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Grothendieck positivity for square root crystals

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Abstract. Normal crystals (also called Stembridge crystals) are a standard tool for proving Schur positivity, since their characters are sums of Schur polynomials. In this extended abstract, we develop an analogous combinatorial framework for symmetric Grothendieck polynomials, the K-theoretic counterparts of Schur polynomials. Specifically, we study certain directed graphs called square root crystals, whose properties mirror classical normal crystals. Our main result is that the character of any normal square root crystal is a sum of symmetric Grothendieck polynomials. This provides a systematic method for establishing positive symmetric Grothendieck expansions, including a new proof of Buch's K-theoretic Littlewood–Richardson rule.

Keywords: Crystals, symmetric Grothendieck polynomials, set-valued tableaux

1 Introduction

Crystals are combinatorial objects that arise naturally in the representation theory of quantum groups. They were introduced independently by Kashiwara [10, 9] and Lusztig [11, 12]. For a historical overview, see [3, §1].

Fix $n \in \mathbb{Z}_{>0}$. In our setting, a *crystal* is a directed graph $\mathcal{B}$ whose edges are labeled by indices in $\{1, 2, \ldots, n-1\}$, together with a weight map $\mathrm{wt} : \mathcal{B} \to \mathbb{Z}^n$ assigning an n-tuple of integers to each vertex. This data satisfies additional axioms described in §2. When $\mathcal{B}$ is finite, its *character* is the weight generating function of its vertices.

We study two families of type $\mathfrak{gl}_n$ crystals that satisfy slightly different axioms. Each family has a distinguished *standard crystal*. The first family consists of the well-studied *(seminormal) $\mathfrak{gl}_n$-crystals*, following the conventions of [3]. Its *standard $\mathfrak{gl}_n$-crystal* $\mathbb{B}_n$, shown in Figure 1, corresponds to the vector representation of $U_q(\mathfrak{gl}_n)$. The second family is made up of certain recently introduced *square root crystals* (abbreviated as *$\sqrt{\mathfrak{gl}_n}$-crystals*) from [14, 17]. The *standard $\sqrt{\mathfrak{gl}_n}$-crystal* is a graph on the nonempty subsets of $[n] := \{1, 2, \ldots, n\}$, shown in Figure 2 for $n = 3$ and defined precisely in Definition 2.4.

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$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n-1} \boxed{n}$$

Figure 1: The *standard $\mathfrak{gl}_n$-crystal*, with weight map $\mathrm{wt}(\boxed{i}) = \mathbf{e}_i \in \mathbb{Z}^n$.

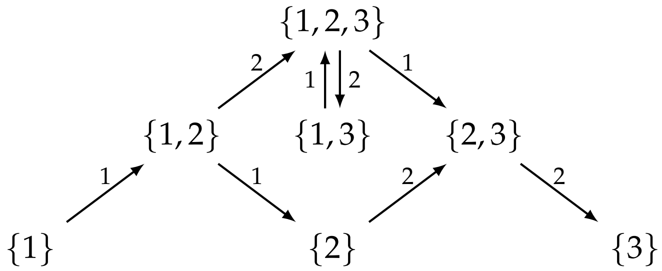

Figure 2: The standard $\sqrt{\mathfrak{gl}_3}$-crystal, with weight map $\mathrm{wt}(S) = \sum_{i \in S} \mathbf{e}_i \in \mathbb{Z}^3$.

The two families share the same associative *tensor product* $\otimes$, which combines two objects $\mathcal{B}$ and $\mathcal{C}$ to form a new crystal $\mathcal{B} \otimes \mathcal{C}$ in the same family (see Definition 2.7). For finite crystals $\mathcal{B}$ and $\mathcal{C}$, we have $\mathsf{ch}(\mathcal{B} \otimes \mathcal{C}) = \mathsf{ch}(\mathcal{B})\mathsf{ch}(\mathcal{C})$.

We are particularly interested in *normal* crystals within these two families.

Definition 1.1. A $\mathfrak{gl}_n$-crystal is *normal* if each of its connected components is isomorphic to a connected component of $\mathbb{B}_n^{\otimes m} = \mathbb{B}_n \otimes \mathbb{B}_n \otimes \cdots \otimes \mathbb{B}_n$ for some $m \in \mathbb{N}$.

Normal $\sqrt{\mathfrak{gl}_n}$-crystals are defined in almost the same way, but using $\mathbb{S}_n$ in place of $\mathbb{B}_n$:

Definition 1.2. A $\sqrt{\mathfrak{gl}_n}$-crystal is *normal* if each of its connected components is isomorphic to a connected component of $\mathbb{S}_n^{\otimes m} = \mathbb{S}_n \otimes \mathbb{S}_n \otimes \cdots \otimes \mathbb{S}_n$ for some $m \in \mathbb{N}$.

Normal $\mathfrak{gl}_n$-crystals have been extensively studied and possess several well-known properties: for instance, they correspond precisely to crystal bases of quantum group representations [3] and can be characterized by the local *Stembridge axioms* [16]. We focus on their Schur-positivity property (1.1), and our main result, Theorem 1.3, establishes an analogous phenomenon for normal $\sqrt{\mathfrak{gl}_n}$-crystals.

We consider two $\mathbb{Z}$-bases of the ring of symmetric polynomials $\mathbb{Z}[x_1, x_2, \dots, x_n]$. The more classical basis is given by the *Schur polynomials* $s_\lambda = s_\lambda(x_1, x_2, \dots, x_n)$, indexed by *partitions* with at most n parts. Each s_λ is homogeneous of degree $|\lambda|$, and it represents the cohomology class of a Schubert variety in the complete flag variety. The second basis consists of the *(signless) symmetric Grothendieck polynomials* $G_\lambda = G_\lambda(x_1, x_2, \dots, x_n)$, which represent the K-theory classes of Schubert varieties in the complete flag variety. [1] Unlike

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[1] In the literature, the term "symmetric Grothendieck polynomial" often refers to $(-1)^{|\lambda|}G_\lambda(-x)$ or $\beta^{-|\lambda|}G_\lambda(\beta x)$ where β is a parameter. A similar remark applies to the polynomials $G_w(x)$ defined later.

Schur polynomials, the G_λ's are not homogeneous, but their lowest-degree component is s_λ. See Definition 3.1 for concrete definitions of both polynomials.

Schur and symmetric Grothendieck polynomials have many further interpretations in representation theory and geometry. Because of these interpretations, it is often meaningful to show that a given polynomial $f \in \mathbb{Z}[x_1, x_2, \dots, x_n]$ is *Schur positive* (resp. *G-positive*): that is, f can be expressed as a $\mathbb{N}$-linear combination of s_λ's (resp. G_λ's).

Normal $\mathfrak{gl}_n$-crystals provide a powerful framework for establishing Schur positivity. The coefficients in the Schur expansion are given by the number of corresponding *highest weight* elements, i.e., the source vertices in the associated crystal graph.

In more detail, let $\mathsf{HW}(\mathcal{B})$ denote the set of such elements in a crystal $\mathcal{B}$. When $\mathcal{B}$ is a finite normal $\mathfrak{gl}_n$-crystal, it is well known that $\mathrm{wt}(b)$ is a partition for every $b \in \mathsf{HW}(\mathcal{B})$ [3, §2.4], and that

$$\mathsf{ch}(\mathcal{B}) = \textstyle\sum_{b \in \mathsf{HW}(\mathcal{B})} s_{\mathrm{wt}(b)} \quad \text{[3, Thms. 3.2 and 8.6].} \tag{1.1}$$

Via this identity, one can show that a generating function over a set of objects is Schur positive by endowing the set with a normal $\mathfrak{gl}_n$-crystal structure and identifying its highest weight elements. For examples of this approach, see [7, 15].

For a $\sqrt{\mathfrak{gl}_n}$-crystal $\mathcal{B}$, it again holds that $\mathrm{wt}(b)$ is partition if $b \in \mathsf{HW}(\mathcal{B})$ [14, Lem. 4.11]. Our main result establishes an analogous character formula for normal $\sqrt{\mathfrak{gl}_n}$-crystals, confirming a conjecture proposed by the first two authors [14, Conj. 4.37].

Theorem 1.3. If $\mathcal{B}$ is a finite normal $\sqrt{\mathfrak{gl}_n}$-crystal, then $\mathsf{ch}(\mathcal{B}) = \sum_{b \in \mathsf{HW}(\mathcal{B})} G_{\mathrm{wt}(b)}$.

Thus, normal $\sqrt{\mathfrak{gl}_n}$-crystals provide a framework for establishing G-positivity. We present several applications of this framework at the end of this extended abstract. This includes new proofs of known results from [1, 2] and one new identity (Corollary 3.9).

2 Normal Crystals

Suppose $\mathcal{B}$ is a nonempty set equipped with maps $\mathrm{wt} : \mathcal{B} \to \mathbb{Z}^n$ and $e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$ for $i \in [n-1]$, where $0 \notin \mathcal{B}$. Define $\varepsilon_i, \varphi_i : \mathcal{B} \to \{0, 1, 2, \dots\} \sqcup \{\infty\}$ to be the functions that measure the maximal number of times one can apply e_i and f_i to a given element of $\mathcal{B}$ without reaching 0. Formally, this means that if $b \in \mathcal{B}$ then

$$\varepsilon_i(b) := \sup\{k \geq 0 : (e_i)^k(b) \neq 0\} \quad \text{and} \quad \varphi_i(b) := \sup\{k \geq 0 : (f_i)^k(b) \neq 0\}.$$

Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ denote the standard basis of $\mathbb{Z}^n$. The following definition is classical.

Definition 2.1 ([3]). The set $\mathcal{B}$ is a *(seminormal) $\mathfrak{gl}_n$-crystal* if for all $i \in [n-1]$ and $b, c \in \mathcal{B}$:

(a) both $\varepsilon_i(b)$ and $\varphi_i(b)$ are finite, and $\varphi_i(b) - \varepsilon_i(b) = \mathrm{wt}(b)_i - \mathrm{wt}(b)_{i+1}$;

(b) $e_i(b) = c$ if and only if $f_i(c) = b$, in which case $\mathrm{wt}(c) = \mathrm{wt}(b) + \mathbf{e}_i - \mathbf{e}_{i+1}$.

The *crystal graph* of $\mathcal{B}$ is the directed graph with vertex set $\mathcal{B}$ and labeled edges $b \xrightarrow{i} c$ whenever $f_i(b) = c \neq 0$. The *highest weight elements* of $\mathcal{B}$ are the source vertices of this graph, i.e., the elements $b \in \mathcal{B}$ with $e_i(b) = 0$ for all i. Two crystals are *isomorphic* if there exists a weight-preserving graph isomorphism between their crystal graphs.

Definition 2.2. The *trivial $\mathfrak{gl}_n$-crystal* $\mathbb{1}_n = \{\varnothing\}$ consists of a single element of weight $0 \in \mathbb{Z}^n$, with $e_i(\varnothing) = f_i(\varnothing) = 0$ for all i. The *standard $\mathfrak{gl}_n$-crystal* $\mathbb{B}_n = \{\boxed{i} : i \in [n]\}$ has the crystal graph shown in Figure 1, with weight map $\mathrm{wt}(\boxed{i}) := \mathbf{e}_i \in \mathbb{Z}^n$.

The following variant of Definition 2.1 was introduced in [14].

Definition 2.3. The set $\mathcal{B}$ is a *$\sqrt{\mathfrak{gl}_n}$-crystal* if for all $i \in [n-1]$ and $b, c \in \mathcal{B}$:

(a) both $\varepsilon_i(b)$ and $\varphi_i(b)$ are finite, and $\frac{\varphi_i(b)-\varepsilon_i(b)}{2} = \mathrm{wt}(b)_i - \mathrm{wt}(b)_{i+1} \in \mathbb{Z}$;

(b) $e_i(b) = c$ iff $b = f_i(c)$, in which case $\mathrm{wt}(c) - \mathrm{wt}(b) = \begin{cases} \mathbf{e}_i & \text{if } \varepsilon_i(b) \in 2\mathbb{Z}, \\ -\mathbf{e}_{i+1} & \text{if } \varepsilon_i(b) \notin 2\mathbb{Z}. \end{cases}$

Crystal graphs, isomorphisms, and highest weight elements for $\sqrt{\mathfrak{gl}_n}$-crystals are defined in the same way as for $\mathfrak{gl}_n$-crystals.

Definition 2.4. The *standard $\sqrt{\mathfrak{gl}_n}$-crystal* $\mathbb{S}_n$ consists of all nonempty subsets $S \subseteq [n]$, with $\mathrm{wt} : \mathbb{S}_n \to \mathbb{N}^n$ given by $\mathrm{wt}(S) = \sum_{i \in S} \mathbf{e}_i$, and crystal operators defined by

$$e_i(S) := \begin{cases} S \sqcup \{i\} & \text{if } S \cap \{i, i+1\} = \{i+1\}, \\ S \setminus \{i+1\} & \text{if } S \cap \{i, i+1\} = \{i, i+1\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$f_i(S) := \begin{cases} S \sqcup \{i+1\} & \text{if } S \cap \{i, i+1\} = \{i\}, \\ S \setminus \{i\} & \text{if } S \cap \{i, i+1\} = \{i, i+1\}, \\ 0 & \text{otherwise.} \end{cases}$$

The crystal graph of $\mathbb{S}_3$ is shown in Figure 2.

Remark 2.5. Any $\sqrt{\mathfrak{gl}_n}$-crystal $\mathcal{B}$ can be converted into a $\mathfrak{gl}_n$-crystal via the following "squaring" operation from [14, §4.1]. Define $\mathcal{B}^{(2)}$ to have the same elements and weight map as $\mathcal{B}$, but with crystal operators e_i^2 and f_i^2, where $e_i(0) = f_i(0) = 0$. The resulting object is always a (seminormal) $\mathfrak{gl}_n$-crystal [14, Prop. 4.4].

The *character* of a finite crystal $\mathcal{B}$ is $\mathrm{ch}(\mathcal{B}) = \sum_{b \in \mathcal{B}} x^{\mathrm{wt}(b)} \in \mathbb{N}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$. If $\mathcal{B}$ is a seminormal $\mathfrak{gl}_n$-crystal, then $\mathrm{ch}(\mathcal{B})$ is symmetric [3, §2.6]. Since $\mathrm{ch}(\mathcal{B}) = \mathrm{ch}(\mathcal{B}^{(2)})$, the same holds when $\mathcal{B}$ is a finite $\sqrt{\mathfrak{gl}_n}$-crystal [14, Prop. 4.4].

Example 2.6. One has $\mathsf{ch}(\mathbb{B}_n) = x_1 + \cdots + x_n$ and $\mathsf{ch}(\mathbb{S}_n) = \sum_{k=1}^{n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}$. We shall see later that these are $s_\lambda(x_1, \ldots, x_n)$ and $G_\lambda(x_1, \ldots, x_n)$ for $\lambda = (1)$.

Suppose $\mathcal{B}$ and $\mathcal{C}$ are both $\mathfrak{gl}_n$-crystals or both $\sqrt{\mathfrak{gl}_n}$-crystals. We define the disjoint union crystal $\mathcal{B} \bigsqcup \mathcal{C}$ in the obvious way. Next, consider the set of formal tensors $\mathcal{B} \otimes \mathcal{C} := \{b \otimes c : b \in \mathcal{B},\, c \in \mathcal{C}\}$. This also has a crystal structure, but the definition is more subtle:

Definition 2.7 ([14])**.** The tensor product $\mathcal{B} \otimes \mathcal{C}$ is a crystal (of respective type $\mathfrak{gl}_n$ or $\sqrt{\mathfrak{gl}_n}$) with weight map $\mathrm{wt}(b \otimes c) = \mathrm{wt}(b) + \mathrm{wt}(c)$ and the crystal operators given by

$$e_i(b \otimes c) = \begin{cases} b \otimes e_i(c) & \text{if } \varepsilon_i(b) \leq \varphi_i(c), \\ e_i(b) \otimes c & \text{if } \varepsilon_i(b) > \varphi_i(c), \end{cases} \quad f_i(b \otimes c) = \begin{cases} b \otimes f_i(c) & \text{if } \varepsilon_i(b) < \varphi_i(c), \\ f_i(b) \otimes c & \text{if } \varepsilon_i(b) \geq \varphi_i(c), \end{cases} \tag{2.1}$$

where we interpret $b \otimes 0 = 0 \otimes c = 0$. This tensor product is associative, for either type of crystal, in the sense that the natural maps $\mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D}) \to (\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D}$ are isomorphisms. Let $\mathcal{B}^{\otimes m} = \mathcal{B} \otimes \mathcal{B} \otimes \cdots \otimes \mathcal{B}$ (m factors), with $\mathcal{B}^{\otimes 0} = \mathbb{1}_n$

Recall our notions of *normal* $\mathfrak{gl}_n$- and $\sqrt{\mathfrak{gl}_n}$-crystals from Definitions 1.1 and 1.2. Both families of normal crystals are closed under tensor products and disjoint unions, but the $\sqrt{\mathfrak{gl}_n}$-family behaves less rigidly than its classical counterpart. Unlike the $\mathfrak{gl}_n$ case, there exist connected normal $\sqrt{\mathfrak{gl}_n}$-crystals with multiple highest weight elements, and non-isomorphic normal $\sqrt{\mathfrak{gl}_n}$-crystals with identical highest weights and characters [14, §4.4].

Despite these pathologies, the characters of normal $\sqrt{\mathfrak{gl}_n}$-crystals display the general positivity property described in Theorem 1.3. The next theorem explains another feature of normal $\mathfrak{gl}_n$-crystals that surprisingly extends to the $\sqrt{\mathfrak{gl}_n}$-case.

For any finite crystal $\mathcal{B}$, define $E_i : \mathcal{B} \to \mathcal{B}$ by $E_i(b) = e_i^{\varepsilon_i(b)}(b)$. Next, let $\mathrm{rect} : \mathcal{B} \to \mathcal{B}$ denote the *rectification* operator $\mathrm{rect} := (E_1 E_2 \cdots E_{n-1}) \cdots (E_1 E_2 E_3)\, (E_1 E_2)(E_1)$.

Theorem 2.8. Suppose $\mathcal{B}$ is a $\mathfrak{gl}_n$- or $\sqrt{\mathfrak{gl}_n}$-crystal. If $\mathcal{B}$ is normal, then rect defines a surjective map $\mathcal{B} \to \mathsf{HW}(\mathcal{B}) = \{b \in \mathcal{B} : e_i(b) = 0 \text{ for all } i\}$.

The sequence of indices in the definition of rect is a reduced word for longest element w_0 in the symmetric group S_n. Replacing this sequence by other reduced words for w_0 gives a family of rectification operations. When $\mathcal{B}$ is a normal $\mathfrak{gl}_n$-crystal, all of these operators are surjections $\mathcal{B} \to \mathsf{HW}(\mathcal{B})$, but this does not hold for normal $\sqrt{\mathfrak{gl}_n}$-crystals.

Using the rect operator, we can state a refinement of our main result, Theorem 1.3. For comparison, we recall another classical theorem for normal $\mathfrak{gl}_n$-crystals:

Theorem 2.9 ([3])**.** Suppose $\mathcal{B}$ is a normal $\mathfrak{gl}_n$-crystal and $c \in \mathsf{HW}(\mathcal{B})$. Then the set $\mathcal{C} = \{b \in \mathcal{B} : \mathrm{rect}(b) = c\}$ is a connected component of $\mathcal{B}$ whose isomorphism class as a $\mathfrak{gl}_n$-crystal depends only on the weight $\lambda = \mathrm{wt}(c)$, which is a partition in $\mathbb{N}^n$.

The character of the $\mathfrak{gl}_n$-crystal $\mathcal{C}$ is the classical *Schur polynomial* $s_\lambda(x_1, \dots, x_n)$; see Definition 3.1. If $\mathcal{B}$ is a normal $\sqrt{\mathfrak{gl}_n}$-crystal, then the fibers $\mathcal{C}$ of $\mathsf{rect} : \mathcal{B} \to \mathsf{HW}(\mathcal{B})$ are no longer always connected components, but we can prove the following substitute:

Theorem 2.10. Suppose $\mathcal{B}$ is a normal $\sqrt{\mathfrak{gl}_n}$-crystal and $c \in \mathsf{HW}(\mathcal{B})$. Then the sum

$$\textstyle\sum_{b \in \mathcal{B} \text{ with } \mathsf{rect}(b) = c} x^{\mathrm{wt}(b)}$$

is a symmetric polynomial that depends only on the weight $\lambda = \mathrm{wt}(c)$, which is a partition in $\mathbb{N}^n$: this gives the *symmetric Grothendieck polynomial* specified in Definition 3.1.

We do not have space in this extended abstract to explain the details of the proof of this result. The main idea is to relate rect when applied to elements of $\mathbb{S}_n^{\otimes m}$ to the *Hecke insertion algorithm* studied in [2].

The prototypical example of a normal $\sqrt{\mathfrak{gl}_n}$-crystal is the tensor power $(\mathbb{1}_n \sqcup \mathbb{S}_n)^{\otimes m}$, whose elements are m-tuples of (possibly empty) subsets of $[n]$. We mention that prior work of the third author [17] gives an explicit combinatorial formula for the crystal operators e_i and f_i on this object. This rule generalizes the well-known *signature rule* [3] for the operators on the $\mathfrak{gl}_n$-crystal $\mathbb{B}_n^{\otimes m}$.

This rule has the following useful corollary. If S_i is a subset of $[n]$ then the tensor $S = S_1 \otimes S_2 \otimes \cdots \otimes S_m$ is an element of the normal $\sqrt{\mathfrak{gl}_n}$-crystal $(\mathbb{1}_n \sqcup \mathbb{S}_n)^{\otimes m}$. Define a word $w(S_i)$ by listing the elements of S_i in order, and form $w(S)$ as the concatenation $w(S_1)w(S_2)\cdots w(S_m)$. For example, if

$$S = \{8,9\} \otimes \{4,5\} \otimes \{5\} \otimes \{1,3,4\} \otimes \{4,5\} \otimes \{5\} \quad \text{then} \quad w(S) = 89455134455.$$

A word $w = w_1 w_2 \cdots w_N$ with $w_i \in [n]$ is a *reverse lattice word* if $\mathrm{wt}(w_k w_{k+1} w_{k+2} \cdots w_N)$ is a partition for all k: here $\mathrm{wt}(w) \in \mathbb{Z}^n$ is defined by setting $\mathrm{wt}(w)_i$ equal to the number of letters in w that are equal to i. The word 431121 is an example while 31221 is not.

Proposition 2.11 ([14])**.** An element S of the $\sqrt{\mathfrak{gl}_n}$-crystal $(\mathbb{1}_n \sqcup \mathbb{S}_n)^{\otimes m}$ is highest weight if and only if $w(S)$ is a reverse lattice word.

3 Applications

In this section, we present several applications of Theorem 1.3 that follow by equipping sets of combinatorial objects with normal $\sqrt{\mathfrak{gl}_n}$-crystal structures.

3.1 Expanding skew symmetric Grothendieck functions

Combined with results in [14, 17], Theorem 1.3 leads to an alternative proof of Buch's combinatorial rule for the positive G_λ-expansion of any *skew symmetric Grothendieck polynomial*, which can be described combinatorially as follows.

The *Young diagram* of a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ is the set of boxes

$$\mathsf{D}_\lambda := \{(i,j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} : 1 \leq j \leq \lambda_i\}. \tag{3.1}$$

If $\mathsf{D}_\lambda \subseteq \mathsf{D}_\nu$, we write $\lambda \subseteq \nu$ and define $\mathsf{D}_{\nu/\lambda} := \mathsf{D}_\nu \setminus \mathsf{D}_\lambda$.

A *set-valued tableau* of shape ν/λ is a filling of the boxes in $\mathsf{D}_{\nu/\lambda}$ with nonempty subsets of $[n]$. The *distributions* of such a tableau are obtained by replacing each set-valued entry with one of its elements. A set-valued tableau T is *semistandard* if all of its distributions have weakly increasing rows and strictly increasing columns. For example,

$$T \ = \ \begin{array}{cccc} & \boxed{134} & \boxed{45} & \boxed{5} \\ \boxed{45} & \boxed{5} & & \\ \boxed{89} & & & \end{array} \tag{3.2}$$

is a semistandard set-valued tableau of shape $\nu/\lambda = (4,2,1)/(1)$. When T is a set-valued tableau, we write $(i,j) \in T$ to indicate that box (i,j) is filled in T, and let T_{ij} denote the entry in that box. For example, for T as in (3.2) we have $T_{22} = \{5\}$ and $T_{31} = \{8,9\}$.

Let $\mathsf{SetTab}_n(\nu/\lambda)$ denote the set of all semistandard set-valued tableaux of shape ν/λ, and write $\mathsf{SetTab}_n(\nu)$ for $\mathsf{SetTab}_n(\nu/\varnothing)$. Define the *weight* of T by

$$\mathsf{wt}(T) := \textstyle\sum_{(i,j)\in T} \sum_{k \in T_{ij}} \mathbf{e}_k \in \mathbb{Z}^n. \tag{3.3}$$

For instance, when $n = 9$, the tableau in (3.2) has $\mathsf{wt}(T) = (1,0,1,3,4,0,0,1,1)$.

Definition 3.1. If $\lambda \subseteq \nu$, then the *symmetric Grothendieck polynomial* of skew shape ν/λ is

$$G_{\nu/\lambda}(x_1, x_2, \ldots, x_n) = \textstyle\sum_{T \in \mathsf{SetTab}_n(\nu/\lambda)} x^{\mathsf{wt}(T)}.$$

We set $G_\nu(x_1, \ldots, x_n) := G_{\nu/\varnothing}(x_1, \ldots, x_n)$, and define $s_\nu(x_1, \ldots, x_n)$ as the homogeneous term of $G_\nu(x_1, \ldots, x_n)$ of lowest degree.

If T is a set-valued tableau with m boxes, then its *column reading word*, denoted $\mathsf{col}(T)$, is the element of $\mathbb{S}_n^{\otimes m}$ obtained by reading the entries of T up each column, moving from left to right. For the tableau in (3.2), this gives

$$\mathsf{col}(T) = \{8,9\} \otimes \{4,5\} \otimes \{5\} \otimes \{1,3,4\} \otimes \{4,5\} \otimes \{5\}$$

Notice that in the notation of Proposition 2.11, we also have $w(\mathsf{col}(T)) = 89455134455$.

We equip $\mathsf{SetTab}_n(\nu/\lambda)$ with a $\sqrt{\mathfrak{gl}_n}$-crystal structure whose weight map is (3.3). The operators f_i and e_i are defined by pulling back the crystal operators on $\mathbb{S}_n^{\otimes m}$ via the map $\mathsf{col}(\cdot)$. By [14, 17], this construction yields a normal $\sqrt{\mathfrak{gl}_n}$-crystal structure on $\mathsf{SetTab}_n(\nu/\lambda)$. By Proposition 2.11, a tableau $T \in \mathsf{SetTab}_n(\nu/\lambda)$ is highest weight if and only if $w(\mathsf{col}(T))$ is a reverse lattice word. Thus, Theorem 1.3 recovers the following:

Corollary 3.2 (Buch [1])**.** If $\lambda \subseteq \nu$ are partitions, and $a^\nu_{\lambda\mu}$ is the number of set-valued tableaux $T \in \mathsf{SetTab}_n(\nu/\lambda)$ for which $w(\mathsf{col}(T))$ is a reverse lattice word of weight μ, then $G_{\nu/\lambda}(x_1, \ldots, x_n) = \sum_\mu a^\nu_{\lambda\mu} G_\mu(x_1, \ldots, x_n)$, where the sum is over all partitions μ.

3.2 The Littlewood–Richardson rule of Buch

As another application of Theorem 1.3, we give an alternate proof of Buch's *Littlewood–Richardson rule*, which describes the decomposition of $G_\lambda(x_1,\dots,x_n)G_\mu(x_1,\dots,x_n)$ for two partitions λ and μ. We equip $\mathsf{SetTab}_n(\lambda) \times \mathsf{SetTab}_n(\mu)$ with a normal $\sqrt{\mathfrak{gl}_n}$-crystal structure given by the tensor product of the crystals on $\mathsf{SetTab}_n(\lambda)$ and $\mathsf{SetTab}_n(\mu)$. Then Theorem 1.3 immediately recovers the following rule.

Corollary 3.3 ([1]). If λ and μ are partitions, and $c^\nu_{\lambda\mu}$ is the number of pairs $(T_1, T_2) \in \mathsf{SetTab}_n(\lambda) \times \mathsf{SetTab}_n(\mu)$ with $w(\mathsf{col}(T_1))w(\mathsf{col}(T_2))$ a reverse lattice word of weight ν, then $G_\lambda(x_1,\dots,x_n)\,G_\mu(x_1,\dots,x_n) = \sum_\nu c^\nu_{\lambda\mu} G_\nu(x_1,\dots,x_n)$, with the sum over all partitions ν.

3.3 Symmetric Grothendieck functions of permutations

There is a generalization of $G_\lambda(x_1,x_2,\dots,x_n)$ indexed by permutations rather than partitions. These polynomials, which arise as the stable limits of K-theoretic representatives for *Schubert varieties* in the complete flag variety [4], are known to be G-positive [2]. We outline here how to derive this positivity property from Theorem 1.3.

Let S_∞ denote the group of permutations of $\mathbb{Z}_{>0}$ with finite support. For each $m \in \mathbb{Z}_{>0}$, we view $S_m \subset S_\infty$ as the subgroup of permutations fixing all points outside $[m]$. The *Demazure product* on S_∞ is the unique associative operation $\circ : S_\infty \times S_\infty \to S_\infty$ defined by $s_i \circ s_i = s_i$ for $s_i := (i, i+1) \in S_\infty$ and $w \circ s_i = ws_i$ for $w \in S_\infty$ with $w(i) < w(i+1)$. A *Hecke word* for $w \in S_\infty$ is a finite sequence of positive integers $i_1 i_2 \cdots i_k$ such that

$$w = s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_k}.$$

A *decreasing Hecke factorization* (DHF) of w is an n-tuple $a = (a^1,\dots,a^n)$ of strictly decreasing words a^i such that $a^1 \circ \cdots \circ a^n \in \mathcal{H}(w)$, the set of Hecke words for w. Define

$$\mathrm{wt}(a) := (\ell(a^1), \ell(a^2), \dots, \ell(a^n)) \in \mathbb{Z}^n, \tag{3.4}$$

and write $\mathsf{Decr}_n(w)$ for the set of all DHFs of w. For each permutation $w \in S_\infty$, define

$$G_w(x_1,x_2,\dots,x_n) := \textstyle\sum_{a \in \mathsf{Decr}_n(w)} x^{\mathrm{wt}(a)}.$$

Example 3.4. Let $w = [1,4,3,2] \in S_4$. Then $\mathsf{Decr}_2(w) = \{(2,32),(32,3),(32,32)\}$, so

$$G_w(x_1,x_2) = x_1x_2^2 + x_1^2x_2 + x_1^2x_2^2 = G_{(2,1)}(x_1,x_2).$$

Choose $m \in \mathbb{Z}_{>0}$ such that $w \in S_{m+1}$. Then all Hecke words of w have letters in $[m]$. We equip $\mathsf{Decr}_n(w)$ with a normal $\sqrt{\mathfrak{gl}_n}$-crystal structure whose weight map is given by (3.4). To describe the operators f_i and e_i, we first embed $\mathsf{Decr}_n(w)$ into $(\mathbb{1}_n \sqcup \mathbb{S}_n)^{\otimes m}$: given $a = (a^1, \dots, a^n) \in \mathsf{Decr}_n(w)$, define

$$\mathsf{svword}(a) := S_1 \otimes S_2 \otimes \cdots \otimes S_m \quad \text{where } S_j = \{i \in [n] : m+1-j \in a^i\}.$$

For example, svword acts on $\mathsf{Decr}_2(w)$ from Example 3.4 as

$$(2,32) \mapsto \{2\} \otimes \{1,2\} \otimes \varnothing, \quad (32,3) \mapsto \{1,2\} \otimes \{1\} \otimes \varnothing, \quad (32,32) \mapsto \{1,2\} \otimes \{1,2\} \otimes \varnothing.$$

We then define f_i and e_i by pulling back the operators on $\mathsf{SetWord}_{m,n}$.

Theorem 3.5. The crystal structure on $\mathsf{Decr}_n(w)$ defined above is a normal $\sqrt{\mathfrak{gl}_n}$-crystal.

The highest weight elements in this crystal can be identified with *increasing tableaux*: fillings of a Young diagram D_λ with entries in $[m]$ whose rows and columns are strictly increasing. The *row reading word* of an increasing tableau T is the word $\mathsf{row}(T)$ obtained by reading the entries row by row from left to right, starting with the bottom row (in English notation). The *reverse row word* $\mathsf{revrow}(T)$ is the reversal of $\mathsf{row}(T)$.

Proposition 3.6. An element $a = (a^1, \dots, a^n) \in \mathsf{Decr}_n(w)$ is a highest-weight element if and only if the following properties hold:

(a) the weight of a is a partition λ; and

(b) writing a^i in row i of D_λ, increasingly from left to right, gives an increasing tableau.

Moreover, the operation in (b) gives a bijection between highest weight elements of $\mathsf{Decr}_n(w)$ and increasing tableaux T such that $\mathsf{revrow}(T) \in \mathcal{H}(w)$.

For example, $a = (32, 3)$ is a highest-weight element since the associated tableau

$$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

is increasing. Combining Theorems 1.3 and 3.5 with Proposition 3.6 recovers this result:

Corollary 3.7 ([2]). If $w \in S_{m+1}$ is a permutation, and $c_{w\lambda}$ is the number of increasing tableaux T of shape λ such that $\mathsf{revrow}(T) \in \mathcal{H}(w)$, then

$$G_w(x_1, \dots, x_n) = \textstyle\sum_\lambda c_{w\lambda} G_\lambda(x_1, \dots, x_n),$$

where the sum is over all partitions λ.

3.4 Generating functions of set-valued decomposition tableaux

Finally, we explain one new identity that can be proved using Theorem 1.3. Assume $\lambda = (\lambda_1 > \lambda_2 > \cdots \geq 0)$ is a *strict* integer partition. A *shifted tableau* of shape λ is a filling of the *shifted diagram* $\mathsf{SD}_\lambda = \{(i, i+j-1) : (i,j) \in \mathsf{D}_\lambda\}$ by numbers in $[n]$.

Let T be a shifted tableau whose rows read left-to-right are *hook words*, meaning integer sequences $w_1 w_2 \cdots w_N$ with $w_1 \geq w_2 \geq \cdots \geq w_m < w_{m+1} < w_{m+2} < \cdots < w_N$ for some $1 \leq m \leq N$. Following [6, 14], we define T to be a *(semistandard) decomposition tableau* if none of the following patterns occur in consecutive rows:

$$\begin{array}{|c|c|c|}\hline a & \cdots & \\ \hline \multicolumn{1}{c|}{} & \cdots & b \\ \cline{2-3}\end{array} \quad \begin{array}{|c|c|c|c|c|}\hline & \cdots & a & \cdots & \\ \hline \multicolumn{1}{c|}{} & \cdots & c & \cdots & b \\ \cline{2-5}\end{array} \text{ for } a \leq b \leq c \quad \text{or} \quad \begin{array}{|c|c|c|}\hline y & \cdots & z \\ \hline \multicolumn{1}{c|}{} & \cdots & x \\ \cline{2-3}\end{array} \quad \begin{array}{|c|c|c|c|c|}\hline & \cdots & y & \cdots & z \\ \hline \multicolumn{1}{c|}{} & \cdots & & \cdots & x \\ \cline{2-5}\end{array} \text{ for } x < y < z.$$

The tableaux here are drawn in English notation. The leftmost boxes are on the main diagonal, and the boxes $\boxed{\cdots}$ with ellipses indicate zero or more intervening columns.

A *set-valued decomposition tableau* of shape λ is a filling of SD_λ by non-empty subsets of $[n]$ whose distributions are all decomposition tableaux. An example for $\lambda = (3,2)$ is

$$T = \begin{array}{|c|c|c|}\hline 45 & 3 & 234 \\ \hline \multicolumn{1}{c|}{} & 12 & 3 \\ \cline{2-3}\end{array}.$$

Let $\mathsf{SetDecTab}_n(\lambda)$ be the family of all set-valued decomposition tableaux of shape λ. We define the *reverse row reading word* $\mathsf{revrow}(T)$ of such a tableau T by reading its rows from right to left, starting with the top row in English notation, and tensoring the set-valued entries together. Also let $\mathrm{wt}(T)$ be as in (3.3). Our example has

$$\mathsf{revrow}(T) = \{2,3,4\} \otimes \{3\} \otimes \{4,5\} \otimes \{3\} \otimes \{1,2\} \quad \text{and} \quad \mathrm{wt}(T) = (1,2,3,2,1).$$

Definition 3.8. For strict partitions λ let $GP^{[\mathsf{dec}]}_\lambda(x_1, \ldots, x_n) := \sum_{T \in \mathsf{SetDecTab}_n(\lambda)} x^{\mathrm{wt}(T)}$.

Cho and Ikeda have conjectured [14, Conj. 3.2] that this polynomial coincides with the *K-theoretic Schur P-function* $GP_\lambda(x_1, \ldots, x_n)$ from [8], which is the weight-generating function for a different family of *(semistandard) set-valued marked shifted tableaux*.

It is known [14, Cor. 3.11] that $GP^{[\mathsf{dec}]}_\lambda(x_1, \ldots, x_n)$ at least has the triangular form $GP_\lambda(x_1, \ldots, x_n) + \sum_{|\mu| > |\lambda|} \mathbb{Z} GP_\mu(x_1, \ldots, x_n)$. Now, we can prove that $GP^{[\mathsf{dec}]}_\lambda$ is G-positive:

Corollary 3.9. If λ is strict then $GP^{[\mathsf{dec}]}_\lambda(x_1, \ldots, x_n) = \sum_\mu g_{\lambda\mu} G_\mu(x_1, \ldots, x_n)$ where the sum is over all partitions μ with at most n parts and $g_{\lambda\mu}$ is the number of tableaux $T \in \mathsf{SetDecTab}_n(\lambda)$ for which $w(\mathsf{revrow}(T))$ is a reverse lattice word of weight μ.

Proof sketch. Prior work of the first two authors [14, Thm. 4.18] shows that $\mathsf{SetDecTab}_n(\lambda)$ has a normal $\sqrt{\mathfrak{gl}_n}$-crystal structure with weight map (3.3) such that $T \in \mathsf{SetDecTab}_n(\lambda)$ is highest weight if and only if $w(\mathsf{revrow}(T))$ is a reverse lattice word. Apply Theorem 1.3 to this crystal. $\square$

We mention that it is known [13, Thm. 3.27] that the polynomial $GP_\lambda(x_1, x_2, \dots, x_n)$ is G-positive. Thus, one approach to proving [14, Conj. 3.2] would be to show that the relevant G-expansion of $GP_\lambda(x_1, x_2, \dots, x_n)$ is also $\sum_\mu g_{\lambda\mu} G_\mu(x_1, x_2, \dots, x_n)$.

Example 3.10. We can use Corollary 3.9 to decompose $GP^{[\mathsf{dec}]}_\lambda(x_1, \dots, x_n)$ when $\lambda = (m)$ has one part. Let $k = \max\{1, m-n+1\}$. One can show as an exercise from [14, §4] that the highest weight elements in the $\sqrt{\mathfrak{gl}_n}$-crystal $\mathsf{SetDecTab}_n((m))$ are the tableaux of shape (m) of the form

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & \cdots & 1 & 2 & 3 & \cdots & j \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & \cdots & 1 & 12 & 3 & 4 & \cdots & j \\ \hline \end{array} \quad \text{with } j \leq n.$$

It then follows from Corollary 3.9 that

$$GP^{[\mathsf{dec}]}_{(m)}(x_1, \dots, x_n) = \textstyle\sum_{i=k}^m G_{(i,1^{m-i})}(x_1, \dots, x_n) + \sum_{i=k+1}^m G_{(i,1^{m+1-i})}(x_1, \dots, x_n). \qquad (3.5)$$

3.5 Future directions

Many questions and problems remain open concerning $\sqrt{\mathfrak{gl}_n}$-crystals. It would be very interesting to find connections between normal square root crystals and representation theory. Another open problem is to determine if there are local axioms in the style of [5, 16] that classify which square root crystals are normal.

We also expect that there are interesting versions of square root crystals for other types. There is at least a theory of (normal) square root crystals [14] associated to the queer Lie superalgebra $\mathfrak{q}_n$. Such crystals have the same relationship to the *$\mathfrak{q}_n$-crystals* introduced in work of Grantcharov et al. [6] as $\sqrt{\mathfrak{gl}_n}$-crystals do to $\mathfrak{gl}_n$-crystals. The first two authors conjectured a "shifted" analogue of Theorem 1.3 for these objects, in which G-positivity is replaced by *GP-positivity*; see [14, Conj. 4.36].

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