\{1,2\}}^{\{1,2\}} = 2t$, $d_{\{2\},\{1,2\}}^{\{1,2,3\}} = \frac{4}{3}$, which means $\varpi_2 \cdot \varpi_{\{1,2\}} = 2t\varpi_{\{1,2\}} + \frac{4}{3}\varpi_{\{1,2,3\}}$.

The above structure constants matrix is sparse and our formula is fast when doing actual computations. We give an example with larger rank in the non-equivariant case.

Example 3.7. *For Lie Type A_9, $I = \{3,6,8\}$, $J = \{1,3,5,6,7\}$, from our formula we have*

$$\varpi_I \cdot \varpi_J = \frac{18}{35}\varpi_{\{1,2,3,4,5,6,7,8\}} + \frac{1}{5}\varpi_{\{1,2,3,5,6,7,8,9\}} + \frac{2}{7}\varpi_{\{1,3,4,5,6,7,8,9\}}.$$

Hence, by Theorem 2.4, we have

$$\begin{aligned} p_I \cdot p_J =& \frac{1}{1! \cdot 1! \cdot 1!} \cdot \frac{1}{1! \cdot 1! \cdot 3!} \varpi_I \varpi_J \\ =& \frac{1}{3!} \left(\frac{18}{35} \varpi_{\{1,2,3,4,5,6,7,8\}} + \frac{1}{5} \varpi_{\{1,2,3,5,6,7,8,9\}} + \frac{2}{7} \varpi_{\{1,3,4,5,6,7,8,9\}} \right) \\ =& \frac{8!}{3!} \frac{18}{35} p_{\{1,2,3,4,5,6,7,8\}} + \frac{3! \cdot 5!}{3!} \frac{1}{5} p_{\{1,2,3,5,6,7,8,9\}} + \frac{7!}{3!} \frac{2}{7} p_{\{1,3,4,5,6,7,8,9\}} \\ =& 3456 p_{\{1,2,3,4,5,6,7,8\}} + 24 p_{\{1,2,3,5,6,7,8,9\}} + 240 p_{\{1,3,4,5,6,7,8,9\}}. \end{aligned}$$

We have the following corollary that our formulas provide an algebraic proof of the positivity of the equivariant structure constants of the Peterson Schubert calculus.

Corollary 3.8. *The structure constant $d_{I,J}^K$ (and the structure constant $c_{I,J}^K$) is a polynomial in t with non-negative coefficients for all $I, J, K \subset \Delta$.*

The following is a simple criterion for when the structure constants are non-zero.

Corollary 3.9. *The structure constant $c_{I,J}^K \neq 0$ (equivalently, $d_{I,J}^K \neq 0$) if and only if*

- *$K \supset I \cup J$, and*
- *For each connected component K_k of K, we have $|K_k| \leq |K_k \cap I| + |K_k \cap J|$.*

Remark 3.10. *The non-equivariant structure constant $m_{I,J}^K \neq 0$ if and only if $K \supset I \cup J$ and $|K_k| = |K_k \cap I| + |K_k \cap J|$ for each connected component K_k of K.*

4 Applications to mixed Eulerian numbers

Firstly, we introduce the mixed Φ-Eulerian numbers for arbitrary Lie types as in [15] and their connections to the structure constants of the Peterson Schubert calculus. We mainly follow the notation as in [12].

Recall that Φ is a crystallographic root system of rank n. Let Λ be the associated integral weight lattice and $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$ be the weight space. The associated Weyl group W acts on the weight space $\Lambda_{\mathbb{R}}$ as a finite real reflection group. Taking a weight $\chi \in \Lambda_{\mathbb{R}}$, the *weight polytope* $P_\Phi(\chi)$ is defined as the convex hull of the Weyl group orbit of χ:

$$P_\Phi(\chi) := \mathrm{ConvexHull}\{w(\chi) \in \Lambda_{\mathbb{R}} \mid w \in W\}.$$

Recall that $\Delta := (\alpha_1, \ldots, \alpha_n)$ denotes the set of simple roots in Φ. In [15], Postnikov gave a formula for the volume of the weight polytope $P_\Phi(\chi)$, with the normalization that the volume of the parallelepiped generated by the simple roots $\alpha_1, \ldots, \alpha_n$ is 1.

Let $\varpi_1, \ldots, \varpi_n$ be the fundamental weights. Suppose $\chi = u_1\varpi_1 + \cdots + u_n\varpi_n$ and consider the associated weight polytope $P_\Phi(\chi)$. Its volume is a homogeneous polynomial V_Φ of degree n in variables $u_1, \ldots, u_n$:

$$V_\Phi(u_1, \ldots, u_n) := \text{volume of } P_\Phi(u_1\varpi_1 + \cdots + u_n\varpi_n). \tag{4.1}$$

Postnikov [15] defined the *mixed Φ-Eulerian numbers* $A^\Phi_{c_1,\ldots,c_n}$, for $c_1, \ldots, c_n \geq 0$ with $c_1 + \cdots + c_n = n$, as the coefficients of the volume polynomial (4.1):

$$V_\Phi(u_1, \ldots, u_n) = \sum_{c_1,\ldots,c_n} A^\Phi_{c_1,\ldots,c_n} \frac{u_1^{c_1}}{c_1!} \cdots \frac{u_n^{c_n}}{c_n!}. \tag{4.2}$$

By this definition, the mixed Φ-Eulerian number $A^\Phi_{c_1,\ldots,c_n}$ is exactly the *mixed volume* of c_1 copies of $P_\Phi(\varpi_1)$, c_2 copies of $P_\Phi(\varpi_2), \ldots$, and c_n copies of $P_\Phi(\varpi_n)$, multiplied by $n!$. Here, the weight polytopes $P_\Phi(\varpi_1), P_\Phi(\varpi_2), \ldots, P_\Phi(\varpi_n)$ are called the *Φ-hypersimplices*. The mixed Φ-Eulerian numbers are known to be non-negative integers; see [15] for more details. When Φ is of type A, these numbers are simply called the *mixed Eulerian numbers*.

Postnikov provided in [15] a combinatorial formula for the mixed Φ-Eulerian numbers in terms of certain binary trees. In [3], Berget–Spink–Tseng studied the log-concavity of matroid h-vectors in relation to the mixed Eulerian numbers, using the fact that the cohomology ring of the type A Permutohedral variety is exactly the Chow ring of the Boolean matroid. In [14], Nadeau–Tewari found a beautiful relation between the mixed Φ-Eulerian numbers and intersection numbers of Schubert varieties and the permutohedral variety for arbitrary Lie types. In [12], Horiguchi showed that the mixed Eulerian numbers can be written as intersection numbers of Schubert divisors in the Peterson variety for an arbitrary Lie type as follows.

Theorem 4.1 ([12, Theorem 1.1])**.** *Let Φ be an irreducible root system of rank n. Let $c_1, \ldots, c_n$ be non-negative integers with $c_1 + \cdots + c_n = n$. Then the mixed Φ-Eulerian number $A^\Phi_{c_1,\ldots,c_n}$ is equal to*

$$A^\Phi_{c_1,\ldots,c_n} = \int_{\mathrm{Pet}_G} \varpi_1^{c_1}\varpi_2^{c_2}\cdots\varpi_n^{c_n},$$

where, as before, Pet_G denotes the Peterson variety associated with the simple algebraic group G, $\varpi_i \in H^2(\mathrm{Pet}_G;\mathbb{Q})$ denotes the first Chern class of the line bundle L_{ϖ_i} on Pet_G, which is also the image of the Schubert class $\sigma_{s_i} \in H^2(G/B;\mathbb{Q})$ under the restriction map $H^2(G/B;\mathbb{Q}) \to H^2(\mathrm{Pet}_G;\mathbb{Q})$.

Remark 4.2. *From (4.2) and the above theorem, it follows that the volume polynomial in (4.1) has the following expression*

$$\mathrm{Vol}\, P_\Phi(u_1\varpi_1 + \cdots + u_n\varpi_n) = \frac{1}{n!}\int_{\mathrm{Pet}_G} (u_1\varpi_1 + \cdots + u_n\varpi_n)^n, \tag{4.3}$$

which is the volume polynomials of the nef divisors $\varpi_1, \ldots, \varpi_n$ on the Peterson variety Pet_G*. While the Peterson variety is singular in general, its rational cohomology ring $H^*(\mathrm{Pet}_G; \mathbb{Q})$ in* (2.5) *admits the structure of the cohomology ring of a (rational smooth) toric orbifold, which satisfies all the Kähler package—the Poincaré duality, the hard Lefschetz theorem, and the Hodge–Riemann bilinear relation. This gives a different proof that* $\mathrm{Vol}\, P_\Phi\,(u_1\varpi_1 + \cdots + u_n\varpi_n)$ *is* Lorentzian *in the sense of [4].*

Using the above theorem, Horiguchi gave a combinatorial model introduced in [1] for the computation of the mixed Eulerian numbers, and derived a type-by-type computation for the mixed Φ-Eulerian numbers for general Lie types by iteratively applying the Monk formula of Drellich in [6]. As an application of our main theorem, we can derive a type-uniform formula for the mixed Φ-Eulerian numbers in arbitrary Lie types as in Theorem 1.5 using Theorem 4.1.

We give some examples using our formula to compute mixed Φ-Eulerian numbers.

Example 4.3. *For Lie type A_8 and $(c_1, \ldots, c_8) = (1,0,2,3,0,0,1,1)$. Then from our formula, it is easy to compute*

$$\begin{aligned} A^{\Phi}_{c_1,\ldots,c_8} &= 8! \left[M_1 M_3^2 M_4^3 M_7 M_8\right]_{\emptyset,\{1,2,3,4,5,6,7,8\}} \\ &= 8! \cdot \frac{41}{70} \\ &= 23616. \end{aligned}$$

Example 4.4. *For Lie type E_6 and $(c_1, \ldots, c_6) = (0,1,0,2,3,0)$. Then from our formula, it is easy to compute*

$$\begin{aligned} A^{\Phi}_{c_1,\ldots,c_6} &= \frac{|W_{E_6}|}{\det(C_{E_6})} \left[M_2 M_4^2 M_5^3\right]_{\emptyset,\{1,2,3,4,5,6\}} \\ &= 2^7 \cdot 3^3 \cdot 5 \cdot \frac{81}{40} \\ &= 34992. \end{aligned}$$

It can be seen from the above examples that our formula avoids the need to discuss the changes in the Lie types case-by-case when considering sub-root systems as in [12, Section 7]. The only thing needed is to substitute $(c_1, \ldots, c_n)$ into our formula to obtain the results.

Acknowledgements

The first author would like to express his gratitude to Hiraku Abe and Haozhi Zeng for helpful discussions. The authors thank Yibo Gao for some useful comments and thank anonymous referees for suggestions which helped improve the exposition.

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