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Positivity in triple Schubert calculus

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Abstract. We prove Samuel's conjecture on certain Graham positivity of the expansion coefficient of two double Schubert polynomials in three sets of variables by establishing a refined version of Graham's positivity theorem. As a corollary, we prove Kirillov's conjecture on the positivity of skew divided difference operators applied to Schubert polynomials.

Keywords: Schubert polynomials, Graham positivity

1 Introduction

Double Schubert polynomials $\{\mathfrak{S}_w(\mathbf{x};\mathbf{y}) \mid w \in S_n\}$, indexed by permutations, are polynomial representatives of Schubert classes in the torus-equivariant cohomology ring of the flag variety $Fl_n(\mathbb{C})$, where $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ are sequences of variables. They can be defined via divided difference operators, and enjoy rich algebraic and combinatorial properties. In particular, they can be computed via different combinatorial models including pipe dreams [5, 13] and bumpless pipe dreams [16].

In this paper, we focus on the coefficients $c^w_{u,v}(\mathbf{y},\mathbf{t})$ from the expansion

$$\mathfrak{S}_u(\mathbf{x};\mathbf{y}) \cdot \mathfrak{S}_v(\mathbf{x};\mathbf{t}) = \sum_{w \in S_\infty} c^w_{u,v}(\mathbf{y},\mathbf{t}) \cdot \mathfrak{S}_w(\mathbf{x};\mathbf{t}). \tag{1.1}$$

In the case $\mathbf{y} = \mathbf{t} = \mathbf{0}$, $c^w_{u,v} := c^w_{u,v}(\mathbf{0},\mathbf{0}) \in \mathbb{N}$ is the *Schubert structure constant* which is the core of Hilbert's fifteenth problem, and finding combinatorial models for these coefficients has sparked tremendous interest throughout the years in the community of algebraic combinatorics and algebraic geometry. In the case where u and v are k-Grassmannian permutations, $\mathfrak{S}_u(\mathbf{x};\mathbf{y})$ and $\mathfrak{S}_v(\mathbf{x};\mathbf{t})$ are the *factorial Schur polynomials*; Molev–Sagan [21] first provided a combinatorial formula for these coefficients and Knutson–Tao [14] provided geometric meanings to the expansion.

Our main result is the following:

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Theorem 1.1 ([22, Conjecture 1.1]). *For $u, v, w \in S_\infty$, $c^w_{u,v}(\mathbf{y}, \mathbf{t}) \in \mathbb{N}[t_i - y_j]_{i,j\geq 1}$.*

Besides the above-mentioned situations, a well-studied case of Theorem 1.1 is that of *separated descents* ([10, 15]), i.e. $\max \mathrm{Des}(u) \leq \min \mathrm{Des}(v)$, where Samuel [22] established a positive formula, and Fan, Guo and the second author [7] found another formula using puzzles in the generality of double Grothendieck polynomials.

Setting $\mathbf{y} = \mathbf{0}$, we obtain Kirillov's conjecture:

Corollary 1.2 ([12, Conjecture 1]). *For $u, v, w \in S_\infty$, $\partial_{w/v}\mathfrak{S}_u(\mathbf{x}) \in \mathbb{N}[\mathbf{x}]$.*

The skew divided difference operators $\partial_{w/v}$, first introduced by Macdonald [19], provide a method for computing Schubert structure constants. These operators have attracted significant attention due to their connections to Fomin–Kirillov algebras [8] . Their generalizations and positivity properties have been extensively studied in works including [3, 4, 18].

Our proof relies on a refined version of Graham's positivity theorem [9] (Theorem 2.4). To apply this result, we establish a new geometric interpretation (Section 2.3) for the coefficients $c^w_{u,v}(\mathbf{y}, \mathbf{t})$, distinct from the one given by Knutson–Tao [14]. A byproduct of this theorem is a geometric explanation for the positivity in Billey's formula (Section 2.2). We end our discussion with a conjecture for Grothendieck polynomials (Section 2.6).

2 Proof of the main theorem

2.1 A refined Graham positivity

Let G be a connected, complex, reductive algebraic group, $B \subset G$ be a Borel subgroup, B^- be its opposite Borel subgroup, $T = B \cap B^-$ be a maximal torus, N (resp., N^-) be the unipotent radical of B (resp., B^-), $W = N_G(T)/T$ be the Weyl group (where $N_G(T)$ is the normalizer subgroup) and Φ be the associated root system, with positive roots Φ^+ and simple roots $\Delta = \{\alpha_1, \ldots, \alpha_r\}$. For $\alpha \in \Phi$, write s_α for the corresponding reflection, and write s_i for s_{α_i} for simplicity. For $w \in W$, its (left) *inversion set* is $I(w) := \{\alpha \in \Phi^+ \mid w^{-1}\alpha \in \Phi^-\}$ and its *non-inversion set* is $J(w) := \{\alpha \in \Phi^+ \mid w^{-1}\alpha \in \Phi^+\} = I(ww_0)$, where w_0 is the longest element in W. For each $w \in W$, we pick a representative of it in the normalizer subgroup $N_G(T)$, and also denote it by w, slightly abusing notation.

The *flag variety G/B* admits a *Bruhat decomposition* $\bigsqcup_{w\in W} B^-wB/B$ into *Schubert cells*. Their closures $\overline{B^-wB/B} \subseteq G/B$ are the *Schubert varieties*. The *Schubert classes*

$$\{[\overline{B^-wB/B}]_T \mid w \in W\}$$

form an $H^*_T(\mathrm{pt})$-basis of $H^*_T(Fl_m(\mathbb{C}))$.

We define the following closed subgroups of G.

Definition 2.1. For $w \in W$, define $N^-(w) := N^- \cap wN^-w^{-1}$ and $B^-(w) = T \cdot N^-(w)$.

By [11, Section 28], as a variety, $N^-(w)$ is isomorphic to the Schubert cell $B^-wB/B \cong \mathbb{C}^{\ell(w_0)-\ell(w)}$ via $x \mapsto xwB/B$. In particular, $N^-(w)$ is connected. Its Lie algebra is

$$\mathfrak{n}^-(w) = \operatorname{span}(F_\alpha \mid \alpha \in J(w)) \subseteq \mathfrak{g} := \operatorname{Lie} G$$

where F_α is a root vector of weight $-\alpha$.

Lemma 2.2. *If $ws_i > w$, then $N^-(ws_i)$ is a normal subgroup of $N^-(w)$.*

Proof. Recall that $[F_\alpha, F_\beta] \in \mathbb{C}^\times F_{\alpha+\beta}$ if $\alpha+\beta$ is a root, and $[F_\alpha, F_\beta] = 0$ otherwise. As $ws_i > w$, we have $J(w) = J(ws_i) \cup \{w\alpha_i\}$ and thus $\mathfrak{n}^-(ws_i) \subset \mathfrak{n}^-(w)$, $N^-(ws_i)$ is a closed subgroup of $N^-(w)$ by [11, Theorem 13.1]. In fact, $\mathfrak{n}^-(ws_i)$ is an ideal of $\mathfrak{n}^-(w)$. To check this, take any $\alpha \in J(ws_i)$ and $\beta \in J(w)$, and it suffices to show $[F_\alpha, F_\beta] \in \mathfrak{n}^-(ws_i)$. If $\alpha + \beta \notin \Phi$, $[F_\alpha, F_\beta] = 0$. Thus, consider $\gamma = \alpha + \beta \in \Phi^+$. If $\beta \in J(ws_i)$, then $\gamma \in J(ws_i)$ as well; and if $\beta = w\alpha_i$, $\gamma \neq w\alpha_i \in J(w)$, so $\gamma \in J(ws_i)$. Indeed, $\mathfrak{n}^-(ws_i)$ is an ideal of $\mathfrak{n}^-(w)$ and by [11, Theorem 13.3], $N^-(ws_i)$ is a normal subgroup of $N^-(w)$. $\square$

Example 2.3. Let $G = GL_3$. We have

w	id	s_1	s_2	s_1s_2	s_2s_1	$s_1s_2s_1$
$N^-(w)$	$\begin{bmatrix}1&0&0\\ *&1&0\\ *&*&1\end{bmatrix}$	$\begin{bmatrix}1&0&0\\ 0&1&0\\ *&*&1\end{bmatrix}$	$\begin{bmatrix}1&0&0\\ *&1&0\\ *&0&1\end{bmatrix}$	$\begin{bmatrix}1&0&0\\ 0&1&0\\ 0&*&1\end{bmatrix}$	$\begin{bmatrix}1&0&0\\ *&1&0\\ 0&0&1\end{bmatrix}$	$\begin{bmatrix}1&0&0\\ 0&1&0\\ 0&0&1\end{bmatrix}$

Note that Lemma 2.2 cannot be generalized to right weak Bruhat order, since being a normal subgroup (denoted by $\trianglelefteq$) is not a transitive relation, for example, $N^-(s_1s_2) \trianglelefteq N^-(s_1) \trianglelefteq N^-(\mathrm{id})$, while $N^-(s_1s_2) \ntrianglelefteq N^-(\mathrm{id})$.

We are now in a position to state a refined version of Graham positivity theorem.

Theorem 2.4. *Let B^- act on a non-singular variety X, and let Y be a $B^-(w)$-invariant effective cycle in X. Then there exist B^--invariant effective cycles $Z_1, \dots, Z_m$ in X such that*

$$[Y]_T \in \sum_{i=1}^m \mathbb{N}[-\alpha]_{\alpha \in I(w)} \cdot [Z_i]_T$$

in $H_T^*(X)$. Recall an effective cycle is a formal $\mathbb{N}$-combination of subvarieties.

Proof. We use induction on $\ell(w)$. When $w = \mathrm{id}$, $B^-(w) = B^-$, so there is nothing to prove. Assume the theorem holds for w. Let us prove the statement for $ws_i > w$. By Lemma 2.2, the pair $B^-(ws_i) \subset B^-(w)$ satisfies the condition (see [2, p. 365]) of [2, Proposition 19.4.4] with $\chi = -w\alpha_i$. So there exist $B^-(w)$-invariant effective cycles Z_1 and Z_2 such that $[Y]_T = [Z_1]_T + \chi[Z_2]_T$. Since $I(ws_i) = I(w) \cup \{w\alpha_i\}$, the inductive step is now established. $\square$

Corollary 2.5. *Let $X = G/B$ be the flag variety. Under the above setting, we have*

$$[Y]_T \in \sum_{w \in W} \mathbb{N}[-\alpha]_{\alpha \in I(w)} \cdot [\overline{B^- wB/B}]_T$$

in $H_T^*(G/B)$.

Proof. Since flag variety has finitely many B^--orbits, any B^--invariant effective cycle over G/B must be a non-negative combination of Schubert classes $[\overline{B^- wB/B}]_T$. □

Remark. When $w = w_0$ is the longest element, $B^-(w_0) = T$ and Theorem 2.4 gives Graham's positivity theorem [9, Theorem 3.2]. Precisely, $I(w_0) = \Phi^+$ is the set of positive roots, and thus the coefficients are positive in negative roots.

Our assumption is stronger than that of Graham's, since a $B^-(w)$-invariant cycle is necessarily T-invariant, yielding a stronger positivity result where roots are restricted to $I(w)$ rather than all positive roots. A similar generalization in this direction appeared in the proof of [1, Theorem 8.7 and Corollary 8.11].

In Section 2.4, we will give a motivating example of the above argument.

Remark. The results in this section admit a generalization by replacing $I(w)$ by any subsets $I \subseteq \Phi^+$ which is an upper ideal for a total order $\leq$ on Φ^+ such that

$$\text{if } \alpha < \beta \text{ such that } \alpha + \beta \in \Phi^+, \text{ then } \alpha < \alpha + \beta.$$

Then the case $I = I(w)$ corresponds to the special case when $<$ is a reflection order. Recall a reflection order is a total order $\leq$ over Φ^+ with

$$\text{if } \alpha < \beta \text{ such that } \alpha + \beta \in \Phi^+, \text{ then } \alpha < \alpha + \beta < \beta.$$

For example, when $G = GL_3$, there are two reflection orders

$$\alpha_1 < \alpha_1 + \alpha_2 < \alpha_2 \qquad \text{or} \qquad \alpha_2 < \alpha_1 + \alpha_2 < \alpha_1.$$

Another interesting case is when $<$ is a linear extension of the dominant order, i.e.

$$\text{if } \alpha < \beta \text{ such that } \alpha + \beta \in \Phi^+, \text{ then } \beta < \alpha + \beta.$$

For example, when $G = GL_3$, there are two extensions of the dominant order

$$\alpha_1 < \alpha_2 < \alpha_1 + \alpha_2 \qquad \text{or} \qquad \alpha_2 < \alpha_1 < \alpha_1 + \alpha_2.$$

In particular, Theorem 2.4 can be extended to group $B_I = T \cdot N_I$ with $N_I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & 0 & 1 \end{bmatrix}$.

2.2 A geometric explanation of positivity in Billey's formula

As an application of Theorem 2.4, we give a geometric explanation of positivity in Billey's formula [6]. See [23] for a great survey on this subject.

The T-fixed points of G/B are $(G/B)^T = \{wB/B \mid w \in W\}$. Define the localization to be the restriction map $\cdot|_w : H_T^*(G/B) \to H_T^*(wB/B) \simeq H_T^*(\mathrm{pt})$. Billey's formula [6] is a combinatorial formula of the localization of Schubert classes $[\overline{B^- uB/B}]_T|_w$ for any $u, w \in W$. Precisely, for a fixed reduced word $w = s_{i_1} \cdots s_{i_\ell}$, the localization

$$[\overline{B^- uB/B}]_T|_w = \sum_J \prod_{j \in J} \beta_j \in H_T^*(\mathrm{pt})$$

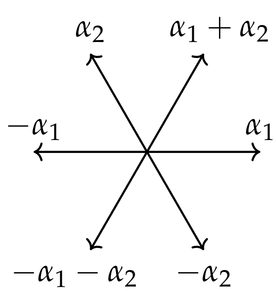
where the sum over all reduced subwords for u, i.e. subsets $J \subseteq \{1, \ldots, \ell\}$ of size $\ell(u)$ such that $\prod_{j \in J} s_{i_j} = u$ and

$$\beta_j = s_{i_1} \cdots s_{i_{j-1}} \alpha_{i_j} \in I(w), \qquad j = 1, \ldots, \ell.$$

Example 2.6. Let us consider $G = GL_3$, and $w = s_1 s_2 s_1$. We have

$$\beta_1 = \alpha_1, \qquad \beta_2 = s_1\alpha_2 = \alpha_1 + \alpha_2, \qquad \beta_3 = s_1 s_2 \alpha_1 = \alpha_2, \qquad I(w) = \{\beta_1, \beta_2, \beta_3\}.$$

The following table shows the formula of $[\overline{B^- uB/B}]_T|_w$ for all $u \in W$.



u	subwords	$[\overline{B^- uB/B}]_T\|_w$
id	$\cancel{s_1}\ \cancel{s_2}\ \cancel{s_1}$	1
s_1	$s_1\ \cancel{s_2}\ \cancel{s_1}$ and $\cancel{s_1}\ \cancel{s_2}\ s_1$	$\alpha_1 + \alpha_2$
s_2	$\cancel{s_1}\ s_2\ \cancel{s_1}$	$\alpha_1 + \alpha_2$
$s_1 s_2$	$s_1\ s_2\ \cancel{s_1}$	$\alpha_1(\alpha_1 + \alpha_2)$
$s_2 s_1$	$\cancel{s_1}\ s_2\ s_1$	$(\alpha_1 + \alpha_2)\alpha_2$
$s_1 s_2 s_1$	$s_1\ s_2\ s_1$	$\alpha_1(\alpha_1 + \alpha_2)\alpha_2$

From its explicit form, we see that

$$[\overline{B^- uB/B}]_T|_w \in \mathbb{N}[\alpha]_{\alpha \in I(w)}. \tag{2.1}$$

We provide a geometric explanation of this positivity using Corollary 2.5.

Recall that the point $w_0 B/B = B^- w_0 B/B$ is invariant under B^-. So the torus fixed point $ww_0 B/B$ is invariant under $wB^- w^{-1}$. This implies that $ww_0 B/B$ is invariant under $B^-(w)$. By Corollary 2.5, we have

$$[ww_0 B/B]_T \in \sum_{u \in W} \mathbb{N}[-\alpha]_{\alpha \in I(w)} \cdot [\overline{B^- uB/B}]_T.$$

Following [20, Section 3] and [2, Section 16.5], for a Weyl group element $w \in W$, the automorphism $gB \mapsto wgB/B$ induces a left action $w^L : H_T^*(G/B) \to H_T^*(G/B)$. We remark that w^L is not $H_T^*(\mathsf{pt})$-linear unless $w = \mathsf{id}$, but it is semilinear with respect to the automorphism of $H_T^*(\mathsf{pt})$ induced by w. Applying w_0^L to the above, we get

$$[w_0 w w_0 B/B]_T \in \sum_{u \in W} \mathbb{N}[-w_0\alpha]_{\alpha \in I(w)} \cdot [\overline{B w_0 u B/B}]_T.$$

As $I(w_0 w w_0) = \{-w_0\alpha \mid \alpha \in I(w)\}$, replacing $w_0 w w_0$ by w and $w_0 u$ by u, we can rewrite

$$[wB/B]_T \in \sum_{u \in W} \mathbb{N}[\alpha]_{\alpha \in I(w)} \cdot [\overline{BuB/B}]_T. \tag{2.2}$$

Now let us take the Poincaré pairing with $[\overline{B^- uB/B}]_T$ on both sides. The left-hand side is $[\overline{B^- uB/B}]_T|_w$, and the right-hand side is the coefficient of $[\overline{BuB/B}]_T$ in (2.2) by [2, Proposition 7.3]. This gives (2.1).

2.3 Application to triple Schubert calculus

We restrict to the case of $G = GL_m(\mathbb{C})$. By [2, Theorem 10.6.4], the class $[\overline{B^- \pi B/B}]_T$ is represented by the *double Schubert polynomial* $\mathfrak{S}_\pi(\mathbf{x};\mathbf{t})$. We will not be working with the algebraic definition of $\mathfrak{S}_\pi(\mathbf{x};\mathbf{t})$'s. For any permutation $\pi \in S_m$, we naturally identify it as $\pi \in S_m \hookrightarrow S_\infty$ via $\pi(k) = k$ for all $k > m$.

Fix permutations u, v. By picking $n \gg 0$, we can assume Equation (1.1) only involves those $w \in S_n$. Let $G := GL_{2n}(\mathbb{C})$ and B, B^-, T be as above. We identify $H_T^*(\mathsf{pt}) = \mathbb{Z}[t_1, \ldots, t_{2n}]$ with $t_i = -\epsilon_i := -c_1(\mathbb{C}_{\epsilon_i})$, where ϵ_i is the character of T corresponding to the i-th diagonal entry. We rename the variables $t_{n+i} = y_i$ for all $i \in [n] := \{1, 2, \ldots, n\}$. Consider a special permutation $\tau \in S_{2n}$ such that $\tau(i) = n+i$, $\tau(n+i) = i$ for all $i \in [n]$. By [2, Section 16.5], $[\tau\overline{B^- uB/B}]_T$ is represented by $\mathfrak{S}_u(\mathbf{x};\tau\mathbf{t}) = \mathfrak{S}_u(\mathbf{x};\mathbf{y})$.

Lemma 2.7. *The intersection $\tau\overline{B^- uB/B} \cap \overline{B^- vB/B}$ is proper and transverse at the generic point.*

Proof. Let $w_0^{(m)} \in S_m$ be the longest permutation $m\ m{-}1 \cdots 1$. For permutations $w, \pi \in S_n$, write $w \times \pi \in S_{2n}$ as the direct sum of w and π; that is, $w \times \pi(i)$ is $w(i)$ if $i \le n$, and is $\pi(i-n)+n$ if $i > n$. Since $v \in S_n$, $s_i v > v$ for $n \le i < 2n$, and thus $\overline{B^- vB/B}$ is invariant under s_i, where $s_i = (i\ i{+}1)$ is the simple transposition. Therefore, we have $\overline{B^- vB/B} = u_0\overline{B^- vB/B}$ where $u_0 = 1 \times w_0^{(n)}$. Similarly, $\tau\overline{B^- uB/B} = \tau u_0 \overline{B^- uB/B}$. As $u_0^{-1}\tau u_0 = w_0^{(2n)}$, the lemma follows from [2, Section 19.3]. □

We are now ready to prove our main theorem.

Proof of Theorem 1.1. By Lemma 2.7, we can rewrite the coefficients of interest via

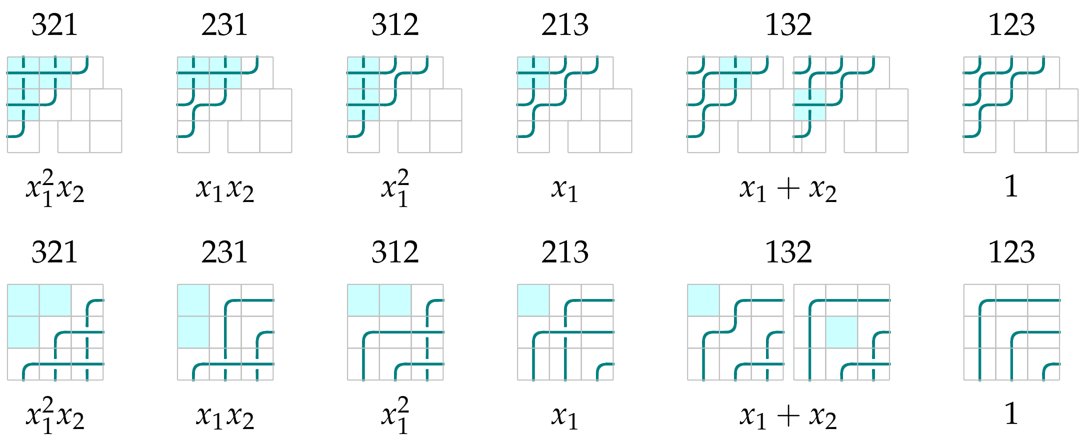
$$[\tau\overline{B^- uB/B} \cap \overline{B^- vB/B}]_T = \sum_{w \in S_n} c_{u,v}^w(\mathbf{y},\mathbf{t}) \cdot [\overline{B^- wB/B}]_T.$$

Since $\tau\overline{B^- uB/B}$ is closed under $\tau N^-\tau^{-1}$ and $\overline{B^- vB/B}$ is closed under N^-, the intersection is closed under $N^- \cap \tau N^- \tau^{-1} =: N^-(\tau)$. By Corollary 2.5, we conclude that $c^w_{u,v}(\mathbf{y}, \mathbf{t}) \in \mathbb{N}[-\alpha]_{\alpha \in I(\tau)}$, where we compute $I(\tau) = \{y_j - t_i \mid 1 \le i, j \le n\}$. □

Example 2.8. Following [7, Lemma 3.3], setting $\mathbf{x} = \mathbf{t}$ in Equation (1.1), we get

$$c^{\mathrm{id}}_{u,v}(\mathbf{y}, \mathbf{t}) = \begin{cases} \mathfrak{S}_u(\mathbf{t}; \mathbf{y}), & v = \mathrm{id}, \\ 0, & v \neq \mathrm{id}. \end{cases}$$

See also [22, Section 5]. By Theorem 1.1, we have $\mathfrak{S}_w(\mathbf{x}; \mathbf{y}) \in \mathbb{N}[x_i - y_j]_{i,j \ge 1}$ and $\mathfrak{S}_w(\mathbf{x}) \in \mathbb{N}[x_i]_{i \ge 1}$. These could be seen directly from the (bumpless) pipe dreams model [5, 16]. For example, for $w \in S_3$:



2.4 A motivating example

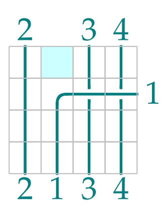
Let us give an illustrating example of the geometric argument. We will need the following double Schubert polynomials:

$$\mathfrak{S}_{s_1}(\mathbf{x}; \mathbf{y}) = x_1 - y_1, \qquad \mathfrak{S}_{s_2 s_1}(\mathbf{x}; \mathbf{y}) = (x_1 - y_1)(x_1 - y_2).$$

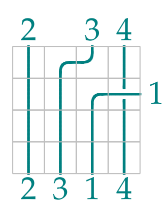
We can compute for $u = v = s_1$

$$\begin{aligned} \mathfrak{S}_{s_1}(\mathbf{x}; \mathbf{y}) \cdot \mathfrak{S}_{s_1}(\mathbf{x}; \mathbf{t}) &= (x_1 - y_1)(x_1 - t_1) = (x_1 - t_2 + t_2 - y_1)(x_1 - t_1) \\ &= (t_2 - y_1) \cdot \mathfrak{S}_{s_1}(\mathbf{x}; \mathbf{t}) + \mathfrak{S}_{s_2 s_1}(\mathbf{x}; \mathbf{t}). \end{aligned}$$

Combinatorially, $u = s_1$ and $v = s_1$ are of separated descents, and the coefficients can also be computed by pipe puzzles [7]



$u^{-1} = 2134$
$v^{-1} = 2134$
$w^{-1} = 2134$



$u^{-1} = 2134$
$v^{-1} = 2134$
$w^{-1} = 2314$

where the shadowed tile has weight $t_2 - y_1$ since it is at the $(1,2)$-entry.

To explain the geometric meaning, we will work with the flag variety Fl_4, and regard y_1, y_2 as equivariant parameters t_3, t_4. We have

$$\begin{aligned}
\mathfrak{S}_{s_1}(\mathbf{x};\mathbf{t}) \text{ represents } & \{V_\bullet \in Fl_4 \mid V_1 \subseteq \operatorname{span}(e_2,e_3,e_4)\}\,,\\
\mathfrak{S}_{s_1}(\mathbf{x};\mathbf{y}) \text{ represents } & \{V_\bullet \in Fl_4 \mid V_1 \subseteq \operatorname{span}(e_1,e_2,e_4)\}\,,\\
\mathfrak{S}_{s_2s_1}(\mathbf{x};\mathbf{t}) \text{ represents } & \{V_\bullet \in Fl_4 \mid V_1 \subseteq \operatorname{span}(e_3,e_4)\}\,.
\end{aligned}$$

As a result,

$$\mathfrak{S}_{s_1}(\mathbf{x};\mathbf{t}) \cdot \mathfrak{S}_{s_1}(\mathbf{x};\mathbf{y}) \text{ represents } \{V_\bullet \in Fl_4 \mid V_1 \subseteq \operatorname{span}(e_2,e_4)\}\,.$$

Let $W = \operatorname{span}(e_2,e_3) \cong \mathbb{P}^1$. Let us consider

$$\begin{tikzcd}
& M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
Fl_4 & & \mathbb{P}(W)
\end{tikzcd} \qquad M = \{(V_\bullet, [v]) \in Fl_4 \times \mathbb{P}(W) \mid V_1 \subseteq \operatorname{span}(v, e_4)\}\,.$$

Then

$$\begin{aligned}
\mathfrak{S}_{s_1}(\mathbf{x};\mathbf{t}) \cdot \mathfrak{S}_{s_1}(\mathbf{x};\mathbf{y}) \text{ represents } & Y := \pi_1(M_2) \subset Fl_4,\\
\mathfrak{S}_{s_1}(\mathbf{x};\mathbf{t}) \text{ represents } & Z_2 := \pi_1(M) \subset Fl_4,\\
\mathfrak{S}_{s_2s_1}(\mathbf{x};\mathbf{t}) \text{ represents } & Z_1 := \pi_1(M_3) \subset Fl_4,
\end{aligned}$$

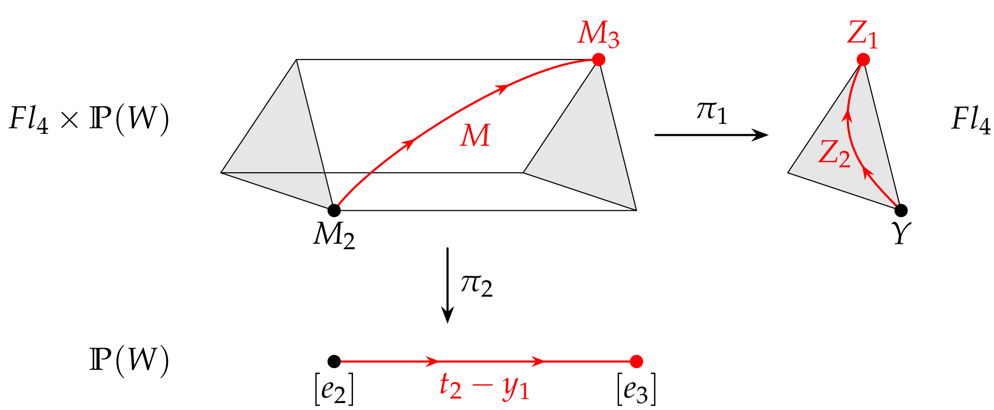
where $M_i \subset M$ is the fiber at the point $[e_i] \in \mathbb{P}(W)$ along π_2 for $i = 2, 3$. Over $\mathbb{P}(W)$,

$$\left.\begin{aligned}
x - t_2 \text{ represents the point } [e_2] \in \mathbb{P}(W)\\
x - y_1 \text{ represents the point } [e_3] \in \mathbb{P}(W)
\end{aligned}\right\} \implies \begin{array}{c} \text{in } H_T^*(\mathbb{P}(W)), \text{ we have} \\ [e_2] = (t_2 - y_1)[\mathbb{P}(W)] + [e_3] \end{array}$$

Applying $\pi_{1*}\pi_2^*$, we get

$$\mathfrak{S}_{s_1}(\mathbf{x};\mathbf{y}) \cdot \mathfrak{S}_{s_1}(\mathbf{x};\mathbf{t}) = (t_2 - y_1) \cdot \mathfrak{S}_{s_1}(\mathbf{x};\mathbf{t}) + \mathfrak{S}_{s_2s_1}(\mathbf{x};\mathbf{t}).$$

The following diagram illustrates the classes.



2.5 Skew divided difference operators

Next we recall some definitions on (skew) divided difference operators.

Definition 2.9 ([19]). The *skew divided difference operator* $\partial_{w/u}$ is characterized by

$$\partial_w(fg) = \sum_{v \in S_\infty} \partial_{w/v}(f)\partial_v(g)$$

for all polynomials f and g. Here, $\partial_w := \partial_{i_1} \cdots \partial_{i_\ell}$ for any reduced word $w = s_{i_1} \cdots s_{i_\ell}$, and $\partial_i(f) = (f - s_i f)/(x_i - x_{i+1})$ is the *divided difference operator*.

See [12, Definition 4] for a more explicit definition of $\partial_{w/u}$.

Proof of Corollary 1.2. By [22, Section 2 p. 5] or [7, Proposition 6.4],

$$\partial_{w/v}\mathfrak{S}_u(\mathbf{x};\mathbf{y}) = c^w_{u,v}(\mathbf{y},\mathbf{x}).$$

Setting $\mathbf{y} = \mathbf{0}$, we obtain that $\partial_{w/v}\mathfrak{S}_u(\mathbf{x}) = c^w_{u,v}(\mathbf{0},\mathbf{x}) \in \mathbb{N}[\mathbf{x}]$ by Theorem 1.1. $\square$

Example 2.10. When $v \lessdot w$ in Bruhat order, i.e. $w = vt_{ab}$ for $a < b$ with $\ell(w) = \ell(v) + 1$, by [19], we have

$$\partial_{w/v} f = \partial_{ab} f = \frac{f - f|_{x_a \leftrightarrow x_b}}{x_a - x_b}.$$

Kirillov [12, Section 8] established his conjecture in this special case. For example, the following table shows $\partial_{ab}\mathfrak{S}_w$ for $w \in S_3$ and $(a,b) = (1,3)$.

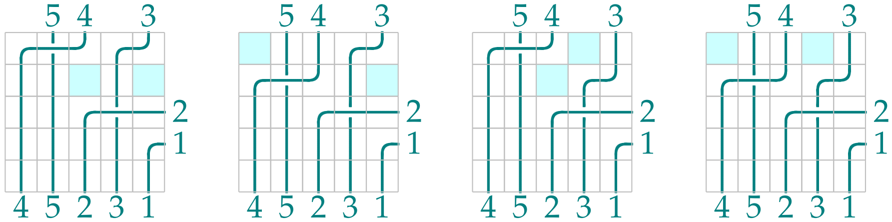
	321	231	312	213	132	123
$\mathfrak{S}_w$	$x_1^2 x_2$	$x_1 x_2$	x_1^2	x_1	$x_1 + x_2$	1
$\partial_{13}\mathfrak{S}_w$	$(x_1 + x_3)x_2$	x_2	$x_1 + x_3$	1	1	0

Note that $\partial_{ab}\mathfrak{S}_w$ is usually not Schubert positive.

Example 2.11. Let $u = 43125$, $v = 14532$ and $w = 53412$. We have

$$\max \mathrm{Des}(u) = 2 \le 3 = \min \mathrm{Des}(v).$$

There are four pipe puzzles [7] for the triple (u, v, w):



$$u^{-1} = 34215,$$
$$v^{-1} = 15423,$$
$$w^{-1} = 45231.$$

The corresponding formula gives:

$$\partial_{w/v}\mathfrak{S}_u(\mathbf{x}) = x_3x_5 + x_1x_5 + x_3x_4 + x_1x_4.$$

2.6 A positivity conjecture for Grothendieck polynomials

Let β be a formal variable and let $\mathfrak{G}_w(\mathbf{x};\mathbf{y})$ be the *double Grothendieck polynomial* for $w \in S_n$. We use the convention consistent with that of [17]. Consider the expansion

$$\mathfrak{G}_u(\mathbf{x};\mathbf{y}) \cdot \mathfrak{G}_v(\mathbf{x};\mathbf{t}) = \sum_{w \in S_\infty} \tilde{c}_{u,v}^{w}(\mathbf{y},\mathbf{t}) \cdot \mathfrak{G}_w(\mathbf{x};\mathbf{t}).$$

We propose the following conjecture, inspired by Theorem 1.1.

Conjecture 2.12. *For $u, v, w \in S_\infty$, $\tilde{c}_{u,v}^{w}(\mathbf{y},\mathbf{t}) \in \mathbb{N}[\beta][t_i \ominus y_j]_{i,j\geq 1}$, where $a \ominus b := \dfrac{a-b}{1+\beta b}$.*

This conjecture was confirmed when u and v are k-Grassmannian permutations by Wheeler and Zinn-Justin [24], and generally for permutations with separated descents by Fan, Guo, and the second author [7].

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